MAU23205 Lecture 20

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Back to Laguerre

Last time I was able to find a change of variable $Y(t) = \begin{bmatrix} t-1 & 1 \\ 1 & 0 \end{bmatrix}$ to convert the coefficient matrix $A(t) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{t} & -\frac{1-t}{t} \end{bmatrix}$ for the Laguerre differential equation of order 1 into the upper triangular matrix $\tilde{A}(t) = \begin{bmatrix} 0 & -\frac{1}{t} \\ 0 & \frac{t-1}{t} \end{bmatrix}$. Where did this $Y(t)$ come from?

In the first column $t-1$ is a solution to the Laguerre equation of order 1

$$tx''(t) + (1-t)x'(t) + x(t) = 0.$$

and 1 is its derivative. It doesn't much matter what we put in the second column. Does this work in general? Where did I get the solution from?

Wronskian reduction of order (1/2)

Suppose $\mathbf{y}_1, \ldots, \mathbf{y}_k$ are linearly independent solutions of $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$. We can then choose functions $\mathbf{y}_{k+1}, \ldots, \mathbf{y}_n$ which make $\mathbf{y}_1, \ldots, \mathbf{y}_n$ linearly dependent. In fact, there's any interval about any point where we can choose $\mathbf{y}_{k+1}, \ldots, \mathbf{y}_n$ constant. They won't generally be solutions to $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$, of course. Let Y be the matrix whose columns are $\mathbf{y}_1, \ldots, \mathbf{y}_n$. Y is invertible. The j'th column of $Y'(t) - A(t)Y(t)$ is $\mathbf{y}_j'(t) - A(t)\mathbf{y}_j(t)$, so is zero if $j \leq k$. If $\tilde{A}(t) = -Y(t)^{-1}(Y'(t) - A(t)Y(t))$ then the first k columns of $\tilde{A}(t)$ are zero. In other words, $\tilde{A}$ is block upper triangular

$$\tilde{A}(t) = \begin{bmatrix} \tilde{A}_{\gamma\gamma} & \tilde{A}_{\gamma\delta} \\ O & \tilde{A}_{\delta\delta} \end{bmatrix}$$

where $\tilde{A}_{\gamma\gamma}$ is the $k \times k$ zero matrix, $\tilde{A}_{\gamma\delta}$ is a $k \times (n-k)$ matrix and $\tilde{A}_{\delta\delta}$ is an $(n-k) \times (n-k)$ matrix.

Wronskian reduction of order (2/2)

As we saw last time, if we can compute $\tilde{W}_{\gamma\gamma}$ and $\tilde{W}_{\delta\delta}$ then we can find $\tilde{W}_{\gamma\delta}(t,r) = \int_t^r \tilde{W}_{\gamma\gamma}(t,s)\tilde{A}_{\gamma\delta}(s)\tilde{W}_{\delta\delta}(s,r)\,ds$. $\tilde{W}_{\delta\gamma}(t,r) = O$ so we have all of $\tilde{W}$. Then $W(t,r) = Y(t)\tilde{W}(t,r)Y(r)^{-1}$. $A_{\gamma\gamma}(t) = O$ so $W_{\gamma\gamma} = I$. The only remaining questions are

- Can we find $\tilde{W}_{\delta\delta}$?
- Can we evaluate the integral?

$\tilde{W}_{\delta\delta}$ is the fundamental matrix for the $(n-k)\times(n-k)$ matrix $\tilde{A}_{\delta\delta}$. We can definitely find it if $k = n-1$, since that's the first order scalar case. This requires evaluating another integral. In practice usually $n = 2$ and $k = 1$. The procedure above allows us to solve the initial value problem $\mathbf{x}(t_0) = \mathbf{x}_0$ for $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{g}(t)$ for any $\mathbf{x}_0$ if we have a single non-zero solution to $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ and can evaluate three integrals.

Equations with polynomial solutions

Certain differential equations are known to have (non-zero) polynomial solutions. The Laguerre equation $xy''(x) + (1-x)y'(x) + ny(x) = 0$ has a polynomial solution of degree n. The Legendre equation $(1-x^2)y''(x) - 2xy'(x) + n(n+1)y(x) = 0$ has a polynomial solution of degree n. The Chebyshev equation $(1-x^2)y''(x) - xy'(x) + n^2y(x) = 0$ has a polynomial solution of degree n. The Hermite equation $y''(x) - xy'(x) + ny(x) = 0$ has a polynomial solution of degree n.

Once you know the solution exists you can find it by a sort of method of undetermined coefficents. Substitute $y(x) = \sum_{j=0}^{n} a_j x^j$ into the equation and solve some linear equations for the coefficients.

Laguerre again (1/4)

Substitute $y(x) = a_0 + a_1 x$ into $xy''(x) + (1-x)y'(x) + y(x) = 0$. $x0 + (1-x)a_1 + a_0 + a_1 x = 0$ This holds if and only if $a_0 + a_1 = 0$, so the general solution is $y(x) = a_1(x-1)$. Our particular solution was obtained by choosing $a_1 = 1$.
We could do the same for any order. For example substitute $y(x) = a_0 + a_1 x + a_2 x^2$ into $xy''(x) + (1-x)y'(x) + 2y(x) = 0$. $2xa_2 + (1-x)(a_1 + 2a_2 x) + 2(a_0 + a_1 x + a_2 x^2) = 0$. This holds if and only if $a_1 + 2a_0 = 0$ and $2a_2 - a_1 + 2a_2 + 2a_1 = 0$. So $a_1 = -2a_0$, $a_2 = -\frac{1}{4}a_1 = \frac{1}{2}a_0$, and $y(x) = a_0\left(1 - 2x + \frac{1}{2}x^2\right)$.
We can choose any non-zero value for a_0 but the conventional choice is $a_0 = 1$.

Laguerre again (2/4)

We might as well do all n at once. Substitute $y(x) = \sum_{j=0}^{\infty} a_j x^j$ into $xy''(x) + (1-x)y'(x) + ny(x) = 0$.

$$xy''(x) = \sum_{j=0}^{\infty} j(j-1)a_j x^{j-1}$$

$$(1-x)y'(x) = \sum_{j=0}^{\infty} j a_j x^{j-1} - \sum_{j=0}^{\infty} j a_j x^j$$

$$ny(x) = \sum_{j=0}^{\infty} n a_j x^j$$

$$xy''(x) + (1-x)y'(x) + ny(x) = \sum_{j=0}^{\infty} j^2 a_j x^{j-1} + \sum_{j=0}^{\infty} (n-j) a_j x^j$$

Laguerre again (3/4)

$$\begin{aligned}
xy''(x) + (1-x)y'(x) + ny(x) &= \sum_{j=0}^{\infty} j^2 a_j x^{j-1} + \sum_{j=0}^{\infty} (n-j) a_j x^j \\
&= \sum_{j=0}^{\infty} (j+1)^2 a_{j+1} x^j + \sum_{j=0}^{\infty} (n-j) a_j x^j
\end{aligned}$$

The second sum should really start from $j = -1$, but that summand is zero. So the differential equation will be satisfied if a_0 is chosen arbitrarily and subsequent coefficients are defined inductively by

$$a_{j+1} = -\frac{n-j}{(j+1)^2} a_j.$$

Note that $a_{n+1} = 0$ and then all subsequent coefficients are 0. This is why there is a solution which is a non-zero polynomial of degree n.

Laguerre again (4/4)

If we take $a_0 = 1$ then the solution of $a_{j+1} = -\frac{n-j}{(j+1)^2} a_j$ is

$$a_j = (-1)^j \frac{n!}{(j!)^2 (n-j)!}.$$

What if n is not an integer? Then x is not a polynomial, but does it make sense? Is it a solution?

$$\frac{a_{j+1} x^j}{a_j x^j} = \frac{j-n}{(j+1)^2} x \to 0$$

as $j \to \infty$ so the power series converges to all x. Term by term differentiation is permissible, so $y(x) = \sum_{j=0}^{\infty} a_j x^j$ satisfies the Laguerre equation. Whether you want to call this an explicit solution is a matter of taste.