MAU23205 Lecture 13

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The structure of the matrix exponential

Suppose A is a square matrix, $p(A) = O$, and $p \neq 0$. p could be the minimal polynomial or the characteristic polynomial or something else. Let $x_1, \ldots, x_m$ be a set of basic solutions, real or complex. Then

$$\exp(tA) = \sum_{1 \leq j,k \leq m} r_{j,k} x_j(t) A^{k-1}.$$

The matrix R is $Y(0)^{-1}$, where $y_{j,k}(t) = x_k^{(j-1)}(t)$.
I'll prove this in Lecture 15. For now we'll just look at examples. Note that you can sum over j first or k first. In the former case you get a sum of (non-basic) solutions times powers of k. In the latter, you get a sum of basic solutions times matrices. Which is more useful depends on what you're doing.

The 2×2 case

In the 2×2 case the characteristic polynomial is quadratic,

$$p_A(z) = z^2 - \operatorname{tr}(A)z + \det(A).$$

If $\Delta_A = \operatorname{tr}(A)^2 - 4\det(A) < 0$ then this p_A factors as

$$p_A(z) = (z - \lambda_+)(z - \lambda_-), \qquad \lambda_\pm = \frac{\operatorname{tr}(A)}{2} \pm i\frac{\sqrt{-\Delta_A}}{2},$$

If $\Delta_A = 0$ then it factors as

$$p_A(z) = (z - \lambda)^2, \qquad \lambda = \frac{\operatorname{tr}(A)}{2}.$$

If $\Delta_A > 0$ then this p_A factors as

$$p_A(z) = (z - \lambda_+)(z - \lambda_-), \qquad \lambda_\pm = \frac{\operatorname{tr}(A)}{2} \pm \frac{\sqrt{\Delta_A}}{2},$$

$\Delta_A = 0$

If $\Delta_A = 0$ then our basic solutions are

$$x_1(t) = \exp\left(\frac{\operatorname{tr}(A)}{2}t\right), \qquad x_2(t) = t\exp\left(\frac{\operatorname{tr}(A)}{2}t\right),$$

The matrix $Y(0)$ is

$$Y(0) = \begin{bmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \operatorname{tr}(A)/2 & 1 \end{bmatrix}$$

The matrix R is

$$R = \begin{bmatrix} 1 & 0 \\ -\operatorname{tr}(A)/2 & 1 \end{bmatrix}$$

Then

$$\begin{aligned} \exp(tA) &= 1x_1(t)I + 0x_1(t)A - \frac{\operatorname{tr}(A)}{2}x_2(t)I + 1x_2(t)A \\ &= \left(1 - \frac{\operatorname{tr}(A)}{2}t\right)\exp\left(\frac{\operatorname{tr}(A)}{2}t\right)I + t\exp\left(\frac{\operatorname{tr}(A)}{2}t\right)A. \end{aligned}$$

A different approach (1/2)

We could have got the same result from the definition of the matrix exponential and a bit of formal calculation. Let $B = A - \frac{\mathrm{tr}(A)}{2} I$.

$$B^2 = A^2 - \mathrm{tr}(A)A + \frac{\mathrm{tr}(A)^2}{4} I.$$

We're in the case $\Delta_A = 0$ so $\frac{\mathrm{tr}(A)^2}{4} = \det(A)$ and

$$B^2 = A^2 - \mathrm{tr}(A)A + \det(A)I = p_A(A) = O$$

by Cayley-Hamilton.

$$\exp(tB) = \sum_{k=0}^{\infty} \frac{t^k}{k!} B^k = \sum_{k=0}^{1} \frac{t^k}{k!} B^k = I + tB = \left(1 - \frac{\mathrm{tr}(A)}{2}\right) I + tA.$$

A different approach (2/2)

$$B = A - \frac{\mathrm{tr}(A)}{2} I, \qquad \exp(tB) = \left(1 - \frac{\mathrm{tr}(A)}{2}\right) I + tA.$$

$$A = B + \frac{\mathrm{tr}(A)}{2} I.$$

$$\exp(tA) = \exp\left(tB + t\frac{\mathrm{tr}(A)}{2} I\right) = \exp(tB) \exp\left(t\frac{\mathrm{tr}(A)}{2} I\right).$$

$$\exp(t\mu I) = \sum_{k=0}^{\infty} \frac{1}{k!} \mu^k t^k I^k = \exp(\mu t) I.$$

$$\exp(tA) = \exp\left(\frac{\mathrm{tr}(A)}{2} t\right) \left[\left(1 - \frac{\mathrm{tr}(A)}{2}\right) I + tA\right].$$

The case $\Delta_A < 0$ (1/3)

If $\Delta_A < 0$ then our complex basic solutions are

$$\begin{aligned} x_1(t) = \exp(\lambda_+ t) &= \exp\left(\frac{\mathrm{tr}(A)}{2}t\right)\exp\left(i\frac{\sqrt{-\Delta_A}}{2}t\right) \\ &= \exp\left(\frac{\mathrm{tr}(A)}{2}t\right)\left(\cos\left(\frac{\sqrt{-\Delta_A}}{2}t\right) + i\sin\left(\frac{\sqrt{-\Delta_A}}{2}t\right)\right) \end{aligned}$$

and

$$x_2(t) = \exp\left(\frac{\mathrm{tr}(A)}{2}t\right)\left(\cos\left(\frac{\sqrt{-\Delta_A}}{2}t\right) - i\sin\left(\frac{\sqrt{-\Delta_A}}{2}t\right)\right)$$

$$Y(0) = \begin{bmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{\mathrm{tr}(A)}{2} + i\frac{\sqrt{-\Delta_A}}{2} & \frac{\mathrm{tr}(A)}{2} - i\frac{\sqrt{-\Delta_A}}{2} \end{bmatrix}$$

The case $\Delta_A < 0$ (2/3)

$$R = \frac{1}{\sqrt{-\Delta}} \begin{bmatrix} \frac{\sqrt{-\Delta_A}}{2} + i\frac{\mathrm{tr}(A)}{2} & -i \\ \frac{\sqrt{-\Delta_A}}{2} - i\frac{\mathrm{tr}(A)}{2} & i \end{bmatrix}$$

$\exp(tA)$ is the sum of four terms,

$$\frac{1}{\sqrt{-\Delta}} \left(\frac{\sqrt{-\Delta_A}}{2} + i\frac{\mathrm{tr}(A)}{2} \right) \exp\left(\frac{\mathrm{tr}(A)}{2} t\right) \left(\cos\left(\frac{\sqrt{-\Delta_A}}{2} t\right) + i \sin\left(\frac{\sqrt{-}}{2}\right.\right.$$

$$\frac{1}{\sqrt{-\Delta}} (-i) \exp\left(\frac{\mathrm{tr}(A)}{2} t\right) \left(\cos\left(\frac{\sqrt{-\Delta_A}}{2} t\right) + i \sin\left(\frac{\sqrt{-\Delta_A}}{2} t\right)\right) A$$

$$\frac{1}{\sqrt{-\Delta}} \left(\frac{\sqrt{-\Delta_A}}{2} - i\frac{\mathrm{tr}(A)}{2} \right) \exp\left(\frac{\mathrm{tr}(A)}{2} t\right) \left(\cos\left(\frac{\sqrt{-\Delta_A}}{2} t\right) - i \sin\left(\frac{\sqrt{-}}{2}\right.\right.$$

$$\frac{1}{\sqrt{-\Delta}} (i) \exp\left(\frac{\mathrm{tr}(A)}{2} t\right) \left(\cos\left(\frac{\sqrt{-\Delta_A}}{2} t\right) - i \sin\left(\frac{\sqrt{-\Delta_A}}{2} t\right)\right) A$$

The case $\Delta_A < 0$ (3/3)

Adding those,

$$\exp(tA) = \exp\left(\frac{\operatorname{tr}(A)}{2}t\right)\left(\cos\left(\frac{\sqrt{-\Delta_A}}{2}t\right) - \frac{\operatorname{tr}(A)}{2}\frac{\sin\left(\frac{\sqrt{-\Delta_A}}{2}t\right)}{\sqrt{-\Delta_A}/2}\right)I$$
$$+ \exp\left(\frac{\operatorname{tr}(A)}{2}t\right)\frac{\sin\left(\frac{\sqrt{-\Delta_A}}{2}t\right)}{\sqrt{-\Delta_A}/2}A$$

Note that the answer is real, as it must be.
As with the case $\Delta_A = 0$ there is a different approach, based on the series definition of the exponential, but I won't give the details.

Avoiding complex calculations

If you don't enjoy complex arithmetic you can avoid it by using a different set of basic solutions.
For real roots λ we still use $\frac{t^k}{k!}\exp(\lambda t)$.
For pairs $\lambda = \kappa \pm i\omega$ of complex conjugate roots we use $\frac{t^k}{k!}\exp(\kappa t)\cos(\omega t)$ and $\frac{t^k}{k!}\exp(\kappa t)\sin(\omega t)$.
As before, k ranges over non-negative integers less than the multiplicity of λ as a root of p.
The number of basic solutions is still equal to the degree of p.
The matrix $Y(0)$ is still given by $y_{jk}(0) = x_k^{(j-1)}(0)$. R is still $Y(0)^{-1}$ and we still have

$$\exp(tA) = \sum_{j=1}^{m}\sum_{k=1}^{m} r_{jk} x_j(t) A^{k-1}.$$

Each summand is now a real matrix valued function.

The case $\Delta_A < 0$ (4/3)

In the 2×2 case the roots are $\lambda = \kappa \pm i\omega$, where

$$\kappa = \frac{\operatorname{tr}(A)}{2} \qquad \omega = \frac{\sqrt{-\Delta_A}}{2}.$$

The real basic solutions are

$$x_1(t) = \exp(\kappa t)\cos(\omega t) \qquad x_2(t) = \exp(\kappa t)\sin(\omega t)$$

The matrices $Y(0)$ and R are

$$Y(0) = \begin{bmatrix} 1 & 0 \\ \kappa & \omega \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 \\ -\kappa/\omega & 1/\omega \end{bmatrix}$$

$$\exp(At) = \exp(\kappa t)\left(\cos(\omega t) - \frac{\kappa}{\omega}\sin(\omega t)\right) I + \frac{1}{\omega}\exp(\kappa t)\sin(\omega t) A.$$