MAU23205 Lecture 16

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20 October 2021

When do all solutions to $\sum_{i=0}^{m} \alpha_i x^{(i)} = 0$ tend to 0?

The general solution to $\sum_{i=0}^{m} \alpha_i x^{(i)}(t) = 0$ is $x(t) = \sum_{j=0}^{m} c_j x_j(t)$, where x_j are the basic solutions. By Lecture 11 these are $q_{\lambda,k}(t) = \frac{t^k}{k!}\exp(\lambda t)$, $r_{\kappa,\omega,k}(t) = \frac{t^k}{k!}\exp(\kappa t)\cos(\omega t)$, and $s_{\kappa,\omega,k}(t) = \frac{t^k}{k!}\exp(\kappa t)\sin(\omega t)$. λ and $\kappa \pm i\omega$ range over the real and complex roots of $p(z) = \sum_{i=0}^{m} \alpha_i z^i$, respectively, and k is less that the multiplicity of the corresponding root.

Under what conditions do all solutions to $\sum_{i=0}^{m} \alpha_i x^{(i)}(t) = 0$ tend to 0 as $t \to +\infty$? If and only if all the basic solutions tend to 0 as $t \to +\infty$, which happens if and only if all roots of p have negative real part.

Can we test this? Yes, if we can factor p. We can also do it without factoring p, using the Routh-Hurwitz Criterion.

Routh-Hurwitz examples (1/2)

Take the polynomial $p(z) = \sum_{i=0}^{m} \alpha_i z^i$ and form the $(m+1) \times (2m+3 \pm 1)/4$ matrix S

$$s_{j,k} = \begin{cases} \alpha_{m-2k+2} & \text{if } j = 1 \\ \alpha_{m-2k+1} & \text{if } j = 2 \\ \frac{s_{j-1,1}s_{j-2,k+1} - s_{j-2,1}s_{j-1,k+1}}{s_{j-1,1}} & \text{if } j > 2 \end{cases}$$

The real parts of the roots of p are all negative if and only if all the entries in the first column are of the same sign.
For example, take $p(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4$. S is

$$\begin{bmatrix} \frac{1}{24} & \frac{1}{2} & 1 \\ \frac{1}{6} & 1 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{2}{3} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

So all roots of p have negative real part.

Routh-Hurwitz examples (2/2)

Indeed the roots of $1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}$ are $-1.729 \pm 0.889i$ and $-0.271 \pm 2.505i$

Or take $p(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$. The first four rows are

$$\begin{bmatrix} \frac{1}{120} & \frac{1}{6} & 1 \\ \frac{1}{24} & \frac{1}{2} & 1 \\ \frac{1}{15} & \frac{4}{5} & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

We can't fill in the fifth row because we'd have to divide by 0! This failure tells you that there's a root whose real part is not negative. $1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$ Indeed the roots of $1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$ are -2.181, $-1.650 \pm 1.694i$, and $0.240 \pm 3.128i$.

When are all solutions to $\sum_{i=0}^{m} \alpha_i x^{(i)} = 0$ bounded?

Under what conditions are all solutions to $\sum_{i=0}^{m} \alpha_i x^{(i)}(t) = 0$ bounded as $t \to +\infty$? If and only if all the basic solutions are bounded $t \to +\infty$, Which happens if and only if all roots of p have non-positive real part and any purely imaginary roots are simple.
Can we test this? Yes, but it's more complicated. Finding repeated roots is easy. They're the roots of $\gcd(p, p')$. We can find and remove purely imaginary roots. $p(iy) = 0$ if and only if p is a common zero of $\sum_{k=0}^{m/2}(-1)^k \alpha_{2k} y^{2k}$ and $\sum_{k=0}^{(m-1)/2}(-1)^k \alpha_{2k+1} y^{2k+1}$. After removing any simple purely imaginary roots we can apply Routh-Hurwitz to check whether the remaining roots have negative real parts.

When do all solutions to $\mathbf{x}'(t) = A\mathbf{x}(t)$ tend to $\mathbf{0}$?

Every solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is of the form

$$\mathbf{x}(t) = \exp(tA)\mathbf{x}_0 = \sum_{1 \le j,k \le m} x_j(t) r_{j,k} A^{k-1} \mathbf{x}_0.$$

If all the basic solutions tend to 0 as $t \to +\infty$ then $\mathbf{x}(t)$ tends $\mathbf{0}$ as $t \to +\infty$. The converse isn't obvious, because we might have a basic solution x_j which doesn't tend to $\mathbf{0}$ but $\sum_{k=1}^{m} r_{j,k} A^{k-1} = O$. This can't happen if p is the minimal polynomial, because then we'd have a non-zero polynomial $q_j(z) = \sum_{k=1}^{m} r_{j,k} z^{k-1}$ of degree less than m such that $q_j(A) = O$. It *can* happen for the characteristic polynomial. We've seen examples.

All solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ tend to $\mathbf{0}$ if and only if all roots of the minimal polynomial have negative real parts. But the characteristic and minimal polynomials have the same set of roots, so all solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ tend to $\mathbf{0}$ if and only if all roots of the characteristic polynomial have negative real parts.

When are all solutions to $\mathbf{x}'(t) = A\mathbf{x}(t)$ bounded?

$$\mathbf{x}(t) = \exp(tA)\mathbf{x}_0 = \sum_{1 \le j,k \le m} x_j(t) r_{j,k} A^{k-1} \mathbf{x}_0.$$

If all the basic solutions are bounded as $t \to +\infty$ then $\mathbf{x}(t)$ is bounded as $t \to +\infty$. The converse isn't obvious, for the same reason as before. Once again, everything is okay if we use the minimal polynomial.

All solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ tend to $\mathbf{0}$ if and only if all roots of the minimal polynomial have non-positive real parts and any purely imaginary roots are simple.

The characteristic and minimal polynomials have the same set of roots, but different multiplicities, so we can't use the characteristic polynomial to answer this question.

An alternate method (1/3)

Suppose C is positive definite symmetric and A is such that all solutions to $\mathbf{x}' = A\mathbf{x}$ tend to $\mathbf{0}$. Define

$$M(t) = \exp(tA)^T C \exp(tA), \qquad B = \int_0^\infty M(t)\,dt$$

The integral exists because $\exp(tA)$ tends to O exponentially fast. $M(t)^T = M(t)$ and if $\mathbf{y} \neq 0$ then

$$\mathbf{y}^T M(t)\mathbf{y} = (\exp(tA)\mathbf{y})^T C \exp(tA)\mathbf{y} > 0$$

so $B = B^T$ and $\mathbf{y}^T B\mathbf{y} > 0$, i.e. B is positive definite symmetric.

$$\begin{aligned} M'(t) &= \frac{d}{dt} \exp(tA)^T C \exp(tA) \\ &= A^T \exp(tA)^T C \exp(tA) + \exp(tA)^T C \exp(tA) A \\ &= A^T M(t) + M(t) A. \end{aligned}$$

An alternate method (2/3)

Integrate $M'(t) = A^T M(t) + M(t)A$ to get

$$-C = A^T B + BA.$$

So if C is positive definite symmetric and A is such that all (basic) solutions to $\mathbf{x}' = A\mathbf{x}$ tend to $\mathbf{0}$. then the matrix equation

$$A^T B + BA + C = O$$

has a positive definite symmetric solution B.
Suppose, conversely, that $A^T B + BA + C = O$ has a positive definite symmetric solution B. If $\mathbf{x}' = A\mathbf{x}$ then

$$\frac{d}{dt}\mathbf{x}(t)^T B\mathbf{x}(t) = \mathbf{x}(t)^T A^T B\mathbf{x}(t) + \mathbf{x}(t)^T BA\mathbf{x}(t) = -\mathbf{x}(t)^T C\mathbf{x}(t)$$

If $\mathbf{x} \neq 0$ then $\mathbf{x}(t)^T B\mathbf{x}(t)$ is positive but strictly decreasing. It must tend to 0, which means $\mathbf{x}(t)$ also tends to $\mathbf{0}$.

An alternate method (3/3)

- If all solutions to $\mathbf{x}' = A\mathbf{x}$ tend to $\mathbf{0}$ as $t \to +\infty$ then for *every* positive definite symmetric matrix C there is a positive definite symmetric solution B to the equation $A^T B + BA + C = O$.
- If for *some* positive definite symmetric matrix C there is a positive definite symmetric solution B to the equation $A^T B + BA + C = O$ then all solutions to $\mathbf{x}' = A\mathbf{x}$ tend to $\mathbf{0}$ as $t \to +\infty$.

This gives a weird Linear Algebra theorem: If the equation $A^T B + BA + C = O$ has a positive definite symmetric solution for some positive definite symmetric C then it has a positive definite symmetric solution for all positive definite symmetric C.
This also gives a way to test whether all solutions to $\mathbf{x}' = A\mathbf{x}$ tend to $\mathbf{0}$: Choose a C and look for a B. $A^T B + BA + C = O$ is a set of linear equations in the entries of B. Solve them and check whether B is positive definite symmetric.