MAU23205 Lecture 24

John Stalker

Trinity College Dublin

12 November 2021

Equilibria of linear systems

The simplest autonomous systems are the linear ones. $\mathbf{x}^*$ is an equilibrium of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$ if and only if $A\mathbf{x}^* + \mathbf{g} = \mathbf{0}$. A here is necessarily square. If A is invertible then there is exactly one equilibrium. If A is not invertible then the will be equilibria only for some $\mathbf{g}$, those in the range of A, but if there is an equilibrium then there are infinitely many, differing by elements of the nullspace (kernel) of A. If $\mathbf{x}^*$ is an equilibrium then

$$\begin{aligned}\frac{d}{dt}\left(\mathbf{x}(t) - \mathbf{x}^*\right) &= \mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g} \\ &= A\mathbf{x}(t) - A\mathbf{x}^* = A\left(\mathbf{x}(t) - \mathbf{x}^*\right)\end{aligned}$$

So $\mathbf{x} - \mathbf{x}^*$ satisfies the corresponding homogeneous equation. In particular, the equation it satisfies doesn't depend on which equilibrium we're looking at. The stability properties are therefore the same for all equilibria of a given linear system.

Stability of linear systems (1/3)

The following were shown to be equivalent in Lecture 17:

1. All solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ are bounded as $t \to +\infty$.
2. The basic solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ are bounded as $t \to +\infty$.
3. $\exp(tA)$ is bounded as $t \to +\infty$.
4. All (complex) roots of the minimal polynomial of A either have negative real part, or have zero real part and multiplicity one.

If $\exp(tA)$ is bounded as $t \to +\infty$ then there is a $C > 0$ such that $\|\exp(tA)\| < C$ for $t \in [0, +\infty)$. If $\epsilon > 0$, $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta = \frac{\epsilon}{C}$ then $\|\mathbf{x}(t) - \mathbf{x}^*\| = \|\exp(tA)(\mathbf{x}_0 - \mathbf{x}^*)\| \leq \|\exp(tA)\|\|\mathbf{x}_0 - \mathbf{x}^*\| < \epsilon$ for all $t \in [0, +\infty)$. So for any $\epsilon > 0$ there is a $\delta > 0$ such that for $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$ the unique solution of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$, $\mathbf{x}(0) = \mathbf{x}_0$, where $\mathbf{g} = -A\mathbf{x}^*$, satisfies $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$ for all $t \in [0, +\infty)$. In other words, $\mathbf{x}^*$ is a stable equilibrium of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$.

Stability of linear systems (2/3)

Since all equilibria of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$ are of the form $\mathbf{x}^*$ where $\mathbf{g} = -A\mathbf{x}^*$, all equilibria of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$ are stable.
Conversely, suppose there is at least one stable equilibrium $\mathbf{x}^*$ of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$ for some $\mathbf{g}$. Taking any $\epsilon > 0$, there is a $\delta > 0$ such that $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$ the unique solution of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$, $\mathbf{x}(0) = \mathbf{x}_0$ satisfies $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$ for all $t \in [0, +\infty)$. For any solution $\mathbf{y}$ to $\mathbf{y}'(t) = A\mathbf{y}(t)$, set $\mathbf{x}(t) = \mathbf{x}^* + \frac{\delta}{2\|\mathrm{y}(0)\|}\mathbf{y}(t)$. Then $\mathbf{x}$ is a solution to $\mathbf{x}'(t) = Ax(t) + \mathbf{g}$ with $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$, so $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$. Then $\|\mathbf{y}(t)\| < 2\frac{\epsilon}{\delta}\|\mathbf{y}(0)\|$ for all $t \in [0, +\infty)$. So all solutions to $\mathbf{y}'(t) = A\mathbf{y}(t)$ are bounded.

Stability of linear systems (3/3)

We can now extend the list from two slides ago. The following are equivalent:

1. All solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ are bounded as $t \to +\infty$.
2. The basic solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ are bounded as $t \to +\infty$.
3. $\exp(tA)$ is bounded as $t \to +\infty$.
4. All (complex) roots of the minimal polynomial of A either have negative real part, or have zero real part and multiplicity one.
5. For any $\mathbf{g}$ and any equilibrium $\mathbf{x}^*$ of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$, $\mathbf{x}^*$ is stable.
6. For some $\mathbf{g}$ there is a stable equilibrium $\mathbf{x}^*$ of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$.

Asymptotic stability of linear systems (1/3)

We also saw in Lecture 17 that the following are equivalent:

1. All solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ tend to $\mathbf{0}$ as $t \to +\infty$.
2. The basic solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ tend to $\mathbf{0}$ as $t \to +\infty$.
3. $\exp(tA)$ tends to O as $t \to +\infty$.
4. All (complex) roots of the minimal polynomial of A have negative real part.
5. All (complex) roots of the characteristic polynomial of A have negative real part.
6. There are positive definite symmetric B and C such that $A^T B + BA + C = O$.
7. For every positive definite symmetric C there is a positive definite symmetric B such that $A^T B + BA + C = O$.
8. There is a quadratic polynomial $V(\mathbf{x}) = \mathbf{x}^T B\mathbf{x}$ in the entries of $\mathbf{x}$ such that $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$ and $V(\mathbf{x}(t))$ is strictly decreasing for all non-zero solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Asymptotic stability of linear systems (2/3)

If all solutions tend to $\mathbf{0}$ then there is certainly a $\delta > 0$ such that all solutions with $\|\mathbf{x}_0 - \mathbf{0}\| < \delta$ tend to $\mathbf{0}$, so $\mathbf{0}$ is asymptotically stable.
Conversely, if $\mathbf{0}$ is asymptotically stable there is a $\delta > 0$ such that all solutions $\mathbf{x}$ with $\|\mathbf{x}_0 - \mathbf{0}\| < \delta$ tend to $\mathbf{0}$, But for any solution $\mathbf{y}$, $\mathbf{x} = \frac{1}{2\delta\|\mathrm{y}(0)\|}\mathbf{y}$ is a solution with $\|\mathbf{x}(0)\| < \delta$, so $\mathbf{x}(t)$ tends to $\mathbf{0}$, and hence so does $\mathbf{y}(t)$. So all solutions tend to $\mathbf{0}$. So we can add "$\mathbf{0}$ is an asymptotically stable equilibrium of $\mathbf{x}'(t) = A\mathbf{x}(t)$" to the list.
Just as with stability, we can go from the behaviour of $\mathbf{0}$ as an equilibrium of $\mathbf{x}'(t) = A\mathbf{x}(t)$ to the behaviour of $\mathbf{x}^*$ as an equilibrium of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$ and back again.

Asymptotic stability of linear systems (3/3)

The following can be added to our list of equivalent conditions from two slides ago:

9. For some $\mathbf{g}$ there is an asymptotically stable equilibrium $\mathbf{x}^*$ of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$.
10. For any $\mathbf{g}$ and any equilibrium $\mathbf{x}^*$ of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$, $\mathbf{x}^*$ is asymptotically stable.
11. For every equilibrium $\mathbf{x}^*$ of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$ there is a quadratic polynomial $V(\mathbf{x}) = (\mathbf{x} - \mathbf{x}^*)^T B(\mathbf{x} - \mathbf{x}^*)$ in the entries of $\mathbf{x}$ such that $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{x}^*$ and $V(\mathbf{x}(t))$ is strictly decreasing for all solutions of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$.
12. For any $\mathbf{g}$ and any equilibrium $\mathbf{x}^*$ of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$, $\mathbf{x}^*$ is strictly stable.
13. For some $\mathbf{g}$ there is an strictly stable equilibrium $\mathbf{x}^*$ of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$.

Strict stability and exponential stability (1/5)

We saw in Lecture 23 that every exponentially stable equilibrium is strictly stable. I also gave an example of a strictly stable equilibrium which is not exponentially stable. That can't happen for linear autonomous systems.

Suppose $\mathbf{x}^*$ is a strictly stable equilibrium of $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$. As we saw last time, that implies that there are positive definite symmetric matrices B and C such that $A^T B + BA + C = O$. Define $U, V \colon \mathbf{R}^n \to \mathbf{R}$ and $S \subseteq \mathbf{R}^n$ by $U(\mathbf{x}) = (\mathbf{x} - \mathbf{x}^*)^T C(\mathbf{x} - \mathbf{x}^*)$ $V(\mathbf{x}) = (\mathbf{x} - \mathbf{x}^*)^T B(\mathbf{x} - \mathbf{x}^*)$ and $S = \{\mathbf{x} \in \mathbf{R}^n \colon U(\mathbf{x}) = 1\}$. Then S is compact, V is continuously differentiable, and

$$
\begin{aligned}
\frac{d}{dt} V(\mathbf{x}(t)) &= (\mathbf{x}'(t) - \mathbf{x}^*)^T B(\mathbf{x}(t) - \mathbf{x}^*) \\
&\quad + (\mathbf{x}(t) - \mathbf{x}^*)^T B(\mathbf{x}'(t) - \mathbf{x}^*) \\
&= (\mathbf{x}(t) - \mathbf{x}^*)^T A^T B(\mathbf{x}(t) - \mathbf{x}^*) \\
&\quad + (\mathbf{x}(t) - \mathbf{x}^*)^T BA(\mathbf{x}(t) - \mathbf{x}^*) \\
&= -(\mathbf{x}(t) - \mathbf{x}^*)^T C(\mathbf{x}(t) - \mathbf{x}^*) = -U(\mathbf{x}(t)).
\end{aligned}
$$

Strict stability and exponential stability (2/5)

V restricted to the compact set S is continuous, and so has a minimum and maximum there. In other words, there are $\mathbf{x}_{\min}, \mathbf{x}_{\max} \in S$ such that for all $\mathbf{y} \in S$

$$V(\mathbf{x}_{\min}) \leq V(\mathbf{y}) \leq V(\mathbf{x}_{\max}).$$

$\mathbf{x}_{\min} \in S$ so $U(\mathbf{x}_{\min}) = (\mathbf{x}_{\min} - \mathbf{x}^*)^T C(\mathbf{x}_{\min} - \mathbf{x}^*) = 1$ and $\mathbf{x}_{\min} - \mathbf{x}^* \neq \mathbf{0}$. B is positive definite, so

$$V(\mathbf{x}_{\min}) = (\mathbf{x}_{\min} - \mathbf{x}^*)^T B(\mathbf{x}_{\min} - \mathbf{x}^*) > 0.$$

For any $\mathbf{x} \neq \mathbf{x}^*$, $U(\mathbf{x}) > 0$, so $\mathbf{y} = \mathbf{x}^* + \frac{1}{\sqrt{U(\mathrm{x})}}(\mathbf{x} - \mathbf{x}^*)$ satisfies

$$U(\mathbf{y}) = (\mathbf{y} - \mathbf{x}^*)^T C(\mathbf{y} - \mathbf{x}^*) = (\mathbf{x} - \mathbf{x}^*)^T \frac{1}{\sqrt{U(\mathbf{x})}} C \frac{1}{\sqrt{U(\mathbf{x})}}(\mathbf{x} - \mathbf{x}^*)$$
$$= \frac{1}{U(\mathbf{x})}(\mathbf{x} - \mathbf{x}^*)^T C(\mathbf{x} - \mathbf{x}^*) = \frac{U(\mathbf{x})}{U(\mathbf{x})} = 1.$$

Strict stability and exponential stability (3/5)

We've just seen that $U(\mathbf{y}) = 1$, so $\mathbf{y} \in S$ and hence

$$0 < V(\mathbf{x}_{\min}) \leq V(\mathbf{y}) \leq V(\mathbf{x}_{\max}).$$

$$V(\mathbf{y}) = (\mathbf{y} - \mathbf{x}^*)^T B(\mathbf{y} - \mathbf{x}^*) = (\mathbf{x} - \mathbf{x}^*)^T \frac{1}{\sqrt{U(\mathbf{x})}} B \frac{1}{\sqrt{U(\mathbf{x})}} (\mathbf{x} - \mathbf{x}^*)$$
$$= \frac{1}{U(\mathbf{x})} (\mathbf{x} - \mathbf{x}^*)^T B(\mathbf{x} - \mathbf{x}^*) = \frac{V(\mathbf{x})}{U(\mathbf{x})}.$$

So

$$V(\mathbf{x}_{\min}) U(\mathbf{x}) \leq V(\mathbf{x}) \leq V(\mathbf{x}_{\max}) U(\mathbf{x})$$

for all $\mathbf{x} \neq \mathbf{x}^*$. The same inequalities hold, trivially, for $\mathbf{x} = \mathbf{x}^*$. So, for any solution $\mathbf{x}$ to $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$,

$$\frac{d}{dt} V(\mathbf{x}(t)) = -U(\mathbf{x}(t)) \leq -\frac{V(\mathbf{x}(t))}{V(\mathbf{x}_{\max})}$$

Strict stability and exponential stability (4/5)

From $\frac{d}{dt}V(\mathbf{x}(t)) = -U(\mathbf{x}(t)) \leq -\frac{V(\mathrm{x}(t))}{V(\mathrm{x}_{\max})}$ it follows that $V(\mathbf{x}(t))\exp\left(\frac{t}{V(\mathrm{x}_{\max})}\right)$ is decreasing, and so

$$V(\mathbf{x}(t)) \leq V(\mathbf{x}(0))\exp\left(-\frac{t}{V(\mathbf{x}_{\max})}\right)$$

for all $t \in [0, +\infty)$. Then

$$\begin{aligned} U(\mathbf{x}(t)) &\leq \frac{V(\mathbf{x}(t))}{V(\mathbf{x}_{\min})} \leq \frac{V(\mathbf{x}(0))}{V(\mathbf{x}_{\min})}\exp\left(-\frac{t}{V(\mathbf{x}_{\max})}\right) \\ &\leq \frac{V(\mathbf{x}_{\max})}{V(\mathbf{x}_{\min})}U(\mathbf{x}(0))\exp\left(-\frac{t}{V(\mathbf{x}_{\max})}\right) \end{aligned}$$

Strict stability and exponential stability (5/5)

We get to choose C. Choose $C = I$, so that

$$U(\mathbf{x}) = (\mathbf{x} - \mathbf{x}^*)^T C(\mathbf{x} - \mathbf{x}^*) = \|\mathbf{x} - \mathbf{x}^*\|^2.$$

With this choice of C,

$$U(\mathbf{x}(t)) \leq \frac{V(\mathbf{x}_{\max})}{V(\mathbf{x}_{\min})} U(\mathbf{x}(0)) \exp\left(-\frac{t}{V(\mathbf{x}_{\max})}\right)$$

becomes

$$\|\mathbf{x}(t) - \mathbf{x}^*\| \leq \sqrt{\frac{V(\mathbf{x}_{\max})}{V(\mathbf{x}_{\min})}} \|\mathbf{x}(0) - \mathbf{x}^*\| \exp\left(-\frac{t}{2V(\mathbf{x}_{\max})}\right)$$

Choosing any $\delta > 0$, $C = \sqrt{\frac{V(\mathrm{x}_{\max})}{V(\mathrm{x}_{\min})}}$, and $\kappa = \frac{1}{2V(\mathrm{x}_{\max})}$, if $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$ then the unique solution to $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{g}$, $\mathbf{x}(0) = \mathbf{x}_0$ satisfies $\|\mathbf{x}(t) - \mathbf{x}^*\| \leq C\|\mathbf{x}_0 - \mathbf{x}^*\| \exp(-\kappa t)$ for all $t \in [0, +\infty)$. So $\mathbf{x}^*$ is an exponentially stable equilibrium.