MAU23205 Lecture 26

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Example (1/3)

Suppose A is an antisymmetric matrix, i.e. $A^T = -A$, such as the 1×1 matrix $A = \begin{bmatrix} 0 \end{bmatrix}$ or the 2×2 matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then

$$
\begin{aligned}
\mathbf{x}^T A\mathbf{x} &= \frac{1}{2}\mathbf{x}^T A\mathbf{x} + \frac{1}{2}\mathbf{x}^T A\mathbf{x} = \frac{1}{2}\mathbf{x}^T A\mathbf{x} + \frac{1}{2}\left(\mathbf{x}^T A\mathbf{x}\right)^T \\
&= \frac{1}{2}\mathbf{x}^T A\mathbf{x} + \frac{1}{2}\mathbf{x}^T A^T\mathbf{x} = \frac{1}{2}\mathbf{x}^T A\mathbf{x} + \frac{1}{2}\mathbf{x}^T(-A)\mathbf{x} \\
&= \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \frac{1}{2}\mathbf{x}^T A\mathbf{x} = 0
\end{aligned}
$$

for all $\mathbf{x}$. Let

$$
\mathbf{F}(\mathbf{x}) = A\mathbf{x} + b\mathbf{x} + c\|\mathbf{x}\|^2\mathbf{x}, \qquad V(\mathbf{x}) = \|\mathbf{x}\|^2 = \mathbf{x}^T\mathbf{x}.
$$

Example (2/3)

$V(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$ is continuously differentiable and $V'(\mathbf{x}) = 2\mathbf{x}^T$.

$$V'(\mathbf{x})\mathbf{F}(\mathbf{x}) = 2\mathbf{x}^T A\mathbf{x} + 2b\mathbf{x}^T\mathbf{x} + 2c\|\mathbf{x}\|^2\mathbf{x}^T\mathbf{x} = 2b\|\mathbf{x}\|^2 + 2c\|\mathbf{x}\|^4.$$

$V(\mathbf{x}^*) = 0$ and $V(\mathbf{x}) = \|\mathbf{x}\|^2 > 0$ for all $\mathbf{x} \in B(\mathbf{x}^*, r) - \{\mathbf{x}^*\}$ if $\mathbf{x}^* = \mathbf{0}$ and $B(\mathbf{x}^*, r)$ is any subset of $r > 0$.
If $b < 0$ then $V'(\mathbf{x})\mathbf{F}(\mathbf{x}) < 0$ for all $\mathbf{x} \in B(\mathbf{x}^*, r) - \{\mathbf{x}^*\}$, where $r = \sqrt{-c/b}$ if $c > 0$ and for any $r > 0$ if $c \leq 0$.
If $b = 0$ and $c < 0$ then $V'(\mathbf{x})\mathbf{F}(\mathbf{x}) < 0$ for all $\mathbf{x} \in B(\mathbf{x}^*, r) - \{\mathbf{x}^*\}$ for any $r > 0$.
In either of these cases, V is a strict Lyapunov function for $\mathbf{x}^* = \mathbf{0}$ and hence $\mathbf{0}$ is a strictly stable equilibrium of $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$.

Example (3/3)

A strict Lyapunov function is necesarily a Lyapunov function, but a Lyapunov function needn't be a strict Lyapunov function. If $b = c = 0$ then our V is a Lyapunov function, but not a strict Lyapunov function. $\mathbf{0}$ is then a stable equilibrium. Is it strictly stable? No. The theorem doesn't tell you, but the calculation does.

$$\frac{d}{dt}V(\mathbf{x}(t)) = V'(\mathbf{x}(t))\mathbf{F}(\mathbf{x}(t)) = 0$$

so $V(\mathbf{x}(t)) = \|\mathbf{x}(t)\|^2$ is constant, which prevents $\lim_{t\to+\infty} \mathbf{x}(t) = \mathbf{0}$ for $\mathbf{x}_0 \neq \mathbf{0}$.

The theorem tells us nothing about instability, but in fact $\mathbf{0}$ is an unstable equilibrium if $b > 0$, or if $b = 0$ and $c > 0$.

The linearisation of $\mathbf{x}'(t) = A\mathbf{x}(t) + b\mathbf{x}(t) + c\|\mathbf{x}(t)\|^2\mathbf{x}(t)$ is $\mathbf{x}'(t) = A\mathbf{x}(t) + b\mathbf{x}(t)$. $\mathbf{0}$ is a strictly stable equilibrium if $b < 0$, stable but not strictly stable if $b = 0$ and unstable if $b > 0$.

Example: Lotka-Volterra (1/6)

Consider the Lotka-Volterra system from the first set of practice problems:

$$x'(t) = ax(t) - bx(t)y(t) \qquad y'(t) = -cy(t) + dx(t)y(t)$$

We take $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Our $\mathbf{F}$ is $\mathbf{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax - bxy \\ -cy + dxy \end{bmatrix}$. Then $\mathbf{F}'\left(\begin{bmatrix} x^* \\ y^* \end{bmatrix}\right)$ is the matrix whose columns are the partial derivatives of $\mathbf{F}$, evaluated at $\begin{bmatrix} x^* \\ y^* \end{bmatrix}$: $A = \mathbf{F}'(\mathbf{x}^*) = \begin{bmatrix} a - by^* & -bx^* \\ dy^* & -c + dx^* \end{bmatrix}$ To determine stability we'll need $\mathbf{tr}(A) = a - c + dx^* - by^*$ and $\det(A) = (a - by^*)(-c + dx^*) - (-bx^*)(dy^*) = -ac + adx^* + bcy^*$.

Example: Lotka-Volterra (2/6)

We're only interested in linearising at equilibria. If $\begin{bmatrix} x^* \\ y^* \end{bmatrix}$ is an equilibrium then our linearised system is

$$\frac{d}{dt}\begin{bmatrix} x(t) - x^* \\ y(t) - y^* \end{bmatrix} = A \begin{bmatrix} x(t) - x^* \\ y(t) - y^* \end{bmatrix}$$

This is the system

$$\begin{aligned} x'(t) &= (a - by^*)(x(t) - x^*) + (-bx^*)(y(t) - y^*)(a - by^*) \\ &= (a - by^*)x(t) - bx^*y(t) - ax^* \\ y'(t) &= dy^*(x(t) - x^*) + (-c + dx^*)(y(t) - y^*) \\ &= dy^*x(t) + (-c + dx^*)y(t) + cy^* \end{aligned}$$

Example: Lotka-Volterra (3/6)

Where are the equilibria? $\mathbf{F}\left(\begin{bmatrix} x^* \\ y^* \end{bmatrix}\right) = \begin{bmatrix} ax^* - bx^*y^* \\ -cy^* + dx^*y^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if and only if $ax^* - bx^*y^* = 0$ and $-cy^* + dx^*y^* = 0$. The equation $ax^* - bx^*y^* = 0$ is satisfied if $x^* = 0$ or if $a - by^* = 0$. The equation $-cy^* + dx^*y^* = 0$ is satisfied if $y^* = 0$ or if $-c + dx^* = 0$.

One possibility is $x^* = 0$ and $y^* = 0$. In that case $\mathrm{tr}(A) = a - c + dx^* - by^* = a - c$ and $\det(A) = -ac + adx^* + bcy^* = -ac$. The characteristic polynomial is $p_A(z) = z^2 - \mathrm{tr}(A)z + \det(A) = (z - a)(z + c)$. The usual assumptions on a, b, c and d are that they are all positive, so one root is positive and one is negative. This equilibrium therefore has a linearisation whose equilibrium is unstable.

Example: Lotka-Volterra (4/6)

Another "possibility" is $x^* = 0$ and $-c + dx^* = 0$. Because of our assumption that $c > 0$, this can't happen.
Similarly, because $a > 0$, we can't have $a - by^* = 0$ and $y^* = 0$.
The only remaining possibility is $a - by^* = 0$ and $-c + dx^* = 0$.
This happens if $x^* = c/d$ and $y^* = a/b$. Then $\mathrm{tr}(A) = 0$ and $\det(A) = ac > 0$. The roots of the characteristic polynomial are purely imaginary. They're of multiplicity 1, so this equilibrium has a linearisation whose equilibrium is stable, but not strictly stable.
So we know the character of the equilibria of the linearisations at the two equilibria. What about the character of those equilibria for the original, non-linear system?

Example: Lotka-Volterra (5/6)

$$V(x,y) = \left(\frac{dx}{c}\right)^{-\sqrt{c/a}} \left(\frac{by}{a}\right)^{-\sqrt{a/c}} \exp\left(\frac{dx+by}{\sqrt{ac}}\right)$$

is a Lyapunov function for the equilibrium $(c/d, a/b)$. You can check using the second derivative test that V has a strict local minimum at $(c/d, a/b)$.

$$V'(x,y) = \begin{bmatrix} \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{bmatrix} = \frac{V(x,y)}{\sqrt{ac}} \begin{bmatrix} \frac{dx-c}{x} & \frac{by-a}{y} \end{bmatrix}$$

$$V'(c/d, a/b) = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial y} \\ \frac{\partial^2 V}{\partial x \partial y} & \frac{\partial^2 V}{\partial y^2} \end{bmatrix} (c/d, a/b) = \frac{1}{\sqrt{ac}} \exp\left(\sqrt{\frac{a}{c}} + \sqrt{\frac{c}{a}}\right) \begin{bmatrix} d^2/c & 0 \\ 0 & b^2/a \end{bmatrix}.$$

$$V'(x,y)\mathbf{F}(x,y) = \frac{V(x,y)}{\sqrt{ac}} \begin{bmatrix} \frac{dx-c}{x}, & \frac{by-a}{y} \end{bmatrix} \begin{bmatrix} (a-by)x \\ (dx-c)y \end{bmatrix} = 0.$$

Example: Lotka-Volterra (6/6)

So $(c/d, a/b)$ is a stable equilibrium for the non-linear system, just as it is for the linearisation.
We can check by hand that $(0, 0)$ is an unstable equilibrium for the non-linear system, just as it is for the linear system.

$$x(t) = \frac{\delta}{2}\exp(at), \qquad y(t) = 0$$

is a solution with $\|(x(0), y(0)) - (0, 0)\| < \delta$ but it is not true that $\|(x(t), y(t)) - (0, 0)\| < \epsilon$ for all $t \geq 0$.