MAU 23205 Lecture 12

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Undetermined coefficients

The solutions to a linear homogenous ordinary differential ordinary differential equation form a vector space, because any linear combination of solutions is a solution, i.e. if x and y are solutions to $\sum_{j=0}^{m} c_j(t)x^{(j)}(t) = 0$ then so is $ax + by$. If c_m has no zeroes this vector space is of dimension m, because the existence and uniqueness theorem tells us that the linear function which takes the function x to the vector $\begin{bmatrix} x(t_0) \\ x'(t_0) \\ \cdots \\ x^{(m-1)}(t_0) \end{bmatrix}$ in $\mathbf{R}^m$ is a bijection. We can write this bijection explicitly in terms of the fundamental solution of the associated first order linear system. Last time we found m linearly independent solutions in the case where the coefficients c_j are constant. These solutions are therefore a basis, which is why they were called basic solutions. We can use this observation to solve initial value problems or boundary value problems by linear algebra.

Undetermined coefficients example (1/2)

Suppose, for example, that we want to solve the initial value problem

$$x''(t) + 2x'(t) + 2x(t) = 0, \quad x(t_0) = x_0, \quad x'(t_0) = v_0.$$

$z^2 + 2z + 2 = (z + 1 - i)(z + 1 + i)$ so the basic solutions are $x_1(t) = \exp(-t)\cos(t)$ and $x_2(t) = \exp(-t)\sin(t)$. Differentiating, $x_1'(t) = \exp(-t)(-\cos(t) - \sin(t))$ and $x_2'(t) = \exp(-t)(\cos(t) - \sin(t))$. We know our solution is of the form $x(t) = a_1x_1(t) + a_2x_2(t)$ so we need

$$x_0 = a_1\exp(-t_0)\cos(t_0) + a_2\exp(-t_0)\sin(t_0),$$
$$v_0 = a_1\exp(-t_0)(-\cos(t_0) - \sin(t_0)) + a_2\exp(-t_0)(\cos(t_0) - \sin(t_0)).$$

These are linear equations for a_1 and a_2 in terms of x_0 and v_0.

Undetermined coefficients example (2/2)

The solution to

$$x_0 = a_1 \exp(-t_0)\cos(t_0) + a_2 \exp(-t_0)\sin(t_0),$$
$$v_0 = a_1 \exp(-t_0)(-\cos(t_0) - \sin(t_0)) + a_2 \exp(-t_0)(\cos(t_0) - \sin(t_0)).$$

is

$$a_1 = \exp(t_0)(\cos(t_0) - \sin(t_0))x_0 - \exp(t_0)\sin(t_0)v_0$$
$$a_2 = \exp(t_0)(\cos(t_0) + \sin(t_0))x_0 + \exp(t_0)\cos(t_0)v_0$$

Substituting into $x(t) = a_1 x_1(t) + a_2 x_2(t)$ gives

$$\begin{aligned} x(t) &= (\exp(t_0)(\cos(t_0) - \sin(t_0))x_0 - \exp(t_0)\sin(t_0)v_0)\exp(-t)\cos(t) \\ &\quad + (\exp(t_0)(\cos(t_0) + \sin(t_0))x_0 + \exp(t_0)\cos(t_0)v_0)\exp(-t)\sin(t) \\ &= \exp(-(t - t_0))(\cos(t - t_0) + \sin(t - t_0))x_0 \\ &\quad + \exp(-(t - t_0))\sin(t - t_0)v_0. \end{aligned}$$

A boundary value problem example (1/4)

You can use undetermined coefficients for boundary value problems as well.
Suppose we want to find a solution to the boundary value problem.

$$y''''(x) + 2y''(x) + y(x) = 0$$
$$y(x_1) = y_1 \quad y'(x_1) = v_1 \quad y(x_2) = y_2 \quad y'(x_2) = v_2$$

The associated polynomial is $z^4 + 2z^2 + 1 = (z - i)^2(z + i)^2$ so the real basic solutions are $\cos(x)$, $\sin(x)$, $x\cos(x)$ and $x \sin x$.
The most general solution is therefore

$$y(x) = a\cos(x) + b\sin(x) + cx\cos(x) + dx\sin(x).$$

We want to choose a, b, c and d such that it satisfies the boundary conditions

$$y(x_1) = y_1 \quad y'(x_1) = v_1 \quad y(x_2) = y_2 \quad y'(x_2) = v_2$$

A boundary value problem example (2/4)

$$
\begin{aligned}
y(x) &= a\cos(x) + b\sin(x) + cx\cos(x) + dx\sin(x),\\
y'(x) &= -a\sin(x) + b\cos(x) + c(\cos(x) - x\sin(x))\\
&\quad + d(\sin(x) + x\cos(x)).
\end{aligned}
$$

$$
\begin{aligned}
a\cos(x_1) + b\sin(x_1) + cx_1\cos(x_1) + dx_1\sin(x_1) &= y_1\\
-a\sin(x_1) + b\cos(x_1) + c(\cos(x_1) - x_1\sin(x_1))&\\
+d(\sin(x_1) + x_1\cos(x_1)) &= v_1\\
a\cos(x_2) + b\sin(x_2) + cx_2\cos(x_2) + dx_2\sin(x_2) &= y_2\\
-a\sin(x_2) + b\cos(x_2) + c(\cos(x_2) - x_2\sin(x_2))&\\
+d(\sin(x_2) + x_2\cos(x_2)) &= v_2
\end{aligned}
$$

A boundary value problem example (3/4)

This is $M \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} y_1 \\ v_1 \\ y_2 \\ v_2 \end{bmatrix}$ where

$$M = \begin{bmatrix} \cos(x_1) & \sin(x_1) & x_1\cos(x_1) & x_1\sin(x_1) \\ -\sin(x_1) & \cos(x_1) & \cos(x_1) - x_1\sin(x_1) & \sin(x_1) + x_1\cos(x_1) \\ \cos(x_2) & \sin(x_2) & x_2\cos(x_2) & x_2\sin(x_2) \\ -\sin(x_2) & \cos(x_2) & \cos(x_2) - x_2\sin(x_2) & \sin(x_2) + x_2\cos(x_2) \end{bmatrix}$$

$\det(M) = (x_1 - x_2)^2 - \sin^2(x_1 - x_2)$ so this has a unique solution, with the obvious exception of when $x_1 = x_2$. If you impose different types of boundary conditions then the condition for there to be a unique solution will change. You can check what happens if I specify $y(x_1)$, $y(x_2)$, $y''(x_1)$ and $y''(x_2)$ instead of $y(x_1)$, $y(x_2)$, $y'(x_1)$ and $y'(x_2)$, for example.

A boundary value problem example (4/4)

This is now a problem of solving linear equations to find a, b, c and d, substituting, and then simplifying. The answer works out to be

$$y(x) = \frac{y_1 g_1(x) + v_1 h_1(x) + y_2 g_2(x) + v_2 h_2(x)}{(x_1 - x_2)^2 - \sin^2(x_1 - x_2)}$$

where

$$\begin{aligned}
g_1(x) &= (x_1 - x_2)\sin(x_1 - x_2)(x - x_2)\sin(x - x_2) \\
&\quad + [(x_1 - x_2)\cos(x_1 - x_2) + \sin(x_1 - x_2)] \\
&\quad\quad [(x - x_2)\cos(x - x_2) - \sin(x - x_2)] \\
h_1(x) &= [(x_1 - x_2)\cos(x_1 - x_2) - \sin(x_1 - x_2)](x - x_2)\sin(x - x_2) \\
&\quad + (x_1 - x_2)\sin(x_1 - x_2)[(x - x_2)\cos(x - x_2) - \sin(x - x_2)]
\end{aligned}$$

and similarly with the 1's and 2's reversed.

Overview

The method is

1. Find the basic solutions as in Lecture 11.
2. Write the general solution as a linear combination of basic solutions
3. Take as many derivatives as needed.
4. Substitute for any initial or boundary conditions to obtain linear (algebraic) equations.
5. Solve them. This step may be complicated, particularly if there are parameters in the conditions, but it's linear algebra.

If you need a lot of derivatives there's a trick to get them.

Derivatives of basic solutions (1/2)

We can easily find the derivatives of the basic solutions.

$$q_{\lambda,k}(t) = \frac{t^k}{k!}\exp(\lambda t) \qquad r_{\kappa,\omega,k}(t) = \frac{t^k}{k!}\exp(\kappa t)\cos(\omega t)$$

$$s_{\kappa,\omega,k}(t) = \frac{t^k}{k!}\exp(\kappa t)\sin(\omega t)$$

$$q'_{\lambda,k}(t) = \begin{cases} \lambda q^{(j)}_{\lambda,k}(t) & \text{if } k = 0, \\ \lambda q^{(j)}_{\lambda,k}(t) + q^{(j)}_{\lambda,k-1}(t) & \text{if } k > 0. \end{cases}$$

$$r'_{\lambda,k}(t) = \begin{cases} \kappa r^{(j)}_{\lambda,k}(t) - \omega s^{(j)}_{\lambda,k}(t) & \text{if } k = 0, \\ \kappa r^{(j)}_{\lambda,k}(t) - \omega s^{(j)}_{\lambda,k}(t) + r^{(j)}_{\lambda,k-1}(t) & \text{if } k > 0. \end{cases}$$

$$s'_{\lambda,k}(t) = \begin{cases} \kappa s^{(j)}_{\lambda,k}(t) + \omega r^{(j)}_{\lambda,k}(t) & \text{if } k = 0, \\ \kappa s^{(j)}_{\lambda,k}(t) + \omega r^{(j)}_{\lambda,k}(t) + s^{(j)}_{\lambda,k-1}(t) & \text{if } k > 0. \end{cases}$$

The derivative each basic solution is a linear combination of basic solutions.

Derivatives of basic solutions (2/2)

If $x_1, \ldots, x_m$ are basic solutions then $x_k' = \sum_{j=1}^{m} d_{j,k} x_j$ for some coefficients $d_{j,k}$, which we can determine using the equations on the preceding slide. We can write this as a matrix equation

$$\frac{d}{dt}\begin{bmatrix} x_1(t) & \cdots & x_m(t) \end{bmatrix} = \begin{bmatrix} x_1(t) & \cdots & x_m(t) \end{bmatrix} D.$$

Higher derivatives of basic solutions are also linear combinations of basic solutions. To find out which linear combinations, it suffices to compute powers of D. If you have a general linear combination $x = \sum_{k=1}^{m} a_k x_k$ then the value of its i'th derivative at t is

$$\begin{bmatrix} x_1(t) & \cdots & x_m(t) \end{bmatrix} D^i \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}.$$