MAU23205 Lecture 29

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Where we left off and what's next

Last time we defined

$$\zeta_{t_0,\mathrm{x}_0,\mathrm{z}}(\mathbf{y}) = \max_{u\in[0,\tau]} \|\mathbf{Z}_{t_0,\mathrm{x}_0,\mathrm{z}}(\mathbf{y}, u)\|$$

where

$$\mathbf{Z}_{t_0,\mathrm{x}_0,\mathrm{z}}(\mathbf{y}, u) = \mathbf{y}(u) - \int_0^u \mathbf{F}(s + t_0, \mathbf{y}(s) + \mathbf{x}_0, \mathbf{z})\, ds$$

and showed that it was continuous. We also showed that $\zeta_{t_0,\mathrm{x}_0,\mathrm{z}}(\mathbf{y}) = 0$ if and only if $\mathbf{x}_{t_0,\mathrm{x}_0,\mathrm{z}}(t) = \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(t - t_0) + \mathbf{x}_0$ is a solution of the initial value problem $\mathbf{x}_{t_0,\mathrm{x}_0,\mathrm{z}}(t_0) = \mathbf{x}_0$, $\mathbf{x}'_{t_0,\mathrm{x}_0,\mathrm{z}}(t) = \mathbf{F}(t, \mathbf{x}_{t_0,\mathrm{x}_0,\mathrm{z}}(t), \mathbf{z})$.
$\zeta_{t_0,\mathrm{x}_0,\mathrm{z}}(\mathbf{y})$ is clearly non-negative for all $\mathbf{y}$. Our next task is to show that it can be made arbitrarily small, i.e. that for every $\epsilon > 0$ there is a $\mathbf{y} \in \mathcal{K}$ such that $\zeta_{t_0,\mathrm{x}_0,\mathrm{z}}(\mathbf{y}) < \epsilon$. For this we'll use the Euler scheme.

Euler scheme (1/5)

The Euler scheme with step size $h > 0$ is

$$\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(0) = \mathbf{0}$$

$$\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh + h) = \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh) + h\mathbf{F}(jh + t_0, \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh) + \mathbf{x}_0, \mathbf{z})$$

and linear in between. We want to show $\zeta_{t_0,\mathrm{x}_0,\mathrm{z}}(\mathbf{y}) < \epsilon$ for h small. Let $\delta = \frac{\delta_{\mathrm{F}}(\epsilon/\tau)}{\sqrt{L^2+1}}$ and suppose $h < \delta$.
If $jh < s \leq \min(jh + h, \tau)$ then

$$\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s) = \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh) + (s - jh)\mathbf{F}(jh + t_0, \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh) + \mathbf{x}_0, \mathbf{z})$$

For all such s we have $|s - jh| \leq h$ so

$$\|\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s) + \mathbf{x}_0, \mathbf{z}) - \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh) + \mathbf{x}_0, \mathbf{z})\| < Lh$$

and hence

$$\|(s + t_0, \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s) + \mathbf{x}_0, \mathbf{z}) - (jh + t_0, \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh) + \mathbf{x}_0, \mathbf{z})\|$$

is less than $\sqrt{L^2 + 1}h$.

Euler scheme (2/5)

Then $\sqrt{L^2+1}h \leq \delta_{\mathrm{F}}(\epsilon/\tau)$ so

$$\|\mathbf{F}(s+t_0, \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s)+\mathbf{x}_0, \mathbf{z}) - \mathbf{F}(jh+t_0, \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh)+\mathbf{x}_0, \mathbf{z})\|$$

is less than ϵ/τ.

$$\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s) = \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh) + (s-jh)\mathbf{F}(jh+t_0, \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh)+\mathbf{x}_0, \mathbf{z})$$

so

$$\begin{aligned}\mathbf{Z}_{t_0,\mathrm{x}_0,\mathrm{z}}(\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}, u) = {} & \mathbf{Z}_{t_0,\mathrm{x}_0,\mathrm{z}}(\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}, jh) \\ & + \int_{jh}^{u} \mathbf{F}(s+t_0, \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s)+\mathbf{x}_0, \mathbf{z})\, ds \\ & - \int_{jh}^{u} \mathbf{F}(jh+t_0, \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh)+\mathbf{x}_0, \mathbf{z})\, ds\end{aligned}$$

Therefore $\|\mathbf{Z}_{t_0,\mathrm{x}_0,\mathrm{z}}(u) - \mathbf{Z}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh)\| < \frac{\epsilon}{\tau}(u-jh)$ when $jh < u \leq \min(jh+h, \tau)$. By induction on j we have $\|\mathbf{Z}_{t_0,\mathrm{x}_0,\mathrm{z}}(u) - \mathbf{Z}_{t_0,\mathrm{x}_0,\mathrm{z}}(0)\| < \frac{\epsilon}{\tau}u$

Euler scheme (3/5)

$\|\mathbf{Z}_{t_0,\mathrm{x}_0,\mathrm{z}}(u) - \mathbf{Z}_{t_0,\mathrm{x}_0,\mathrm{z}}(0)\| < \frac{\epsilon}{\tau} u$. But $\mathbf{Z}_{t_0,\mathrm{x}_0,\mathrm{z}}(0) = \mathbf{0}$ and $u \leq \tau$ so $\|\mathbf{Z}_{t_0,\mathrm{x}_0,\mathrm{z}}(u)\| < \epsilon$ for all $u \in [0, \tau]$ and $(t_0, \mathbf{x}_0, \mathbf{z}) \in D$. Therefore $\zeta_{t_0,\mathrm{x}_0,\mathrm{z}}(\mathbf{y}) < \epsilon$, as promised.

We're not done yet! We still need to show that $\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}} \in \mathcal{K}$.

$$\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s) = \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh) + (s - jh)\mathbf{F}(jh + t_0, \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh) + \mathbf{x}_0, \mathbf{z})$$

for $jh < s \leq jh + h$ and $\|\mathbf{F}(jh + t_0, \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh) + \mathbf{x}_0, \mathbf{z})\| \leq L$ so

$$\|\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s) - \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh)\| \leq L(s - jh)$$

and, by induction on j

$$\|\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s) - \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(0)\| \leq Ls.$$

But $\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(0)$ and $s \leq \tau$ so

$$\|\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s)\| \leq L\tau \leq \delta_x.$$

Euler scheme (4/5)

$$\|\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s)\| \leq L\tau \leq \delta_{\mathrm{x}}.$$

This means that $\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}$ is a function from $[0,\tau]$ to $\bar{B}(\mathbf{0},\delta_x)$, which is one of the requirements for $\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}$ to belong to $\mathcal{K}$. We need it to be a *continuous* function as well, but this is clear from the definition.

Also, we define $\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}$ only for $s \in [0,\tau]$ and $(t_0,\mathbf{x}_0,\mathbf{z}) \in D$, so $0 \leq s \leq \tau \leq \delta_t$, $|t_0 - \underline{t}| \leq \delta_t$, $\|\mathbf{x}_0 - \underline{\mathbf{x}}\| \leq \delta_{\mathrm{x}}$ and $\|\mathbf{z} - \underline{\mathbf{z}}\| \leq \delta_{\mathrm{z}}$.

This means that

$$(s+t_0, \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s+t_0, t_0, \mathbf{x}_0)+\mathbf{x}_0, \mathbf{z}) \in \bar{B}(\underline{t}, 2\delta)\times\bar{B}(\underline{\mathbf{x}}, 2\delta_{\mathrm{x}})\times\bar{B}(\underline{\mathbf{z}}, \delta_{\mathrm{z}}) = K$$

This is important because that's where we have quantitative information on $\mathbf{F}$.

Strictly speaking, this part of the argument should be moved into the induction on j and checked at every step.

Euler scheme (5/5)

Also from

$$\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s) = \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh) + (s - jh)\mathbf{F}(jh + t_0, \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh) + \mathbf{x}_0, \mathbf{z})$$

and

$$\begin{aligned}\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s) = \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}&(jh + h)\\ &- (jh + h - s)\mathbf{F}(jh + t_0, \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(jh) + \mathbf{x}_0, \mathbf{z})\end{aligned}$$

it follows that $\|\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s^\flat) - \mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}(s^\sharp)\| \leq L|s^\flat - s^\sharp|$.
So $\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}} \in \mathcal{K}$ and $\zeta_{t_0,\mathrm{x}_0,\mathrm{z}}(\mathbf{y}_{t_0,\mathrm{x}_0,\mathrm{z}}) < \epsilon$ for all $h < \delta$.

Arzelà-Ascoli

The Arzelà-Ascoli Theorem gives necessary and sufficient conditions for a space of continuous functions to be compact. There are many versions but the one we want is

> *Suppose X and Y are compact metric spaces and $\mathcal{A}$ is a subset of the metric space of functions from X to Y. Then $\mathcal{A}$ is compact if and only if $\mathcal{A}$ is closed and (uniformly) equicontinuous.*

Y is compact so all functions from X to Y are bounded. So the set of functions is indeed a metric space with metric

$$d(f,g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

For continuous functions the supremum is actually a maximum since X is compact and d is continuous. The word "uniformly" is redundant, but sometimes useful.

$\mathcal{K}$ is compact (1/2)

We defined $\mathcal{K}$ to be the space of continuous functions $\mathbf{y}$ from $[0, \tau]$ to $\bar{B}(\mathbf{x}, \delta_x)$ such that

$$\|\mathbf{y}(s^\flat) - \mathbf{y}(s^\sharp)\| \leq L|s^\flat - s^\sharp|$$

for all $s^\flat, s^\sharp \in [0, \tau]$. The condition that $\mathbf{y}$ is continuous is redundant since it follows from the inequality. Take $\delta = \epsilon/(L+1)$. $\mathcal{K}$ is a closed subset. If $\mathbf{y} \notin \mathcal{K}$ then there are $s^\flat, s^\sharp \in [0, \tau]$ such that

$$\|\mathbf{y}(s^\flat) - \mathbf{y}(s^\sharp)\| > L|s^\flat - s^\sharp|.$$

Let

$$r = \frac{\|\mathbf{y}(s^\flat) - \mathbf{y}(s^\sharp)\| - L|s^\flat - s^\sharp|}{2}.$$

If $\tilde{\mathbf{y}} \in B(\mathbf{y}, r)$ then $\|\tilde{\mathbf{y}}(s^\flat) - \mathbf{y}(s^\flat)\| < r$ and $\|\tilde{\mathbf{y}}(s^\sharp) - \mathbf{y}(s^\sharp)\| < r$.

$\mathcal{K}$ is compact (2/2)

$$\begin{aligned}\|\mathbf{y}(s^\flat) - \mathbf{y}(s^\sharp)\| &\le \|\tilde{\mathbf{y}}(s^\flat) - \tilde{\mathbf{y}}(s^\sharp)\| + \|\tilde{\mathbf{y}}(s^\flat) - \mathbf{y}(s^\flat)\| + \|\tilde{\mathbf{y}}(s^\sharp) - \mathbf{y}(s^\sharp)\| \\ &< \|\tilde{\mathbf{y}}(s^\flat) - \tilde{\mathbf{y}}(s^\sharp)\| + 2r \\ &< \|\tilde{\mathbf{y}}(s^\flat) - \tilde{\mathbf{y}}(s^\sharp)\| + \|\mathbf{y}(s^\flat) - \mathbf{y}(s^\sharp)\| - L|s^\flat - s^\sharp|\end{aligned}$$

So $\|\tilde{\mathbf{y}}(s^\flat) - \tilde{\mathbf{y}}(s^\sharp)\| > L|s^\flat - s^\sharp|$. In other words, if $\mathbf{y} \notin \mathcal{K}$ then there is an $r > 0$ such that if $\tilde{\mathbf{y}} \in B(\mathbf{y}, r)$ then $\tilde{\mathbf{y}} \notin \mathcal{K}$. So the complement of $\mathcal{K}$ is open and $\mathcal{K}$ is closed.
For and ϵ, let $\delta = \epsilon/(L+1)$. If $\mathbf{y} \in \mathcal{K}$, $s^\flat, s^\sharp \in [0, \tau]$ and $d_{[0,\tau]} = (s^\flat, s^\sharp)|s^\flat - s^\sharp| < \delta$ then

$$d_{\bar{B}(0,\delta_x)}(\mathbf{y}(s^\flat), \mathbf{y}(s^\sharp)) = \left\|\mathbf{y}(s^\flat) - \mathbf{y}(s^\sharp)\right\| \le L|s^\flat - s^\sharp| \le L\delta < \epsilon.$$

So $\mathcal{K}$ is uniformly equicontinuous and hence compact.

The existence theorem

For every $(t_0, \mathbf{x}_0, \mathbf{z}) \in D$ we have that $\zeta_{t_0,x_0,z}$ is continuous on the compact metric space $\mathcal{K}$ and therefore has a minimum. This minimum is non-negative, but also non-positive, so must be zero. In other words, there is a $\mathbf{y}_{t_0,x_0,z} \in \mathcal{K}$ such that $\zeta_{t_0,x_0,z}(\mathbf{y}_{t_0,x_0,z}) = 0$. We've already seen that if $\zeta_{t_0,x_0,z}(\mathbf{y}_{t_0,x_0,z}) = 0$ and $\mathbf{x}_{t_0,x_0,z}$ is defined by

$$\mathbf{x}_{t_0,x_0,z}(t) = \mathbf{y}_{t_0,x_0,z}(t - t_0) + \mathbf{x}_0$$

then $\mathbf{x}_{t_0,x_0,z}(t)$ solves the initial value problem

$$\mathbf{x}'_{t_0,x_0,z}(t) = \mathbf{F}(t, \mathbf{x}_{t_0,x_0,z}(t), \mathbf{z}), \quad \mathbf{x}_{t_0,x_0,z}(t_0) = \mathbf{x}_0.$$

This solution is valid at least for $t - t_0 \in [0, \tau]$, i.e. for $t \in [t_0, t_0 + \tau]$. This τ is the same for all $(t_0, \mathbf{x}_0, \mathbf{z}) \in D$, i.e. for all $(t_0, \mathbf{x}_0, \mathbf{z})$ sufficiently close to $(\underline{t}, \underline{\mathbf{x}}, \underline{\mathbf{z}})$.