Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Science, Technology, Engineering and Mathematics

School of Mathematics

SF Maths **Semester 1 2021-2022**
JS TSM

MAU23205 Ordinary Differential Equations

Monday 13 December 2021 RDS Simmonscourt 14:00 — 16:00

Prof. John Stalker

Instructions to candidates:

Additional instructions for this examination:

Credit will be given for the best 3 questions answered.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (20 points)

   (a) (4 points) What is the order of the system

   $$x'(t) = x(t) - x(t)y(t)? \qquad y'(t) = x(t)y(t) - y(t)$$

   Is it autonomous? Is it linear? If so, is it homogeneous?

   (b) (6 points) Show that

   $$V(x, y) = xy \exp(-x - y)$$

   is an invariant of the system above.

   (c) (10 points) Prove that all solutions to this system with initial conditions $x_0, y_0 > 0$ are bounded.

2. (20 points)

   (a) (10 points) Compute $\exp(tA)$ where

$$A = \begin{bmatrix} -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & 2 \\ 4 & -2 & -1 & 4 \\ 2 & -2 & -1 & 3 \end{bmatrix}.$$

   Hint: What is A^2?

   (b) (10 points) Do all solutions to

$$x'''(t) + 2x''(t) + 2x'(t) + 3x(t) = 0$$

   tend to zero as t tends to $+\infty$? Why or why not?

3. (20 points) The Legendre equation of order two

$$(1-x^2)y''(x) - 2xy'(x) + 6y(x) = 0$$

has a non-zero polynomial solution of degree 2.

(a) (5 points) Find such a solution.

(b) (5 points) Find an equivalent first order system.

(c) (10 points) Find the Green's function for this equation.

4. (20 points) All of these questions relate to the autonomous system

$$x'(t) = x(t)^2 + y(t)^2 - 2x(t) \qquad y'(t) = x(t)^2 + y(t)^2 - 2y(t)$$

   (a) (2 points) Find the equilibria.

   (b) (4 points) Find the linearisation at each equilibrium.

   (c) (6 points) Which equilibria, if any, are stable? Which, if any, are strictly stable?

   (d) (8 points) Find a strict Lyapunov function for any strictly stable equilibria.