MAU23205 Lecture 18

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Back to the general linear case

The solution to $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{g}(t)$, $\mathbf{x}(t_0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = W(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} W(t, s)\mathbf{g}(s)\, ds,$$

where $W = \lim_{m\to\infty} W_m$, $W_0 = I$ and

$$W_{m+1}(t, r) = I + \int_{t}^{r} A(s) W_m(s, r)\, ds.$$

The first few are

$$W_0(t, r) = I \qquad W_1(t, r) = I + \int_{r}^{t} A(s_1)\, ds_1$$

$$W_2(t, r) = I + \int_{r}^{t} A(s_1)\, ds_1 + \int_{r}^{t} \int_{r}^{s_2} A(s_2)A(s_1)\, ds_1\, ds_2$$

Rewriting W (1/4)

In general $W_m = \sum_{k=0}^{m} U_k$ where

$$U_k(t,r) = \int_r^t \int_r^{s_k} \cdots \int_r^{s_2} A(s_k)A(s_{k-1})\cdots A(s_1)\, ds_1 \cdots ds_{k-1}\, ds_k\,.$$

The order in which matrices are applied to a (column) vector is right to left, i.e. $A(s_1)$ then $A(s_2)$, etc.
$r \leq s_1 \leq s_2 \leq \cdots \leq s_{k-1} \leq s_k \leq t$ so the multiplications occur in order of "time", i.e. in increasing order of the argument of A. We can change the order of integration by permuting the variables with any permutation of $\{1, 2, \ldots, k\}$. For example,

$$\begin{aligned} U_2(t,r) &= \int_t^r \int_r^{s_2} A(s_2)A(s_1)\, ds_1\, ds_2 = \int_{r \leq s_1 \leq s_2 \leq t} A(s_2)A(s_1) \\ &= \int_t^r \int_{s_1}^t A(s_2)A(s_1)\, ds_2\, ds_1 = \int_t^r \int_{s_2}^t A(s_1)A(s_2)\, ds_1\, ds_2\,. \end{aligned}$$

At the end I renamed the variables.

Rewriting W (2/4)

$$U_2(t,r) = \int_t^r \int_r^{s_2} A(s_2)A(s_1)\,ds_1\,ds_2 = \int_t^r \int_{s_2}^t A(s_1)A(s_2)\,ds_1\,ds_2.$$

We can average these:

$$U_2(t,r) = \frac{1}{2}\int_t^r \int_r^{s_2} A(s_2)A(s_1)\,ds_1\,ds_2 + \frac{1}{2}\int_t^r \int_{s_2}^t A(s_1)A(s_2)\,ds_1\,ds_2.$$

Define P_2 by $P_2(s_1,s_2) = \begin{cases} A(s_2)A(s_1) & \text{if } s_1 \le s_2 \\ A(s_1)A(s_2) & \text{if } s_2 \le s_1 \end{cases}$

This is called the time-ordered product of the A's. Then

$$U_2(t,r) = \frac{1}{2}\int_r^t \int_r^t P(s_1,s_2)\,ds_1\,ds_2.$$

Rewriting W (3/4)

I did this for $k = 2$, but we can do the same for any k.

$$U_k(t, r) = \frac{1}{k!} \int_r^t \cdots \int_r^t \int_r^t P(s_1, s_2, \ldots, s_k)\, ds_1\, ds_2 \cdots ds_k$$

where

$$P(s_1, s_2, \ldots, s_k) = A(\sigma_k) \cdots A(\sigma_2) A(\sigma_1)$$

and σ_1, σ_2, ..., σ_k are s_1, s_2, ..., s_k reordered in such a way that $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k$.

Suppose $A(p)A(q) = A(q)A(p)$ for all p, q. Then the reordering was redundant.

$$P(s_1, s_2, \ldots, s_k) = A(s_k) \cdots A(s_2) A(s_1).$$

$$U_k(t, r) = \frac{1}{k!} \int_r^t \cdots \int_r^t \int_r^t A(s_k) \cdots A(s_2) A(s_1) ds_1\, ds_2 \cdots ds_k$$

Rewriting W (4/4)

If $A(p)A(q) = A(q)A(p)$ for all p, q then

$$\begin{aligned} U_k(t,r) &= \frac{1}{k!}\int_r^t \cdots \int_r^t \int_r^t A(s_k)\cdots A(s_2)A(s_1)ds_1\, ds_2 \cdots ds_k \\ &= \frac{1}{k!}\int_r^t A(s_k)\, ds_k \cdots \int_r^t A(s_2)\, ds_2 \int_r^t A(s_1)\, ds_1 \\ &= \frac{1}{k!}\left(\int_r^t A(s)\, ds\right)^k . \end{aligned}$$

$$W_m(t,r) = \sum_{k=0}^{m} \frac{1}{k!}\left(\int_r^t A(s)\, ds\right)^k .$$

$$W(t,r) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\int_r^t A(s)\, ds\right)^k = \exp\left(\int_r^t A(s)\, ds\right) .$$

Special cases

If $A(p)A(q) = A(q)A(p)$ for all p, q then

$$W(t, r) = \exp\left(\int_r^t A(s)\,ds\right).$$

Under what conditions is $A(p)A(q) = A(q)A(p)$ for all p, q?

- If A is constant. In that case $\int_r^t A(s)\,ds = (t - r)A$ and $W(t, r) = \exp((t - r)A)$. This is the case we spent the last two weeks considering.
- If A is 1×1. 1×1 matrices always commute. The solution to $x'(t) = a(t)x(t) + g(t)$, $x(t_0) = x_0$ is

$$x(t) = w(t, t_0)x_0 + \int_{t_0}^t w(t, s)g(s)\,ds$$

where

$$w(t, r) = \exp\left(\int_r^t a(s)\,ds\right).$$

Warning!!!

$$W(t,r) = \exp\left(\int_r^t A(s)\,ds\right).$$

is *not true* in general! It's true in the constant coefficient case and the scalar (1×1) case, but not in general.

$$A(t) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{t^2} & -\frac{1}{t} \end{bmatrix}$$

is a counterexample.

$$\int_r^t A(s)\,ds = \begin{bmatrix} 0 & t-r \\ \frac{1}{t} - \frac{1}{t} & \log(r/t) \end{bmatrix}$$

Its exponential is not $\begin{bmatrix} \cos(\log(t/r)) & r\sin(\log(t/r)) \\ -\frac{1}{t}\sin(\log(t/r)) & \frac{r}{t}\cos(\log(t/r)) \end{bmatrix}$, which is the correct $W(t,r)$.

First order linear scalar example

What it the solution to $x'(t) = 2tx(t) + \exp(t^2)$, $x(t_0) = x_0$?
The solution to $x'(t) = a(t)x(t) + g(t)$, $x(t_0) = x_0$ is

$$x(t) = w(t, t_0)x_0 + \int_{t_0}^{t} w(t, s)g(s)\, ds$$

where

$$w(t, r) = \exp\left(\int_r^t a(s)\, ds\right).$$

Here $a(t) = 2t$ and $g(t) = \exp(t^2)$.

$$w(t, r) = \exp\left(\int_r^t 2s\, ds\right) = \exp\left(t^2 - r^2\right).$$

$$\begin{aligned} x(t) &= \exp\left(t^2 - t_0^2\right) x_0 + \int_{t_0}^{t} \exp\left(t^2 - s^2\right) \exp\left(s^2\right)\, ds \\ &= \exp\left(t^2 - t_0^2\right) x_0 + (t - t_0) \exp\left(t^2\right). \end{aligned}$$

Green's functions (1/3)

$\sum_{j=0}^{m} \alpha_j(t) x^{(j)}(t) = f(t)$ with $\alpha_m = 1$ is equivalent to $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{g}(t)$ where

$$a_{j,k}(t) = \begin{cases} 0 & \text{if } j < m, k \neq j+1, \\ 1 & \text{if } j < m, k = j+1, \\ -\alpha_{k-1}(t) & \text{if } j = m. \end{cases}$$

$$g_j(t) = \begin{cases} 0 & \text{if } j < m, \\ f(t) & \text{if } j = m. \end{cases}$$

The solution to $\mathbf{x}(t_0) = \mathbf{x}_0$, $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{g}(t)$ is

$$\mathbf{x}(t) = W(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} W(t, s)\mathbf{g}(s)\, ds.$$

$$\frac{\partial W}{\partial t}(t, s) = A(t)W(t, s), \qquad \frac{\partial W}{\partial s}(t, s) = -W(t, s)A(s).$$

Green's functions (2/3)

$$\frac{\partial W}{\partial t}(t,s) = A(t)W(t), \qquad \frac{\partial W}{\partial s}(t,s) = -W(t)A(s).$$

$$\frac{\partial w_{j,l}}{\partial t}(t,s) = \sum_{k=1}^{m} a_{j,k}(t) w_{k,l}(t,s),$$

$$\frac{\partial w_{i,k}}{\partial s}(t,s) = -\sum_{j=1}^{m} w_{i,j}(t,s) a_{j,k}(s).$$

$$a_{j,k}(t) = \begin{cases} 0 & \text{if } j < m,\ k \neq j+1, \\ 1 & \text{if } j < m,\ k = j+1, \\ -\alpha_{k-1}(t) & \text{if } j = m. \end{cases}$$

If $j < m$ then $\frac{\partial w_{j,l}}{\partial t}(t,s) = w_{j+1,l}(t,s)$. If $k > 1$ then $\frac{\partial w_{i,k}}{\partial s}(t,s) = -w_{i,k-1}(t,s) + \alpha_{k-1}(s) w_{m,k}(t,s)$.

Green's functions (3/3)

If $j < m$ then $\frac{\partial w_{j,k}}{\partial t}(t,s) = w_{j+1,k}(t,s)$. If $k > 1$ then $\frac{\partial w_{j,k}}{\partial s}(t,s) = -w_{j,k-1}(t,s) + \alpha_{k-1}(s) w_{m,k}(t,s)$. Equivalently, $w_{j+1,k}(t,s) = \frac{\partial w_{j,k}}{\partial t}(t,s)$ and $w_{j,k-1}(t,s) = \alpha_{k-1}(s) w_{m,k}(t,s) - \frac{\partial w_{j,k}}{\partial s}(t,s)$. If we know $G(s,t) = w_{1,m}$ then we can compute all the other entries in $W(s,t)$ from these formulae. This G is called the *Green's function* for the equation $\sum_{j=0}^{m} \alpha_j(t) x^{(j)}(t) = f(t)$ From the general solution formula we see that it's the solution to the initial value problem $x(s) = 0$, $x'(s) = 0$, ..., $x^{(m-1)}(s) = 1$ for the homogeneous equation $\sum_{j=0}^{m} \alpha_j(t) x^{(j)}(t) = 0$. Also, the solution to the initial value problem $x(t_0) = 0$, $x'(t_0) = 0$, ..., $x^{(m-1)}(t_0) = 0$ for the inhomogeneous equation $\sum_{j=0}^{m} \alpha_j(t) x^{(j)}(t) = f(t)$ is $x(t) = \int_{t_0}^{t} G(t,s) f(s)\, ds$.