MAU23205 2021-2022 Practice Problem Set 2

Solutions

1. The existence and uniqueness theorem shows that the initial value problem

$$x(t_0) = x_0 \qquad y(t_0) = y_0 \qquad z(t_0) = z_0$$

for the system

$$x'(t) = y(t)z(t) \qquad y'(t) = -x(t)z(t) \qquad z'(t) = -k^2x(t)z(t)$$

has, for each k, a unique solution which depends continuously not only on t but also on the initial values t_0, x_0, y_0 and z_0. Explain how to use the existence and uniqueness theorem to get continuous dependence on k as well.

Hint: Apply it to a larger system.

Solution: We just replace it by the equivalent initial value problem

$$w(t_0) = k \qquad x(t_0) = x_0 \qquad y(t_0) = y_0 \qquad z(t_0) = z_0$$

$$w'(t) = 0 \quad x'(t) = y(t)z(t) \quad y'(t) = -x(t)z(t) \quad z'(t) = -w(t)^2x(t)z(t)$$

The initial value problem for this extended system has a unique solution which depends continuously on its initial values, one of which is the value of k. You get a solution to the original system by ignoring w, but the remaining functions x, y and z still depend continuously on k.

2. Find numerical approximations to the system

$$x'(t) = -y(t) \qquad y'(t) = x(t)$$

using the second order predictor-corrector method instead of Euler. Do the approximate solutions remain bounded at t tends to infinity for a fixed step size h?

Solution: The prediction step is

$$\hat{x}_{j+1} = \tilde{x}_j - h\tilde{y}_j, \qquad \hat{y}_{j+1} = \tilde{y}_j + h\tilde{x}_j.$$

The correction step is

$$\tilde{x}_{j+1} = \tilde{x}_j - \frac{h}{2}\tilde{y}_j - \frac{h}{2}\hat{y}_{j+1}, \qquad \tilde{y}_{j+1} = \tilde{y}_j + \frac{h}{2}\tilde{x}_j + \frac{h}{2}\hat{x}_{j+1}.$$

Substituting the previous equations into these,

$$\tilde{x}_{j+1} = \left(1 + \frac{h^2}{2}\right)\tilde{x}_j - h\tilde{y}_j, \qquad \tilde{y}_{j+1} = \left(1 + \frac{h^2}{2}\right)\tilde{y}_j + h\tilde{x}_j.$$

From this it follows that

$$\tilde{x}_{j+1}^2 + \tilde{y}_{j+1}^2 = \left(1 + \frac{h^4}{4}\right)\left(\tilde{x}_j^2 + \tilde{y}_j^2\right)$$

so the solutions cannot be bounded as j tends to infinity for fixed h, which is the same ast t tending to infinity.

3. Suppose that A is an $n \times n$ matrix.

   (a) Show that A and A^T have the same characteristic polynomial.
   Solution: The determinant of a matrix is the same as the determinant of its transpose, so

   $$p_{A^T}(z) = \det\left(zI - A^T\right) = \det\left((zI - A)^T\right) = \det\left(zI - A\right) = p_A(z).$$

   (b) Show that A and A^T have the same minimal polynomial.
   Solution: Suppose that

   $$p(z) = z^m + \alpha_{m-1} z^{m-1} + \cdots + c_1 z + c_0$$

   is the minimal polynomial of A. Then

   $$p(A) = A^m + \alpha_{m-1} A^{m-1} + \cdots + c_1 A + c_0 I = O.$$

   Taking the transpose and using the fact the transpose of a sum is the sum of the transposes,

   $$p(A) = (A^m)^T + \alpha_{m-1}\left(A^{m-1}\right)^T + \cdots + c_1 A^T + c_0 I^T = O^T.$$

   The zero matrix and the identity matrix are unchanged by taking transposes, so we can replace O^T and I^T above by O and I. Doing this and using the fact that the transpose of a product is the product of the transposes in reverse order,

   $$\left(A^T\right)^m + \alpha_{m-1}\left(A^T\right)^{m-1} + \cdots + c_1\left(A^T\right) + c_0 I = O.$$

   The left hand side is $p\left(A^T\right)$ so

   $$p\left(A^T\right) = O.$$

   From this it doesn't follow immediately that p is the minimal polynomial of A^T, merely that it is a multiple of the minimal polynomial of A^T. In other words, if q is the minimal polynomial of A^T then p/q is not just a rational function but actually a polynomial.
   On the other hand the same argument with A and A^T swapped and p and q swapped shows that q/p is a polynomial. The only polynomials whose reciprocals are also polynomials are the non-zero constants. Since minimal polynomials are by definition monic the non-zero constant p/q must be 1. In other words, $p = q$.