MAU23205 Lecture 30

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An example (1/3)

Last time we showed existence of solutions to the initial value problem $\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}(t), \mathbf{z})$, $\mathbf{x}(t_0) = \mathbf{x}_0$ if $\mathbf{F}$ is continuous. The following example shows that we can't expect uniqueness or continuous dependence on initial conditions without more assumptions on $\mathbf{F}$.

$$x'(t) = |x(t)|^{1/2}$$

is a separable first order scalar equation. For $x > 0$ we can write it as

$$\frac{dx}{x^{1/2}} = dt.$$

Integrating both sides,

$$2x^{1/2} - 2x_0^{1/2} = t - t_0$$

so

$$x(t) = \left(x_0^{1/2} - \frac{1}{2} t_0 + \frac{1}{2} t \right)^2$$

This solution is valid for all $t \geq t_0$

An example (2/3)

For $x < 0$ we have

$$\frac{dx}{(-x)^{1/2}} = dt$$
$$-2(-x)^{1/2} + 2(-x_0)^{1/2} = t - t_0$$
$$x(t) = -\left((-x_0)^{1/2} + \frac{1}{2}t_0 - \frac{1}{2}t\right)^2$$

This function is defined for all $t \geq t_0$ but is only a valid solution for $t_0 \leq t \leq t_0 + 2(-x_0)^{1/2}$. We can extend it from there to all $t \geq t_0$ though by setting it equal to 0 for all $t > t_0 + 2(-x_0)^{1/2}$. This works because both $x(t)$ and $x'(t)$ approach the same limit as we approach $t_0 + 2(-x_0)^{1/2}$ from either the left or the right.

An example (3/3)

For $x_0 = 0$ we have the solution $x(t) = 0$, but also the solutions

$$x(t) = \begin{cases} 0 & \text{if } t_0 \leq t \leq u, \\ \left(\frac{1}{2}t - \frac{1}{2}u\right)^2 & \text{if } t > 0 \end{cases}$$

for all $u \geq t_0$. So solutions aren't unique. They also don't depend continuously on the initial conditions. As a function of x_0 we have

$$x(t_0 + \tau) = \begin{cases} \left(x_0^{1/2} + \frac{1}{2}\tau\right)^2 & \text{if } x_0 > 0 \\ 0 & \text{if } -\frac{1}{4}\tau^2 \leq x_0 < 0 \\ -\left((-x_0)^{1/2} - \frac{1}{2}\tau\right)^2 & \text{if } x_0 < -\frac{1}{4}\tau^2 \end{cases}$$

The value of $x(t_0 + \tau)$ for $x_0 = 0$ depends on the choice of the solution. We can arrange for it to have any value between 0 and $\frac{1}{4}\tau^2$. No matter what value we choose, $x(t_0 + \tau)$ will not be a continuous function of x_0 near 0.

Cauchy sequences and completeness

A sequence $\alpha\colon \mathbf{N} \to Z$ where Z is a metric space with metric d_Z is called a *Cauchy sequence* if for all $\epsilon > 0$ there is a $k \in \mathbf{N}$ such that for all $m, n \geq k$ we have

$$d_Z(\alpha_m, \alpha_n) < \epsilon.$$

Convergent sequences are always Cauchy sequences. If all Cauchy sequences are convergent then Z is said to be *complete*.
Examples: $\mathbf{R}$ is complete but $\mathbf{Q}$ is not.
Suppose X and Y are metric spaces and Y is complete. Let Z be the set of bounded continuous functions from X to Y with the sup metric. Then Z is complete. This is not obvious.

The Banach Fixed Point Theorem

If Z is a metric space then a function $\varphi \colon Z \to Z$ is said to be a *contraction* if there is a constant $c < 1$ such that

$$d_Z(\varphi(x), \varphi(y)) \leq c d_Z(x, y)$$

for all $x, y \in Z$. Contractions are continuous, with $\delta = \epsilon$. The Banach Fixed Point Theorem is the following:

> *If Z is a complete metric space and $\varphi \colon Z \to Z$ is a contraction then there is a unique $z \in Z$ such that $\varphi(z) = z$.*

The proof is fairly easy. Choose any $a \in Z$ and define a sequence α by $\alpha_0 = a$, $\alpha_{j+1} = \varphi(\alpha_j)$. α is Cauchy and hence converges to some $z \in Z$. Taking limits in $\alpha_{j+1} = \varphi(\alpha_j)$ gives $z = \varphi(z)$. That gives the existence of a fixed point. To get the uniqueness, assume $\varphi(w) = w$ and $\varphi(z) = z$. Then

$$d_Z(w, z) = d_Z(\varphi(w), \varphi(z)) \leq c d_Z(w, z)$$

so $(1 - c) d_Z(w, z) \leq 0$. $1 - c > 0$ and $d_Z(w, z) \geq 0$ so $d_Z(w, z) = 0$. Then $w = z$.

The function $\Phi_{t_0,\mathbf{x}_0,\mathbf{z}}$

Suppose $\mathbf{F}$ is not just continuous but continuously differentiable in $B(\underline{t}, r_t) \times B(\underline{\mathbf{x}}, r_x) \times B(\underline{\mathbf{z}}, r_z)$. Let $M = \max_K \|\mathbf{F}'\|$ with $K = \bar{B}(\underline{t}, 2\delta_t) \times \bar{B}(\underline{\mathbf{x}}, 2\delta_x) \times \bar{B}(\underline{\mathbf{z}}, \delta_z)$, as before. Set $\tau = \min(\delta_t, \delta_x/L, 1/2M)$ and let $\mathcal{B}$ be the set of continuous functions $\mathbf{y}$ from $[0, \tau]$ to $\bar{B}(\mathbf{0}, \delta_x)$. Define $\Phi_{t_0,x_0,z} \colon \mathcal{B} \to \mathcal{B}$ by

$$\Phi_{t_0,x_0,z}(\mathbf{y})(u) = \int_0^u \mathbf{F}(s + t_0, \mathbf{y}(s) + \mathbf{x}_0, \mathbf{z})\, ds$$

for $(t_0, \mathbf{x}_0, \mathbf{z}) \in D = \bar{B}(\underline{t}, \delta_t) \times \bar{B}(\underline{\mathbf{x}}, \delta_x) \times \bar{B}(\underline{\mathbf{z}}, \delta_z)$. Then

$$\begin{aligned}\Phi_{t_0,x_0,z}(\mathbf{y}) = \mathbf{y} \quad &\Leftrightarrow \quad \mathbf{y}(u) = \int_0^u \mathbf{F}(s + t_0, \mathbf{y}(s) + \mathbf{x}_0, \mathbf{z})\, ds \\ &\Leftrightarrow \quad \mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}(t), \mathbf{z}) \quad \mathbf{x}(t_0) = \mathbf{x}_0\end{aligned}$$

where $\mathbf{x}$ and $\mathbf{y}$ are related by

$$\mathbf{x}(t) = \mathbf{y}(t - t_0) + \mathbf{x}_0, \qquad \mathbf{y}(s) = \mathbf{x}(s + t_0) - \mathbf{x}_0.$$

A useful inequality (1/3)

To get existence *and uniqueness* it suffices to show that $\Phi_{t_0,\mathrm{x}_0,\mathrm{z}}$ is a contraction on $\mathcal{B}$.

$$\Phi_{t_0^\flat,\mathrm{x}_0^\flat,\mathrm{z}^\flat}(\mathbf{y}^\flat)(u) - \Phi_{t_0^\sharp,\mathrm{x}_0^\sharp,\mathrm{z}^\sharp}(\mathbf{y}^\sharp)(u) = \tilde{\Phi}(1,u) - \tilde{\Phi}(0,u)$$

where

$$\begin{aligned}
\tilde{\Phi}(p,u) &= \int_0^u \tilde{\mathbf{F}}(p,s)\,ds \\
\tilde{\mathbf{F}}(p,s) &= \mathbf{F}(\tilde{t}(p,s), \tilde{\mathbf{x}}(p,s), \tilde{\mathbf{z}}(p,s)) \\
\tilde{t}(p,s) &= s + pt_0^\flat + (1-p)t_0^\sharp \\
\tilde{\mathbf{x}}(p,s) &= p\mathbf{y}^\flat(s) + (1-p)\mathbf{y}^\sharp(s) + p\mathbf{x}_0^\flat + (1-p)\mathbf{x}_0^\sharp \\
\tilde{\mathbf{z}}(p,s) &= p\mathbf{z}^\flat + (1-p)\mathbf{z}^\sharp
\end{aligned}$$

A useful inequality (2/3)

$$
\begin{aligned}
\frac{\partial \tilde{\Phi}}{\partial p}(p, u) &= \int_0^u \frac{\partial \tilde{\mathbf{F}}}{\partial p}(p, s)\, ds \\
\frac{\partial \tilde{\mathbf{F}}}{\partial p}(p, s) &= \frac{\partial \mathbf{F}}{\partial t}(\tilde{t}(p, s), \tilde{\mathbf{x}}(p, s), \tilde{\mathbf{z}}(p, s))\frac{\partial \tilde{t}}{\partial p}(p, s) \\
&\quad + \frac{\partial \mathbf{F}}{\partial \mathbf{x}}(\tilde{t}(p, s), \tilde{\mathbf{x}}(p, s), \tilde{\mathbf{z}}(p, s))\frac{\partial \tilde{\mathbf{x}}}{\partial p}(p, s) \\
&\quad + \frac{\partial \mathbf{F}}{\partial \mathbf{z}}(\tilde{t}(p, s), \tilde{\mathbf{x}}(p, s), \tilde{\mathbf{z}}(p, s))\frac{\partial \tilde{\mathbf{z}}}{\partial p}(p, s) \\
\frac{\partial \tilde{t}}{\partial p}(p, s) &= t_0^\flat - t_0^\sharp \\
\frac{\partial \tilde{\mathbf{x}}}{\partial p}(p, s) &= \mathbf{y}^\flat(s) - \mathbf{y}^\sharp(s) + \mathbf{x}_0^\flat - \mathbf{x}_0^\sharp \\
\frac{\partial \tilde{\mathbf{z}}}{\partial p}(p, s) &= \mathbf{z}^\flat - \mathbf{z}^\sharp
\end{aligned}
$$

A useful inequality (3/3)

$$\left\| \frac{\partial \tilde{\mathbf{F}}}{\partial p}(p, s) \right\| \leq M \|(t_0^\flat, \mathbf{x}_0^\flat, \mathbf{z}^\flat) - (t_0^\sharp, \mathbf{x}_0^\sharp, \mathbf{z}^\sharp)\| + M \|\mathbf{y}^\flat(s) - \mathbf{y}^\sharp(s)\|$$

$$\left\| \frac{\partial \tilde{\Phi}}{\partial p}(p, u) \right\| \leq M\tau \|(t_0^\flat, \mathbf{x}_0^\flat, \mathbf{z}^\flat) - (t_0^\sharp, \mathbf{x}_0^\sharp, \mathbf{z}^\sharp)\| + M\tau \sup_{s \in [0,\tau]} \|\mathbf{y}^\flat(s) - \mathbf{y}^\sharp(s)\|$$

$\tilde{\Phi}(1, u) - \tilde{\Phi}(0, u) = \int_0^1 \frac{\partial \tilde{\Phi}}{\partial p}(p, u)\, dp$ so

$$\begin{aligned}
&\sup_{u \in [0,\tau]} \|\Phi_{t_0^\flat, x_0^\flat, z_0^\flat}(\mathbf{y}^\flat)(u) - \Phi_{t_0^\sharp, x_0^\sharp, z_0^\sharp}(\mathbf{y}^\sharp)(u)\| \\
&\quad = \sup_{u \in [0,\tau]} \|\tilde{\Phi}(1, u) - \tilde{\Phi}(0, u)\| \\
&\quad \leq M\tau \|(t_0^\flat, \mathbf{x}_0^\flat, \mathbf{z}^\flat) - (t_0^\sharp, \mathbf{x}_0^\sharp, \mathbf{z}^\sharp)\| \\
&\qquad + M\tau \sup_{s \in [0,\tau]} \|\mathbf{y}^\flat(s) - \mathbf{y}^\sharp(s)\|
\end{aligned}$$

Existence and uniqueness

$$d\left(\Phi_{t_0^\flat,\mathrm{x}_0^\flat,\mathrm{z}_0^\flat}(\mathbf{y}^\flat),\Phi_{t_0^\sharp,\mathrm{x}_0^\sharp,\mathrm{z}_0^\sharp}(\mathbf{y}^\sharp)\right)\le\frac{1}{2}\|(t_0^\flat,\mathbf{x}_0^\flat,\mathbf{z}^\flat)-(t_0^\sharp,\mathbf{x}_0^\sharp,\mathbf{z}^\sharp)\|+\frac{1}{2}d(\mathbf{y}^\flat,\mathbf{y}^\sharp)$$

If $(t_0^\flat,\mathbf{x}_0^\flat,\mathbf{z}^\flat)=(t_0^\sharp,\mathbf{x}_0^\sharp,\mathbf{z}^\sharp)=(t_0,\mathbf{x}_0,\mathbf{z})$ then this reduces to

$$d\left(\Phi_{t_0,\mathrm{x}_0,\mathrm{z}_0}(\mathbf{y}^\flat),\Phi_{t_0,\mathrm{x}_0,\mathrm{z}_0}(\mathbf{y}^\flat)\right)\le\frac{1}{2}d(\mathbf{y}^\flat,\mathbf{y}^\sharp)$$

which shows that $\Phi_{t_0,\mathrm{x}_0,\mathrm{z}_0}$ is a contraction. The Banach Fixed Point Theorem therefore gives the existence and uniqueness of solutions to the initial value problem.

Continuous dependence on initial conditions, parameters

To get continuous dependence on initial conditions and parameters, let $\mathbf{x}^\flat$ be the solution to the IVP for initial conditions and parameters $(t_0^\flat, \mathbf{x}_0^\flat, \mathbf{z}^\flat)$ and $\mathbf{x}^\sharp$ the solution corresponding to $(t_0^\sharp, \mathbf{x}_0^\sharp, \mathbf{z}^\sharp)$. In other words, $\mathbf{y}^\flat = \Phi_{t_0^\flat, x_0^\flat, z_0^\flat}(\mathbf{y}^\flat)$, $\mathbf{y}^\sharp = \Phi_{t_0^\sharp, x_0^\sharp, z_0^\sharp}(\mathbf{y}^\sharp)$.

$$d\left(\Phi_{t_0^\flat, x_0^\flat, z_0^\flat}(\mathbf{y}^\flat), \Phi_{t_0^\sharp, x_0^\sharp, z_0^\sharp}(\mathbf{y}^\sharp)\right) \le \frac{1}{2}\|(t_0^\flat, \mathbf{x}_0^\flat, \mathbf{z}^\flat) - (t_0^\sharp, \mathbf{x}_0^\sharp, \mathbf{z}^\sharp)\| + \frac{1}{2}d(\mathbf{y}^\flat, \mathbf{y}^\sharp)$$

becomes

$$d(\mathbf{y}^\flat, \mathbf{y}^\sharp) \le \frac{1}{2}\|(t_0^\flat, \mathbf{x}_0^\flat, \mathbf{z}^\flat) - (t_0^\sharp, \mathbf{x}_0^\sharp, \mathbf{z}^\sharp)\| + \frac{1}{2}d(\mathbf{y}^\flat, \mathbf{y}^\sharp)$$

so

$$d(\mathbf{y}^\flat, \mathbf{y}^\sharp) \le \|(t_0^\flat, \mathbf{x}_0^\flat, \mathbf{z}^\flat) - (t_0^\sharp, \mathbf{x}_0^\sharp, \mathbf{z}^\sharp)\|.$$

Therefore $d(\mathbf{x}^\flat, \mathbf{x}^\sharp) \le \|(t_0^\flat, \mathbf{x}_0^\flat, \mathbf{z}^\flat) - (t_0^\sharp, \mathbf{x}_0^\sharp, \mathbf{z}^\sharp)\|$.