MAU23205 2021-2022 Practice Problem Set 1
Solutions

1. Assume that $\alpha, \beta, \gamma, \delta > 0$. Show that

$$I = \left(\frac{\delta x}{\gamma}\right)^{-\sqrt{\gamma/\alpha}} \left(\frac{\beta y}{\alpha}\right)^{-\sqrt{\alpha/\gamma}} \exp\left(\frac{\delta x + \beta y}{\sqrt{\alpha\gamma}}\right)$$

is an invariant of the system

$$\frac{dx}{dt} = \alpha x - \beta xy \qquad \frac{dy}{dt} = -\gamma y + \delta xy$$

Solution: Differentiating,

$$\begin{aligned}
\frac{dI}{dt} &= -\frac{\delta}{\sqrt{\alpha\gamma}} \left(\frac{\delta x}{\gamma}\right)^{-1-\sqrt{\gamma/\alpha}} \left(\frac{\beta y}{\alpha}\right)^{-\sqrt{\alpha/\gamma}} \exp\left(\frac{\delta x + \beta y}{\sqrt{\alpha\gamma}}\right) \frac{dx}{dt} \\
&\quad - \frac{\beta}{\sqrt{\alpha\gamma}} \left(\frac{\delta x}{\gamma}\right)^{-\sqrt{\gamma/\alpha}} \left(\frac{\beta y}{\alpha}\right)^{-1-\sqrt{\alpha/\gamma}} \exp\left(\frac{\delta x + \beta y}{\sqrt{\alpha\gamma}}\right) \frac{dy}{dt} \\
&\quad + \left(\frac{\delta x}{\gamma}\right)^{-\sqrt{\gamma/\alpha}} \left(\frac{\beta y}{\alpha}\right)^{-\sqrt{\alpha/\gamma}} \exp\left(\frac{\delta x + \beta y}{\sqrt{\alpha\gamma}}\right) \frac{\delta}{\sqrt{\alpha\gamma}} \frac{dx}{dt} \\
&\quad + \left(\frac{\delta x}{\gamma}\right)^{-\sqrt{\gamma/\alpha}} \left(\frac{\beta y}{\alpha}\right)^{-\sqrt{\alpha/\gamma}} \exp\left(\frac{\delta x + \beta y}{\sqrt{\alpha\gamma}}\right) \frac{\beta}{\sqrt{\alpha\gamma}} \frac{dy}{dt} \\
&= \frac{I}{\sqrt{\alpha\gamma}} \left(\frac{\delta x - \gamma}{x}\frac{dx}{dt} + \frac{\beta y - \alpha}{y}\frac{dy}{dt}\right) \\
&= \frac{I}{\sqrt{\alpha\gamma}} \left((\delta x - \gamma)(\alpha - \beta y) + (\beta y - \alpha)(-\gamma + \delta x)\right) = 0
\end{aligned}$$

2. The differential equation

$$\frac{dy}{dx} = -\frac{6x + 2y + 5}{2x + 2y + 4}$$

has a quadratic invariant, i.e. an invariant of the form

$$I = ax^2 + bxy + cy^2 + dx + ey + f.$$

   (a) In fact there are infinitely many quadratic invariants, but find at least one non-zero invariant.
   Solution: By the usual rules for differentiation,

$$\frac{dI}{dx} = 2ax + by + bx\frac{dy}{dx} + 2cy\frac{dy}{dx} + d + e\frac{dy}{dx} = (bx + 2cy + e)\frac{dy}{dx} + 2ax + by + d$$

This will clearly be zero if

$$\frac{dy}{dx} = -\frac{2ax + by + d}{bx + 2cy + e}.$$

Examining the differential equation, we see that choosing

$$a = 3 \quad b = 2 \quad d = 5 \quad c = 1 \quad e = 4$$

accomplishes this. Any multiple of these values would work equally well, provided the same multiple is chosen for all of them. There are no restrictions on f so we might as well choose $f = 0$. This leads to the invariant

$$I(x, y) = 3x^2 + 2xy + y^2 + 5x + 4y.$$

(b) Use this invariant to solve the initial value problem

$$y(0) = -4.$$

Solution: Since I is invariant

$$I(x, y(x)) = I(0, y(0)) = I(0, -4) = 0.$$

So $y(x)$ satisfies the quadratic equation

$$3x^2 + 2xy(x) + y(x)^2 + 5x + 4y(x) = 0.$$

A better way to write this equation is by grouping the powers of $y(x)$:

$$y(x)^2 + (2x + 4)y(x) + 3x^2 + 5x = 0.$$

You can solve this by the quadratic formula or by completing the square. Completing the square gives

$$[y(x) + x + 2]^2 + 2x^2 + x - 4 = 0$$

or

$$(y(x) + x + 2)^2 + 2\left(x + \frac{1}{4}\right)^2 + x - \frac{31}{8} = 0.$$

This has a solution only when

$$\left|x + \frac{1}{4}\right| \leq \frac{\sqrt{31}}{4},$$

i.e. when

$$-\frac{1}{4} - \frac{\sqrt{31}}{4} \leq x \leq -\frac{1}{4} + \frac{\sqrt{31}}{4}.$$

In that case

$$y(x) + x + 2 = \pm\sqrt{4 - x - 2x^2}.$$

The initial conditions force us to choose the minus sign, so

$$y(x) = -x - 2 - \sqrt{4 - x - 2x^2}.$$

This solution is only valid for

$$-\frac{1}{4} - \frac{\sqrt{31}}{4} < x < -\frac{1}{4} + \frac{\sqrt{31}}{4}.$$

because the function fails to be differentiable at the endpoints of the interval.