MAU23205 2021-2022 Practice Problem Set 3
Solutions

1. Verify that

$$W(t,s) = \begin{bmatrix} \cos(\log(t/s)) & s\sin(\log(t/s)) \\ -\frac{1}{t}\sin(\log(t/s)) & \frac{s}{t}\cos(\log(t/s)) \end{bmatrix}$$

is the fundamental solution for

$$A(t) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{t^2} & -\frac{1}{t} \end{bmatrix}$$

in the sense of Lecture 10.
Solution: As shown in Lecture 10 there is a unique W such that

(a) $\frac{\partial W}{\partial t}(t,s) = A(t)W(t,s)$,
(b) $W(t,s)$ is invertible, with $W(t,s)^{-1} = W(s,t)$.
(c) $W(t,t) = I$,
(d) $\frac{\partial W}{\partial s}(t,s) = -W(t,s)A(s)$,
(e) $W(t,s)W(s,r) = W(t,r)$, and

It therefore suffices to show that our W satisfies these conditions.

(a)

$$\begin{aligned} \frac{\partial W}{\partial t}(t,s) &= \begin{bmatrix} -\frac{1}{t}\sin(\log(t/s)) & \frac{s}{t}\cos(\log(t/s)) \\ \frac{1}{t^2}\sin(\log(t/s)) - \frac{1}{t^2}\cos(\log(t/s)) & -\frac{s}{t^2}\cos(\log(t/s)) - \frac{s}{t^2}\sin(\log(t/s)) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{t^2} & -\frac{1}{t} \end{bmatrix} \begin{bmatrix} \cos(\log(t/s)) & s\sin(\log(t/s)) \\ -\frac{1}{t}\sin(\log(t/s)) & \frac{s}{t}\cos(\log(t/s)) \end{bmatrix} \\ &= A(t)W(t,s). \end{aligned}$$

(b)

$$\begin{aligned} W(t,s)W(s,t) &= \begin{bmatrix} \cos(\log(t/s)) & s\sin(\log(t/s)) \\ -\frac{1}{t}\sin(\log(t/s)) & \frac{s}{t}\cos(\log(t/s)) \end{bmatrix} \begin{bmatrix} \cos(\log(s/t)) & t\sin(\log(s/t)) \\ -\frac{1}{s}\sin(\log(s/t)) & \frac{t}{s}\cos(\log(s/t)) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\log(t/s)) & s\sin(\log(t/s)) \\ -\frac{1}{t}\sin(\log(t/s)) & \frac{s}{t}\cos(\log(t/s)) \end{bmatrix} \begin{bmatrix} \cos(\log(t/s)) & -t\sin(\log(t/s)) \\ \frac{1}{s}\sin(\log(t/s)) & \frac{t}{s}\cos(\log(t/s)) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

So $W(t,s)$ is invertible and $W(s,t)$ is its inverse.

(c)

$$W(t,t) \begin{bmatrix} \cos(\log(t/t)) & t\sin(\log(t/t)) \\ -\frac{1}{t}\sin(\log(t/t)) & \frac{t}{t}\cos(\log(t/t)) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d)

$$
\begin{aligned}
\frac{\partial W}{\partial s}(t,s) &= \begin{bmatrix} \frac{1}{s}\sin(\log(t/s)) & \sin(\log(t/s)) - \cos(\log(t/s)) \\ \frac{1}{st}\cos(\log(t/s)) & \frac{1}{t}\cos(\log(t/s)) + \frac{1}{t}\sin(\log(t/s)) \end{bmatrix} \\
&= -\begin{bmatrix} \cos(\log(t/s)) & s\sin(\log(t/s)) \\ -\frac{1}{t}\sin(\log(t/s)) & \frac{s}{t}\cos(\log(t/s)) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\frac{1}{s^2} & -\frac{1}{s} \end{bmatrix} \\
&= -W(t,s)A(s).
\end{aligned}
$$

(e)

$$
\begin{aligned}
W(t,s)W(s,r) &= \begin{bmatrix} \cos(\log(t/s)) & s\sin(\log(t/s)) \\ -\frac{1}{t}\sin(\log(t/s)) & \frac{s}{t}\cos(\log(t/s)) \end{bmatrix} \begin{bmatrix} \cos(\log(s/r)) & r\sin(\log(s/r)) \\ -\frac{1}{s}\sin(\log(s/r)) & \frac{r}{s}\cos(\log(s/r)) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\log(t/s) + \log(s/r)) & r\sin(\log(t/s) + \log(s/r)) \\ -\frac{1}{t}\sin(\log(t/s) + \log(s/r)) & \frac{r}{t}\cos(\log(t/s) + \log(s/r)) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\log(t/r)) & s\sin(\log(t/r)) \\ -\frac{1}{t}\sin(\log(t/r)) & \frac{r}{t}\cos(\log(t/r)) \end{bmatrix} \\
&= W(t,r)
\end{aligned}
$$

2. Express the solution to the initial value problem $x(t_0) = x_0$ $x'(t_0) = y_0$ for the differential equation

$$t^2 x''(t) + t x'(t) + x(t) = f(t)$$

in terms of t_0, x_0, y_0 and f.
Note: You may use the previous problem, even if you didn't manage to solve it. Your solution will involve an integral. Rewatch Lecture 9 if you don't know where to start.
Solution: Set $y = x'$, so $x'(t) = y(t)$ and $y'(t) = -\frac{1}{t}^2 x(t) - \frac{1}{t}y(t) + \frac{1}{t^2}f(t)$, or, in matrix form,

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{t^2} & -\frac{1}{t} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{t^2}f(t) \end{bmatrix}.$$

This is of the form $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{g}(t)$ so according to Lecture 9 the solution to the initial value problem $\mathbf{x}(t_0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = W(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} W(t,s)\mathbf{g}(s)$$

where W is the fundamental solution corresponding to A. This was given in the previous problem as

$$W(t,s) = \begin{bmatrix} \cos(\log(t/s)) & s\sin(\log(t/s)) \\ -\frac{1}{t}\sin(\log(t/s)) & \frac{s}{t}\cos(\log(t/s)) \end{bmatrix}$$

so

$$
\begin{aligned}
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} \cos(\log(t/t_0)) & t_0\sin(\log(t/t_0)) \\ -\frac{1}{t}\sin(\log(t/t_0)) & \frac{t_0}{t}\cos(\log(t/t_0)) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\
&\quad + \int_{t_0}^{t} \begin{bmatrix} \cos(\log(t/s)) & s\sin(\log(t/s)) \\ -\frac{1}{t}\sin(\log(t/s)) & \frac{s}{t}\cos(\log(t/s)) \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{s^2}f(s) \end{bmatrix} ds.
\end{aligned}
$$

We only care about the first row

$$x(t) = x_0 \cos(\log(t/t_0)) + y_0 t_0 \sin(\log(t/t_0)) + \int_{t_0}^{t} \frac{1}{s} \sin(\log(t/s)) f(s)\, ds.$$

3. Find a set of basic solutions to the equation

$$x^{(7)} + 3x^{(6)} + 5x^{(5)} + 7x^{(4)} + 7x''' + 5x'' + 3x' + x = 0.$$

Note:

$$z^7 + 3z^6 + 5z^5 + 7z^4 + 7z^3 + 5z^2 + 3z + 1 = (z+1)^3(z^2+1)^2.$$

Solution: The complex roots are -1 with multiplicity 3 and and $\pm i$ with multiplicity 2. The basic solutions are therefore $\exp(-t)$, $t\exp(-t)$, $\frac{t^2}{2}\exp(-t)$, $\cos(t)$, $\sin(t)$, $t\cos(t)$ and $t\sin(t)$.