MAU23205 Lecture 17

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A second order example (1/4)

Do all solutions of $x'' + 2x' + 2x = 0$ tend to 0 as t tends to ∞? The equivalent first order system is $\mathbf{x}'(t) = A\mathbf{x}(t)$, where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

with characteristic polynomial $p_A(z) = z^2 + 2z + 2$. This factors as $p_A(z) = (z + 1 - i)(z + 1 + i)$. The roots, $-1 + i$ and $-1 - i$, have negative real part, so all solutions tend to 0.
Alternatively we can choose a positive definite symmetric C and solve $A^T B + BA + C = O$. We might as well choose

$$C = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A second order example (2/4)

The general 2×2 symmetric matrix is

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}.$$

$$A^T B = \begin{bmatrix} -2b_{12} & -2b_{22} \\ b_{11} - 2b_{12} & b_{12} - 2b_{22} \end{bmatrix} \qquad BA = \begin{bmatrix} -2b_{12} & b_{11} - 2b_{12} \\ -2b_{22} & b_{12} - 2b_{22} \end{bmatrix}$$

$$A^T B + BA + C = \begin{bmatrix} -4b_{12} + 1 & b_{11} - 2b_{12} - 2b_{22} \\ b_{11} - 2b_{12} - 2b_{22} & 2b_{12} - 4b_{22} + 1 \end{bmatrix}$$

We need $-4b_{12} + 1 = 0$, $2b_{12} - 4b_{22} + 1 = 0$ and $b_{11} - 2b_{12} - 2b_{22} = 0$. The solution is $b_{12} = 1/4$, $b_{22} = 3/8$ and $b_{11} = 5/4$, so

$$B = \begin{bmatrix} 5/4 & 1/4 \\ 1/4 & 3/8 \end{bmatrix}.$$

A second order example (3/4)

The preceding calculation gives $\frac{d}{dt}\mathbf{x}(t)^T B\mathbf{x}(t) = -\mathbf{x}(t)^T C\mathbf{x}(t)$, where $B = \begin{bmatrix} 5/4 & 1/4 \\ 1/4 & 3/8 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}$, i.e.
$\frac{d}{dt}\left(\frac{5}{4}x(t)^2 + \frac{1}{2}x(t)x'(t) + \frac{3}{8}x'(t)^2\right) = -\left(x(t)^2 + x'(t)^2\right) < 0$,
and hence $\mathbf{x}(t)^T B\mathbf{x}(t)$ is strictly decreasing if x is not the zero solution. To conclude that all solutions tend to zero we need that $\mathbf{x}^T B\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$, in other words that B is positive definite.

Is B positive definite? All the entries are positive, but this is neither necessary nor sufficient for positive definiteness!

There are several ways to check positive definiteness.

Sylvester's criterion is that the leading principal minors, i.e. the 1×1 submatrix in the the upper left corner, the 2×2 submatrix in the upper left corner, etc., all have positive determinant. In this case the leading principal minors are $[5/4]$ and $\begin{bmatrix} 5/4 & 1/4 \\ 1/4 & 3/8 \end{bmatrix}$, with determinants $5/4 > 0$ and $13/32 > 0$, so B is positive definite.

A second order example (4/4)

An alternate criterion is based on Gaussian elimination (row reduction), which puts matrices in row echelon form, with the first non-zero entry in each row farther to the right than the previous ones, using three permitted types of operations:

- swap two rows
- multiply a row by a non-zero scalar
- add a scalar multiple of one row to another

If you can put a matrix in row echelon form using only the third type, and all the diagonal entries are positive, then the original matrix was positive definite and vice versa. This criterion is more complicated to explain, but easier to implement. Adding $-1/5$ of the first row of B to the second gives the row echelon form $\begin{bmatrix} 5/4 & 1/4 \\ 0 & 13/40 \end{bmatrix}$. $5/4 > 0$ and $13/40 > 0$, so B is positive definite.

A third order example (1/4)

Do all solutions to $x'''(t) + 3x''(t) + 6x'(t) + 6x(t) = 0$ tend to 0 as t tends to ∞? The equivalent first order matrix equation is $\mathbf{x}'(t) = A\mathbf{x}(t)$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -6 & -3 \end{bmatrix}.$$

We might as well choose $C = I$. Let

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}.$$

We want to solve $A^T B + BA + C = O$, so we compute the left hand side.

A third order example (2/4)

$$A^T B = \begin{bmatrix} -6b_{13} & -6b_{23} & -6b_{33} \\ b_{11} - 6b_{13} & b_{12} - 6b_{23} & b_{13} - 6b_{33} \\ b_{12} - 3b_{13} & b_{22} - 3b_{23} & b_{23} - 3b_{33} \end{bmatrix}$$

$$BA = \begin{bmatrix} -6b_{13} & b_{11} - 6b_{13} & b_{12} - 3b_{13} \\ -6b_{23} & b_{12} - 6b_{23} & b_{22} - 3b_{23} \\ -6b_{33} & b_{13} - 6b_{33} & b_{23} - 3b_{33} \end{bmatrix}$$

So $A^T B + BA + C$ is

$$\begin{bmatrix} -12b_{13} + 1 & b_{11} - 6b_{13} - 6b_{23} & b_{12} - 3b_{13} - 6b_{33} \\ b_{11} - 6b_{13} - 6b_{23} & 2b_{12} - 12b_{23} + 1 & b_{13} + b_{22} - 3b_{23} - 6b_{33} \\ b_{12} - 3b_{13} - 6b_{33} & b_{13} + b_{22} - 3b_{23} - 6b_{33} & 2b_{23} - 6b_{33} + 1 \end{bmatrix}$$

We need to find $b_{11}, \ldots, b_{33}$ which make the entries zero.

A third order example (3/4)

$$-12b_{13} + 1 = 0 \quad b_{11} - 6b_{13} - 6b_{23} = 0$$
$$b_{12} - 3b_{13} - 6b_{33} = 0 \quad 2b_{12} - 12b_{23} + 1 = 0$$
$$b_{13} + b_{22} - 3b_{23} - 6b_{33} = 0 \quad 2b_{23} - 6b_{33} + 1 = 0$$

Add $1/2$ times the first equation, -2 times the third equation and 2 times the sixth equation to the fourth equation, and then divide it by -8. Divide the first equation by -12.

$$b_{13} - 1/12 = 0 \quad b_{11} - 6b_{13} - 6b_{23} = 0$$
$$b_{12} - 3b_{13} - 6b_{33} = 0 \quad b_{23} - 7/16 = 0$$
$$b_{13} + b_{22} - 3b_{23} - 6b_{33} = 0 \quad 2b_{23} - 6b_{33} + 1 = 0$$

Use the first and fourth equations to eliminate b_{13} and b_{23} from the remaining equations.

$$b_{13} - 1/12 = 0 \quad b_{11} - 25/8 = 0 \quad b_{12} - 6b_{33} - 1/4 = 0$$
$$b_{23} - 7/16 = 0 \quad b_{22} - 6b_{33} - 59/48 = 0 \quad -6b_{33} + 15/8 = 0$$

A third order example (4/4)

$$b_{13} - 1/12 = 0 \quad b_{11} - 25/8 = 0 \quad b_{12} - 6b_{33} - 1/4 = 0$$
$$b_{23} - 7/16 = 0 \quad b_{22} - 6b_{33} - 59/48 = 0 \quad -6b_{33} + 15/8 = 0$$

Subtract the sixth equation from the third and fifth equations, and then divide it by -6.

$$b_{13} - 1/12 = 0 \quad b_{11} - 25/8 = 0 \quad b_{12} - 17/8 = 0$$
$$b_{23} - 7/16 = 0 \quad b_{22} - 149/48 = 0 \quad b_{33} - 5/16 = 0$$

So

$$B = \begin{bmatrix} 25/8 & 17/8 & 1/12 \\ 17/8 & 149/8 & 7/16 \\ 1/12 & 7/16 & 5/16 \end{bmatrix}$$

Is B positive definite? Yes. So all solutions tend to 0 as $t \to \infty$.

Why not use the roots of the characteristic polynomial?

The characteristic polynomial is $p_A(z) = z^3 + 3z^2 + 6z + 6$. The roots can be found from the cubic formula. They are

$$-1 - \sqrt[3]{\sqrt{2}+1} + \sqrt[3]{\sqrt{2}-1}$$

and

$$-1 + \frac{1}{2}\sqrt[3]{\sqrt{2}+1} - \frac{1}{2}\sqrt[3]{\sqrt{2}-1} \pm \frac{i}{2}\sqrt{3}\left(\sqrt[3]{\sqrt{2}+1} + \sqrt[3]{\sqrt{2}-1}\right).$$

The calculation which leads to this is not pretty. Can you look at this and see whether the real parts are negative? If we had a fourth order equation we'd have to use the quartic formula, which is much worse. If we had a fifth order equation we'd have to use the quintic formula, which doesn't exist, or numerical root finding, which is tricky.