MAU23205 Lecture 11

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Linear constant coefficient scalar differential equations

The general linear homogeneous constant coefficient scalar differential equation of order m is $\sum_{j=0}^{m} c_j x^{(j)}(t) = 0$, i.e.

$$c_m x^{(m)}(t) + c_{m-1} x^{(m-1)}(t) + \cdots + c_1 x'(t) + c_0 x(t) = 0$$

with $c_m \neq 0$. Dividing by c_m gives $\sum_{j=0}^{m} \alpha_j x^{(j)}(t) = 0$, i.e.

$$x^{(m)}(t) + \alpha_{m-1} x^{(m-1)}(t) + \cdots + \alpha_1 x'(t) + \alpha_0 x(t) = 0,$$

which is equivalent, so we might as well consider only equations with leading coefficient 1. Define $q_{\lambda,k}(t) = \frac{t^k}{k!} \exp(\lambda t)$. When is $q_{\lambda,k}$ a solution of this equation? This is easy to answer for $k = 0$. $q_{\lambda,0}(t) = \exp(\lambda t)$, $q'_{\lambda,0}(t) = \lambda q_{\lambda,0}(t)$, and $q^{(j)}_{\lambda,0}(t) = \lambda^j q_{\lambda,0}(t)$, so $\sum_{j=0}^{m} \alpha_j q^{(j)}_{\lambda,0}(t) = \sum_{j=0}^{m} \alpha_j \lambda^j q_{\lambda,0}(t) = p(\lambda) q_{\lambda,0}(t)$. This is zero if and only if λ is a root of p. If p has m distinct roots then we have m distinct solutions. In fact they're linearly independent solutions.

The Fundamental Theorem of Algebra

Every monic polynomial p of degree m with complex coefficients has a factorisation

$$p(z) = \prod_{j=1}^{l} (z - \lambda_j)^{d_j}.$$

$\sum_{j=1}^{l} d_j = m.$

This gives us l solutions, $q_{\lambda_1,0}, \ldots, q_{\lambda_l,0}$. If $d_1 = \cdots = d_l = 1$, i.e. if all the roots are simple, then $l = m$. There are two problems:

- Some of the roots may be complex, even if the coefficients of p are real.
- There may be repeated roots.

You won't be faced with both problems at once unless the degree is at least 4.

Repeated roots (1/2)

For repeated roots we need $q_{\lambda,k}$ for $k > 0$. $q_{\lambda,k}(t) = \frac{t^k}{k!}\exp(\lambda t)$

$$q'_{\lambda,k}(t) = \lambda q_{\lambda,k}(t) + q_{\lambda,k-1}(t)$$

$$q^{(j)}_{\lambda,k}(t) = \sum_{i=0}^{\min(j,k)} \frac{j!}{i!(j-i)!}\lambda^{j-i} q_{\lambda,k-i}(t)$$

$$\sum_{j=0}^{m} \alpha_j q^{(j)}_{\lambda,k}(t) = \sum_{j=0}^{m} \sum_{i=0}^{\min(j,k)} \frac{j!}{i!(j-i)!}\alpha_j \lambda^{j-i} q_{\lambda,k-i}(t)$$

$$\sum_{j=0}^{m} \alpha_j q^{(j)}_{\lambda,k}(t) = \sum_{i=0}^{k} \sum_{j=i}^{m} \frac{j!}{i!(j-i)!}\alpha_j \lambda^{j-i} q_{\lambda,k-i}(t)$$

$$\sum_{j=0}^{m} \alpha_j q^{(j)}_{\lambda,k}(t) = \sum_{i=0}^{k} \frac{p^{(i)}(\lambda)}{i!} q_{\lambda,k-i}(t)$$

Repeated roots (2/2)

$$\sum_{j=0}^{m} \alpha_j q_{\lambda,k}^{(j)}(t) = \sum_{i=0}^{k} \frac{p^{(i)}(\lambda)}{i!} q_{\lambda,k-i}(t)$$

If λ is a root of order greater than k, i.e. if $p^{(i)}(\lambda) = 0$ for $0 \leq i \leq k$, then

$$\sum_{j=0}^{m} \alpha_j q_{\lambda,k}^{(j)}(t) = 0$$

In other words, $q_{\lambda,k}$ is a solution to

$$\sum_{j=0}^{m} \alpha_j x^{(j)}(t) = 0.$$

There are d_j solutions for each j, $q_{\lambda_j,k}$ for all $0 \leq k < d_j$, and $\sum_{j=1}^{l} d_j = m$, so we now have m solutions. They could still be complex though.

Examples

What are some solutions to $x'''(t) - x'(t) = 0$? The associated polynomial is $p(z) = z^3 - z$. It factors as $(z+1)z(z-1)$ with simple roots $\lambda_1 = -1$, $\lambda_2 = 0$ and $\lambda_3 = 1$. We therefore have solutions $\exp(-1t)$, $\exp(0t)$ and $\exp(1t)$. Normally we'd write these as $\exp(-t)$, 1 and $\exp(t)$.

What are some solutions to $x''''(t) - 2x''(t) + x(t) = 0$? The associated polynomial is $p(z) = z^4 - 2z^2 + 1$. It factors as $(z+1)^2(z-1)^2$ with repeated roots $\lambda_1 = -1$ with $d_1 = 2$ and $\lambda_2 = 1$ with $d_2 = 2$. We therefore have solutions $\exp(-t)$, $t\exp(-t)$, $\exp(t)$ and $t\exp(t)$.

What are some solutions to $x''''(t) + 2x''(t) + x(t) = 0$? The associated polynomial is $p(z) = z^4 + 2z^2 + 1$. It factors as $(z+i)^2(z-i)^2$ with repeated roots $\lambda_1 = -i$ with $d_1 = 2$ and $\lambda_2 = i$ with $d_2 = 2$. We therefore have solutions $\exp(-it)$, $t\exp(-it)$, $\exp(it)$ and $t\exp(it)$. $\exp(\pm it) = \cos(t) \pm i\sin(t)$.

Complex roots (1/2)

If

$$\sum_{j=0}^{m} c_j(t)x^{(j)}(t) = 0$$

then

$$\sum_{j=0}^{m} \overline{c_j}(t)\bar{x}^{(j)}(t) = 0$$

and vice versa. If the coefficients are all real then x is a solution to $\sum_{j=0}^{m} c_j(t)x^{(j)}(t) = 0$ if and only if $\bar{x}$ is. So if x is a solution then so is its real part $\frac{x+\bar{x}}{2}$ and its imaginary part $\frac{x-\bar{x}}{2i}$. We assume throughout this module that our equations have real coefficients. For example, $\exp(-it)$, $t\exp(-it)$, $\exp(it)$ and $t\exp(it)$ are solutions to $x''''(t) + 2x''(t) + x(t) = 0$, so $\cos(t)$, $\sin(t)$, $t\cos(t)$ and $t\sin(t)$ are solutions.

Complex roots (2/2)

More generally, if $\lambda = \kappa + i\omega$ is a root of order d of p then

$$q_{\lambda,k}(t) = \frac{t^k}{k!}\exp(\lambda t) = \frac{t^k}{k!}\exp(\kappa t)\cos(\omega t) + i\frac{t^k}{k!}\exp(\kappa t)\sin(\omega t)$$

is a solution to $\sum_{j=0}^{m} \alpha_j x^{(j)}(t) = 0$ for all $0 \leq k \leq d$. So

$$r_{\kappa,\omega,k}(t) = \frac{t^k}{k!}\exp(\kappa t)\cos(\omega t) \qquad s_{\kappa,\omega,k}(t) = \frac{t^k}{k!}\exp(\kappa t)\sin(\omega t)$$

are also solutions.

$\lambda = \kappa + i\omega$ is a root of p if and only if $\bar{\lambda} = \kappa - i\omega$ is a root of p, and both roots are of the same order. Changing the sign of ω leaves $r_{\kappa,\omega,k}$ and changes the sign of $s_{\kappa,\omega,k}$, so it doesn't give us interesting new solutions. We therefore always choose only one from each pair of complex roots.

Basic solutions

To any linear constant coefficient scalar differential equation $\sum_{j=0}^{m} c_j x^{(j)}(t) = 0$ we associate the polynomial

$$p(z) = \sum_{j=0}^{m} \alpha_j z^j$$

where $\alpha_j = c_j / c_m$. This factors as $p(z) = \prod_{j=1}^{l} (z - \lambda_j)^{d_j}$. To each real root λ of order d we associate the d solutions $q_{\lambda,k}$ for $0 \leq k < d$. To each pair $\kappa + i\omega$, $\kappa - i\omega$ of complex roots of order d we associate $2d$ solutions $r_{\kappa,\omega,k}$ and $s_{\kappa,\omega,k}$ for $0 \leq k < d$. These m solutions are all real valued functions. They are linearly independent, although this isn't obvious. They form a basis for the real vector space of all real valued solutions. For this reason I'll call them *basic solutions*.

Example

What are the basic solutions to the differential equation $x^{(9)}(t) + x^{(6)}(t) - x^{(3)}(t) - x(t) = 0$? The associated polynomial is

$$z^9 + z^6 - z^3 - 1 = (z-1)(z+1)^2(z^2 - z + 1)^2(z^2 + z + 1)$$

The real roots give us the basic solutions $\exp(t)$, $\exp(-t)$ and $t\exp(-t)$. $z^2 - z + 1 = \left(z - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(z - \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$. This pair of complex roots, each of order 2, gives us the four basic solutions $\exp\left(\frac{1}{2}t\right)\cos\left(\frac{\sqrt{3}}{2}t\right)$, $\exp\left(\frac{1}{2}t\right)\sin\left(\frac{\sqrt{3}}{2}t\right)$, $t\exp\left(\frac{1}{2}t\right)\cos\left(\frac{\sqrt{3}}{2}t\right)$ and $t\exp\left(\frac{1}{2}t\right)\sin\left(\frac{\sqrt{3}}{2}t\right)$.
$z^2 + z + 1 = \left(z + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(z + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$. This pair of simple complex roots gives us the two basic solutions $\exp\left(-\frac{1}{2}t\right)\cos\left(\frac{\sqrt{3}}{2}t\right)$ and $\exp\left(-\frac{1}{2}t\right)\sin\left(\frac{\sqrt{3}}{2}t\right)$.

Why are the basic solutions linearly independent?

Suppose that $x_1, x_2, \ldots, x_m$ are the basic solutions. Suppose that a linear combination of them is zero. In other words, suppose that $\sum_{k=1}^{m} \beta_k x_k(t) = 0$ for all t. Define $y_{j,k} = x_k^{(j-1)}$ for $1 \leq j \leq m$. $\sum_{k=1}^{m} \beta_k y_{j,k}(t) = 0$ for all t. In other words $Y(t)\beta = \mathbf{0}$. In particular, $Y(0)\beta = \mathbf{0}$. If all roots are real and simple then $x_k(t) = \exp(\lambda_k t)$ and $y_{j,k}(0) = \lambda_k^{j-1}$. In other words, $Y(0)$ is a Vandermonde matrix. We know those are invertible, so $Y(0)\beta = \mathbf{0}$ implies $\beta = \mathbf{0}$. In other words, the only linear combination which is zero is the one with all coefficients zero. So $x_1, x_2, \ldots, x_m$ are linearly independent.

Showing that $Y(0)$ is still invertible when there are repeated or complex roots is more complicated, but it can be done, although I won't do it. Of course for any given equation you can check directly that $Y(0)$ is invertible.