MAU23205 Lecture 27

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Stability comparison with linearisation

Theorem A: Suppose $\mathbf{F}$ is continuous in $B(\mathbf{x}^*, r)$ for some $r > 0$, $\mathbf{F}(\mathbf{x}^*) = \mathbf{0}$ and $\mathbf{F}$ is differentiable at $\mathbf{x}^*$. If all eigenvalues of $\mathbf{F}'(\mathbf{x}^*)$, i.e. roots of its characteristic or minimal polynomial, have negative real part then $\mathbf{x}^*$ is a strictly stable equilibrium of $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$.

I'll prove this in a moment.

Theorem B: Suppose $\mathbf{F}$ is continuous in $B(\mathbf{x}^*, r)$ for some $r > 0$, $\mathbf{F}(\mathbf{x}^*) = \mathbf{0}$ and $\mathbf{F}$ is differentiable at $\mathbf{x}^*$. If some eigenvalue of $\mathbf{F}'(\mathbf{x}^*)$ has positive real part then $\mathbf{x}^*$ is an unstable equilibrium of $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$.

I won't prove this, but you are allowed to assume it when doing problems.

The case where no eigenvalue has positive real part but some eigenvalue is purely imaginary is not addressed by either theorem. In this case knowledge of $\mathbf{F}'(\mathbf{x}^*)$ is not enough to determine stability, as the example $\mathbf{F}(\mathbf{x}) = A\mathbf{x} + b\mathbf{x} + c\|\mathbf{x}\|^2\mathbf{x}$ from last lecture shows.

Proof of Theorem A (1/4)

Set $A = \mathbf{F}'(\mathbf{x}^*)$ and $C = I$. All the eigenvalues of A have negative real part so there is a symmetric positive definite B such that $A^T B + BA + C = O$. Define

$$V(\mathbf{x}) = (\mathbf{x} - \mathbf{x}^*)^T B(\mathbf{x} - \mathbf{x}^*).$$

$V(\mathbf{x}) > V(\mathbf{x}^*) = 0$ if $\mathbf{x} \neq \mathbf{x}^*$ because B is positive definite.

$$V'(\mathbf{x}) = 2(\mathbf{x} - \mathbf{x}^*)^T B.$$

$\mathbf{F}$ is differentiable at $\mathbf{x}^*$, i.e. for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\|\mathbf{x} - \mathbf{x}^*\| < \delta$ then

$$\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}^*) - \mathbf{F}'(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \leq \epsilon\|\mathbf{x} - \mathbf{x}^*\|.$$

$$\mathbf{F}(\mathbf{x}) - A(\mathbf{x} - \mathbf{x}^*) \leq \epsilon\|\mathbf{x} - \mathbf{x}^*\|.$$

$$V'(\mathbf{x})\mathbf{F}(\mathbf{x}) = 2(\mathbf{x} - \mathbf{x}^*)^T B\left(A(\mathbf{x} - \mathbf{x}^*) + \mathbf{F}(\mathbf{x}) - A(\mathbf{x} - \mathbf{x}^*)\right).$$

Proof of Theorem A (2/4)

$$V'(\mathbf{x})\mathbf{F}(\mathbf{x}) = 2(\mathbf{x}-\mathbf{x}^*)^T BA(\mathbf{x}-\mathbf{x}^*) + 2(\mathbf{x}-\mathbf{x}^*)^T B\left(\mathbf{F}(\mathbf{x}) - A(\mathbf{x}-\mathbf{x}^*)\right).$$

$$\begin{aligned}
2(\mathbf{x}-\mathbf{x}^*)^T BA(\mathbf{x}-\mathbf{x}^*) &= (\mathbf{x}-\mathbf{x}^*)^T BA(\mathbf{x}-\mathbf{x}^*) \\
&\quad + (\mathbf{x}-\mathbf{x}^*)^T BA(\mathbf{x}-\mathbf{x}^*) \\
&= ((\mathbf{x}-\mathbf{x}^*)^T BA(\mathbf{x}-\mathbf{x}^*))^T \\
&\quad + (\mathbf{x}-\mathbf{x}^*)^T BA(\mathbf{x}-\mathbf{x}^*) \\
&= (\mathbf{x}-\mathbf{x}^*)^T A^T B(\mathbf{x}-\mathbf{x}^*) \\
&\quad + (\mathbf{x}-\mathbf{x}^*)^T BA(\mathbf{x}-\mathbf{x}^*) \\
&= (\mathbf{x}-\mathbf{x}^*)^T (A^T B + BA)(\mathbf{x}-\mathbf{x}^*) \\
&= -(\mathbf{x}-\mathbf{x}^*)^T C(\mathbf{x}-\mathbf{x}^*) \\
&= -(\mathbf{x}-\mathbf{x}^*)^T (\mathbf{x}-\mathbf{x}^*) = -\|\mathbf{x}-\mathbf{x}^*\|^2
\end{aligned}$$

Proof of Theorem A (3/4)

$$V'(\mathbf{x})\mathbf{F}(\mathbf{x}) = 2(\mathbf{x}-\mathbf{x}^*)^T BA(\mathbf{x}-\mathbf{x}^*)+2(\mathbf{x}-\mathbf{x}^*)^T B\left(\mathbf{F}(\mathbf{x}) - A(\mathbf{x}-\mathbf{x}^*)\right).$$

$$2(\mathbf{x}-\mathbf{x}^*)^T BA(\mathbf{x}-\mathbf{x}^*) = -\|\mathbf{x}-\mathbf{x}^*\|^2$$

$$\begin{aligned}\left\|2(\mathbf{x}-\mathbf{x}^*)^T B\left(\mathbf{F}(\mathbf{x}) - A(\mathbf{x}-\mathbf{x}^*)\right)\right\| &\le 2\|\mathbf{x}-\mathbf{x}^*\|\|\mathbf{F}(\mathbf{x}) - A(\mathbf{x}-\mathbf{x}^*)\| \\ &\le 2\epsilon\|\mathbf{x}-\mathbf{x}^*\|^2\end{aligned}$$

$$-2\epsilon\|\mathbf{x}-\mathbf{x}^*\|^2 \le 2(\mathbf{x}-\mathbf{x}^*)^T B\left(\mathbf{F}(\mathbf{x}) - A(\mathbf{x}-\mathbf{x}^*)\right) \le 2\epsilon\|\mathbf{x}-\mathbf{x}^*\|^2$$

$$-(1+2\epsilon)\|\mathbf{x}-\mathbf{x}^*\|^2 \le V'(\mathbf{x})\mathbf{F}(\mathbf{x}) \le -(1-2\epsilon)\|\mathbf{x}-\mathbf{x}^*\|^2$$

Proof of Theorem A (4/4)

$$-(1+2\epsilon)\|\mathbf{x}-\mathbf{x}^*\|^2 \leq V'(\mathbf{x})\mathbf{F}(\mathbf{x}) \leq -(1-2\epsilon)\|\mathbf{x}-\mathbf{x}^*\|^2$$

We get to choose ϵ so choose $\epsilon = 1/3$ so

$$V'(\mathbf{x})\mathbf{F}(\mathbf{x}) \leq -\frac{1}{3}\|\mathbf{x}-\mathbf{x}^*\|^2 < 0$$

if $\mathbf{x} \neq \mathbf{x}^*$. We also needed $\|\mathbf{x}-\mathbf{x}^*\| < \delta$. $V'(\mathbf{x})\mathbf{F}(\mathbf{x}) < 0$ for $\mathbf{x} \in B(\mathbf{x}^*, r) - \{\mathbf{x}^*\}$ where $r = \delta$. Also $V(\mathbf{x}) > V(\mathbf{x}^*)$ for $\mathbf{x} \in B(\mathbf{x}^*, r) - \{\mathbf{x}^*\}$. V is therefore a strict Lyapunov function. By the Lyapunov Theorem then $\mathbf{x}^*$ is a strictly stable equilibrium.

One more stability example: the pendulum (1/5)

The behaviour of a pendulum is described by the second order differential equation

$$m\ell\theta''(t) + mg\sin(\theta(t)) = \varphi(\theta(t), \theta'(t)).$$

θ is the angular displacement from the downward vertical position, $m > 0$ is the mass, $\ell > 0$ is the length of the arm, $g > 0$ is the acceleration of gravity and φ is the sum of all forces other than gravity, e.g. friction, air resistance, etc. Two common approximations are $\sin\theta = \theta$, the small amplitude assumption, and $\varphi = 0$, the no dissipation assumption. I want to avoid making either assumption, although I still want to allow $\varphi = 0$.
As usual, we'll reduce to a first order system, by taking $\omega = \theta'$.

$$\theta'(t) = \omega(t) \qquad \omega'(t) = -\frac{g}{\ell}\sin(\theta(t)) + \frac{1}{m\ell}\varphi(\theta(t), \omega(t)).$$

One more stability example: the pendulum (2/5)

If $\varphi = 0$ then energy is conserved, i.e.

$$E(\theta, \omega) = \frac{m\ell^2}{2}\omega^2 - mg\ell(\cos\theta)$$

is an invariant.

We now drop the assumption that $\varphi = 0$. φ is meant to describe forces like friction, which reduce the energy of the system, or at least don't increase it, so the natural assumption is

$$\omega\varphi(\theta, \omega) \leq 0$$

for all θ and ω. E is now no longer an invariant but it is a Lyapunov function. It has a strict local minimum at $(0, 0)$ and

$$\begin{aligned} E'(\theta, \omega)\mathbf{F}(\theta, \omega) &= -mg\ell\omega(t)\sin(\theta(t)) + \ell\omega(t)\varphi(\theta(t), \omega(t)) \\ &\quad + mg\ell\omega(t)\sin(\theta(t)) = \ell\omega(t)\varphi(\theta(t), \omega(t)) \leq 0 \end{aligned}$$

One more stability example: the pendulum (3/5)

I'll assume that φ is differentiable. From $\omega\varphi(\theta,\omega) \leq 0$ it follows that $\varphi(\theta,\omega) \geq 0$ for $\omega < 0$ and $\varphi(\theta,\omega) \leq 0$ for $\omega > 0$. It follows that $\varphi(\theta,0) = 0$. Also $\partial\varphi(\theta,0)/\partial\theta = 0$ and $\partial\varphi(\theta,0)/\partial\omega \leq 0$
The equilibria are the points (θ^*,ω^*) where

$$\omega^* = 0 \qquad -\frac{g}{\ell}\sin(\theta^*) + \frac{1}{m\ell}\varphi(\theta^*,\omega^*) = 0$$

The points $(\theta^*,\omega^*) = (0,0)$ and $(\theta^*,\omega^*) = (\pi,0)$ clearly satisfy those conditions. Conversely, if $\omega^* = 0$ and $-\frac{g}{\ell}\sin(\theta^*) + \frac{1}{m\ell}\varphi(\theta^*,\omega^*) = 0$ then $-\frac{g}{\ell}\sin(\theta^*) = 0$ and hence $(\theta^*,\omega^*) = (n\pi,0)$, where n is an integer. There's no meaningful distinction between angles which differ by an integer multiple of 2π, so $(0,0)$ and $(\pi,0)$ are the only equilibria we need to consider.

One more stability example: the pendulum (4/5)

We now linearise the equations. With $\mathbf{x} = \begin{bmatrix} \theta \\ \omega \end{bmatrix}$, $\mathbf{x}^* = \begin{bmatrix} \theta^* \\ \omega^* \end{bmatrix}$ and $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} \omega \\ -\frac{g}{\ell}\sin(\theta) + \frac{1}{m\ell}\varphi(\theta,\omega) \end{bmatrix}$ our equation is $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$ and our equliibrium is $\mathbf{x}^*$.

$$A = \mathbf{F}'(\mathbf{x}^*) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos(\theta^*) + \frac{1}{2m\ell}\frac{\partial\varphi}{\partial\theta}(\theta^*,\omega^*) & \frac{1}{2m\ell}\frac{\partial\varphi}{\partial\omega}(\theta^*,\omega^*) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ (-1)^{n+1}\frac{g}{\ell} & \frac{1}{2m\ell}\frac{\partial\varphi}{\partial\omega}(n\pi,0) \end{bmatrix}$$

at $(\theta^*,\omega^*) = (n\pi, 0)$.
$p_A(z) = z^2 - \frac{1}{2m\ell}\frac{\partial\varphi}{\partial\omega}(n\pi,0)z + (-1)^n\frac{g}{\ell}$. For $n = 1$ this has two real roots, one of which is negative, so the equilibrium $(\pi, 0)$ is unstable, as expected. For $n = 0$ and $\frac{\partial\varphi}{\partial\omega}(0,0) < 0$ this has roots with negative real part, so the equilibrium $(0, 0)$ is strictly stable by Theorem A.

One more stability example: the pendulum (5/5)

For $n = 0$ and $\frac{\partial \varphi}{\partial \omega}(0,0) = 0$ this has two distinct purely imaginary roots, so Theorems A and B tell us nothing. We can still use Lyapunov's theorem, with $V = E$, to get stability. E isn't a strict Lyapunov function so we don't get strict stability. We shouldn't expect it because the case $\varphi = 0$ is compatible with $\frac{\partial \varphi}{\partial \omega}(0,0) = 0$. The idea of using $V = E$ comes from the physics of the problem. It works as well as we can expect for $\frac{\partial \varphi}{\partial \omega}(0,0) = 0$. It does not work very well for $\frac{\partial \varphi}{\partial \omega}(0,0) < 0$. It's still only a Lyapunov function, not a strict Lyapunov function. We proved Theorem A by finding a strict Lyapunov function, but that function was a quadratic polynomial with no relation to the energy and no particular physical meaning. If I make the usual choice of $C = I$ then even the units don't make sense!