MAU23205 Lecture 22

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Autonomous systems

A first order system where the derivatives of the dependent variables are expressed in terms of the dependent variables and the parameters, if any, but *not* the independent variable, is called autonomous.

I was careful not to say "a system where the derivatives are independent of the independent variable", because that could be confusing. A simple example, from Lecture 1, is

$$\frac{dx}{dt} = -x - y \qquad \frac{dy}{dt} = +x - y$$

The right hand sides of these equations are expressions in x and y, not involving t. We've solved this system, and every solution is of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \exp(-t) \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Autonomous systems

The derivatives are

$$\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \exp(-t) \begin{bmatrix} -\cos(t) - \sin(t) & -\cos(t) + \sin(t) \\ \cos(t) - \sin(t) & -\cos(t) - \sin(t) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

Unless $x_0 = y_0 = 0$, the derivatives dx/dt and dy/dt are not independent of t.

This is one of those things which is clearer in abstract form. The general first order system is one of the form

$$\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}(t), \mathbf{z})$$

where $\mathbf{z}$ are the parameters. This system is autonomous if and only if $\mathbf{F}(t, \mathbf{x}, \mathbf{z})$ is independent of t. That's quite different from saying that $\mathbf{x}'(t)$ is independent of t.

Linear systems

The example above was a linear homogeneous constant coefficient system. Such systems are always autonomous. Conversely, a linear system which is autonomous is necessarily constant coefficient. It isn't necessarily homogeneous, but the inhomogeneous term, if it's present, has to be constant.
The general first order linear system is

$$\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}(t), \mathbf{z})$$

with $\mathbf{F}(t, \mathbf{x}) = A(t)\mathbf{x} + \mathbf{g}(t)$. A necessary and sufficient condition for $\mathbf{F}$ to be independent of t is for A and $\mathbf{g}$ to be independent of t.
Not all autonomous systems are linear. A non-linear example, also from Lecture 1 is

$$x'(t) = y(t)z(t) \quad y'(t) = -x(t)z(t) \quad z'(t) = -k^2x(t)y(t)$$

Equilibria

$\mathbf{x}^*$ is called an equilibrium of the autonomous system

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t), \mathbf{z})$$

for the parameter value $\mathbf{z}$ if $\mathbf{F}(\mathbf{x}^*, \mathbf{z}) = \mathbf{0}$. If this is the case then $\mathbf{x}(t) = \mathbf{x}^*$ is a solution, and in fact is a constant solution. Conversely, the value of any constant solution is an equilibrium. So equilibria and constant solutions are nearly the same thing, but strictly speaking the former are vectors while the latter are vector valued functions. The unique equilibrium of the system $x' + x + y = 0$, $y' - x + y = 0$ is $(0, 0)$. The system

$$x'(t) = y(t)z(t) \quad y'(t) = -x(t)z(t) \quad z'(t) = -k^2x(t)y(t)$$

has many equilibria. Every point the x axis y axis or z axis is an equilibrium. If $k = 0$ then every point on the xy plane is also an equilibrium.

A topical example

Another autonomous system is the SIR model from epidemiology.

$$s'(t) = -bs(t)i(t), \quad i'(t) = bs(t)i(t) - ki(t) \quad r'(t) = ki(t).$$

s, i and r represent the susceptible, infected and "recovered" proportions of the population. Some people use "removed" in place of "recovered". $s + i + r$ is an invariant, as it should be. b represents the transmission rate and k the recovery rate. It makes sense to assume both are positive. What are the equilibria?

$$-bs^*i^* = 0, \quad bs^*i^* - ki^* = 0 \quad ki^* = 0.$$

A necessary and sufficient condition for these to be satisfied is $i^* = 0$. Not all of these equilibria are stable though.

Stability

There are a number of different, related, notions of stability. Suppose $\mathbf{x}^*$ is an equilibrium of $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x})$. $\mathbf{x}^*$ is called

- stable if for all $\epsilon > 0$ there is a $\delta > 0$ such that if $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$ then the initial value problem $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$ has a solution and every solution can be extended to a solution on $[0, \infty)$, and for all such solutions $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$ for all $t \in [0, \infty)$,
- unstable if it is not stable,
- asymptotically stable if there is a $\delta > 0$ such that if $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$ then the initial value problem $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$ has a solution and every solution can be extended to a solution on $[0, \infty)$, and for all such solutions $\lim_{t\to+\infty} \mathbf{x}(t) = \mathbf{x}^*$,
- strictly stable if it is stable and asymptotically stable,
- exponentially stable if there are a $\delta > 0$, a $C > 0$ and a $\kappa > 0$ such if $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$ then $\|\mathbf{x}(t) - \mathbf{x}^*\| \leq C\|\mathbf{x}_0 - \mathbf{x}_*\| \exp(-\kappa t)$ for all $t \in [0, \infty)$,

Comments

If **F** is even vaguely reasonable then some of the statements on the previous solution can be simplified. If **F** is continuous then the condition that the IVP has a solution follows from the existence part of the existence and uniqueness theorem. If **F** is continuously differentiable then we can replace references to "every solution" and "all such solutions" with "that solution", by the uniqueness part of the existence and uniqueness theorem.

Warning: Not everyone uses these terms in precisely the same way. Most people use "asymptotically stable" to refer to what I've called "strictly stable" and have no term for what I've called "asymptotically" stable. That's mostly fine, because because systems which are asymptotically stable but not strictly stable rarely come up, and aren't very useful.

Some people use "stable" to refer to what I've called "strictly stable" or even for what I've called "exponentially stable". They have no terms for weaker notions of stability. Those people are best avoided.

Linearisation

The linearisation of the autonomous system $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x})$ at the equilibrium $\mathbf{x}^*$ is the system

$$\mathbf{x}'(t) = \mathbf{F}\left(\mathbf{x}^*\right) + \mathbf{F}'\left(\mathbf{x}^*\right)\left(\mathbf{x}(t) - \mathbf{x}^*\right).$$

Here $\mathbf{F}'$ is the matrix valued derivative of a vector valued function of a vector argument, as in Lecture 7. I'm assuming here that it exists and is continuous near $\mathbf{x}_*$. It's the matrix whose j'th row, k'th column is $\partial F_k/\partial x_j$, where F_k is the k'th column of $\mathbf{F}$. Since $\mathbf{x}^*$ is an equilibrium of $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x})$ the term $\mathbf{F}(\mathbf{x}^*)$ on the right is zero. The only reason I've written it at all is so that you can recognise $\mathbf{F}(\mathbf{x}^*) + \mathbf{F}'(\mathbf{x}^*)(\mathbf{x}(t) - \mathbf{x}^*)$ as the first order Taylor expansion of $\mathbf{F}$ about $\mathbf{x}^*$ evaluated at $\mathbf{x}(t)$. Two alternate ways to write the linearisation are

$$\mathbf{x}'(t) = \mathbf{F}'\left(\mathbf{x}^*\right)\mathbf{x}(t) - \mathbf{F}'\left(\mathbf{x}^*\right)\mathbf{x}^*$$

$$\frac{d}{dt}\left(\mathbf{x}(t) - \mathbf{x}^*\right) = \mathbf{F}'\left(\mathbf{x}^*\right)\left(\mathbf{x}(t) - \mathbf{x}^*\right).$$

Intuitive meaning

An equilibrium is stable if all solutions which start out sufficiently close to equilibrium stay close to equilibrium. How close they need to start out is determined by how close you want them to stay.

It's strictly stable if solutions which start out sufficiently close to equilibrium also tend to equilibrium.

The other notions of stability, and the linearisation, are mostly useful for providing necessary or sufficient conditions either for stability or for strict stability.

The physical example to keep in mind is a pendulum. There are two equilibria. One with the weight below the pivot point and one with the weight above the pivot point. The first should be stable and the second should be unstable. The stable equilibrium should be strictly stable if there are dissipative forces, like air friction, but should not be strictly stable if there are no such forces.