MAU23205 2021-2022 Assignment 1 Due 11 October 2021

Solutions

1. The system

$$\frac{dx}{dt} = x - y - x^2 - 2xy - y^2 \qquad \frac{dy}{dt} = x - y + x^2 + 2xy + y^2$$

has an invariant which is a cubic polynomial in x and y. Find it.

Solution: The general cubic polynomial in x and y is

$$U(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j.$$

By the chain rule

$$\frac{d}{dt}U(x(t), y(t)) = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt}.$$

$$\frac{\partial U}{\partial x} = 3ax^2 + 2bxy + cy^2 + 2ex + fy + h$$

and

$$\frac{\partial U}{\partial y} = bx^2 + 2cxy + 3dy^2 + fx + 2gy + i$$

while dx/dt and dy/dt are given by the differential equations. Substituting,

$$\begin{aligned}\frac{d}{dt}U(x(t), y(t)) = {} & r_{4,0}x^4 + r_{3,1}x^3y + r_{2,2}x^2y^2 + r_{1,3}xy^3 + r_{0,4}y^4 \\ & + r_{3,0}x^3 + r_{2,1}x^2y + r_{1,2}xy^2 + r_{0,3}y^3 \\ & + r_{2,0}x^2 + r_{1,1}xy + r_{0,2}y^2 + r_{1,0}x + r_{0,1}y + r_{0,0}\end{aligned}$$

where

$$\begin{aligned}
r_{4,0} &= -3a + b & r_{3,1} &= -6a + 2c \\
r_{2,2} &= -3a - 3b + 3c + 3d & r_{1,3} &= -2b + 6d \\
r_{0,4} &= -c + 3d & r_{3,0} &= 3a + b - 2e + f \\
r_{2,1} &= -3a + b + 2c - 4e + f + 2g & r_{1,2} &= -2b - c + 3d - 2e - f + 4g \\
r_{0,3} &= -c - 3d - f + 2g & r_{2,0} &= 2e + f - h + i \\
r_{1,1} &= -2e + 2g - 2h + 2i & r_{0,2} &= -f - 2g - h + i \\
r_{1,0} &= h + i & r_{0,1} &= -h - i \\
r_{0,0} &= 0
\end{aligned}$$

For U to be an invariant we should have $\frac{d}{dt}U(x(t), y(t)) = 0$, so we want all these coefficients $r_{i,j}$ to be zero. That gives us 15 equations for the ten quantities a, b, $\ldots$, j, but many of them are redundant. From the first and second equations we get $b = 3a$ and $c = 3a$. The third equation then implies $d = a$. The fourth and fifth equations give no new information. The sixth equation then tells us that $f = -6a + 2e$. Substituting the values obtained so far into the seventh equation we find $g = e$. The eighth and ninth equations give no new information. Shifting to the other end, the fifteenth equation obviously gives no new information while the fourteenth tells us that $i = -h$. The thirteenth equation gives no new information. We have three equations left, the tenth through twelfth. Substituting what we've obtained so far into those equations gives the equations $-6a + 4e - 2h = 0$, $-4h = 0$ and $6a - 4e - 2h = 0$. From these we see that $h = 0$ and $e = \frac{3}{2}a$. The various coefficients in U are therefore $b = 3a$, $c = 2a$, $d = a$, $e = \frac{3}{2}a$, $f = -3a$, $g = \frac{3}{2}a$, $h = 0$ and $i = 0$. a and j can be chosen arbitrarily, except that we have to choose $a \neq 0$ to get a genuine cubic rather than just a constant. We may as well choose $j = 0$ for simplicity and $a = 2$ to avoid denominators. This gives

$$U = 2x^3 + 6x^2y + 6xy^2 + 2y^z + 3x^2 - 6xy + 3y^2.$$

2. The differential equation

$$p''(z) = 6p(z)^2 - \frac{1}{2}g_2$$

arises in the theory of elliptic functions. Do not attempt to solve it. Instead use the general existence and uniqueness theorem to show that solutions to the initial value problem $p(z_0) = p_0$, $p'(z_0) = q_0$ depend continuously on p_0, q_0 *and* g_2.
Hint: This requires replacing the equation with an appropriate system.
Solution: We apply the existence and uniqueness theorem to the initial value problem

$$p(z_0) = p_0, \qquad q(z_0) = q_0 \qquad r(z_0) = g_2$$

for the system

$$p'(z) = q, \qquad q'(z) = 6p(z)^2 - \frac{1}{2}r(z), \qquad r'(z) = 0.$$

Any solution to this problem gives a solution to the original one by forgetting q and r. This solution depends continuously on p_0, q_0 and g_2 because they are initial conditions in the initial value problem for the system.

3. Rewrite

$$tx''(t) + (1-t)x'(t) + \lambda x(t) = f(t)$$

as $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{g}(t)$ for appropriate A and $\mathbf{g}$.
Solution: Divide by t and write $\mathbf{x} = \begin{bmatrix} x \\ x' \end{bmatrix}$. Then

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ -\frac{\lambda}{t} & \frac{t-1}{t} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ \frac{f(t)}{t} \end{bmatrix}.$$