MAU23205 Lecture 10

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6 October 2021

Where we left off last time

We saw that we can solve the initial value problem $\mathbf{x}(t_0) = \mathbf{x}_0$ for the linear differential equation $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{g}(t)$ if the sequence of matrix valued functions defined by

$$W_0(t, r) = I \qquad W_{k+1}(t, r) = I + \int_r^t A(s) W_k(s, r)\, ds$$

has a limit. If this limit is W then the solution is

$$\mathbf{x}(t) = W(t, t_0)\mathbf{x}_0 + \int_{t_0}^t W(t, s)\mathbf{g}(s)\, ds.$$

This is a sum of two parts: $W(t, t_0)\mathbf{x}_0$ solves the linear homogeneous equation $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ with initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$, and $\int_{t_0}^t W(t, s)\mathbf{g}(s)\, ds$ solves the linear inhomogeneous equation $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{g}(t)$ with initial data $\mathbf{x}(t_0) = \mathbf{0}$.

Our goal today is to prove the sequence converges.

Convergence (1/3)

We only expect a reasonable theory when A is continuous in an interval J, so I'll assume this from now on. I'll also assume J is closed.

In addition to the Ws it's convenient, temporarily, to introduce some Vs.

$$V_0(t,r) = I \qquad V_{k+1}(t,r) = I - \int_r^t V_k(s,r)A(s)\,dr.$$

Note that $V_k(t,t) = W_k(t,t) = I$ for all k.

$$V_{k+1}(t,r) - V_k(t,r) = -\int_r^t \left[V_k(s,r) - V_{k-1}(s,r)\right] A(s)\,ds,$$

$$W_{k+1}(t,r) - W_k(t,r) = \int_r^t A(s)\left[W_k(s,r) - W_{k-1}(s,r)\right]\,ds.$$

except in the case $k=0$, where we have to remove the quantity in brackets.

Convergence (2/3)

$$
\begin{aligned}
\|V_{k+1}(t,r) - V_k(t,r)\| &= \left\| \int_r^t [V_k(s,r) - V_{k-1}(s,r)]\, A(s)\, ds \right\| \\
&\le \int_r^t \|[V_k(s,r) - V_{k-1}(s,r)]\, A(s)\|\; ds \\
&\le \int_r^t \|V_k(s,r) - V_{k-1}(s,r)\|\, \|A(s)\|\; ds \\
&\le \max_J \|A\| \int_r^t \|V_k(s,r) - V_{k-1}(s,r)\|\; ds
\end{aligned}
$$

If $\|V_k(t,r) - V_{k-1}(t,r)\| \le \frac{|t-r|^k (\max_J \|A\|)^k}{k!}$ for all $t, r \in J$ then $\|V_{k+1}(t,r) - V_k(t,r)\| \le \frac{|t-r|^{k+1} (\max_J \|A\|)^{k+1}}{(k+1)!}$ for all $t, r \in J$. The base case holds as well, so the result holds for all k by induction.

Convergence (3/3)

Similarly, $\|W_{k+1}(t,r) - W_k(t,r)\| \leq \frac{|t-r|^{k+1}(\max_J \|A\|)^{k+1}}{(k+1)!}$ for all $t, r \in J$.

$$W_m(t,r) = I + \sum_{k=0}^{m-1} \left[W_{k+1}(t,r) - W_k(t,r)\right].$$

By the triangle inequality and induction

$$\|W_m(t,r)\| \leq \sum_{k=0}^{m} \frac{|t-r|^k \left(\max_J \|A\|\right)^k}{k!}.$$

The comparison test with $\exp(|t-r| \max_J \|A\|)$ shows that the sequence converges uniformly on $J \times J$ to a limit which is bounded by $\exp(|t-r| \max_J \|A\|)$. The same holds for the Vs.

Properties of V and W (1/3)

The integral of the limit of a uniformly convergent sequence is the limit of the integrals,
so from

$$V_{k+1}(t,r) = I - \int_r^t V_k(s,r)A(s)\,dr$$

$$W_{k+1}(t,r) = I + \int_r^t A(s)W_k(s,r)\,dr$$

we get

$$V(t,r) = I - \int_r^t V(s,r)A(s)\,dr$$

$$W(t,r) = I + \int_r^t A(s)W(s,r)\,dr.$$

By the Fundamental Theorem of Calculus

$$V'(t,r) = -V(t,r)A(t), \qquad W'(t,r) = A(t)W(t,r).$$

The $'$'s denote derivatives with respect to the first argument.

Properties of V and W (2/3)

Let $U(t,s,r) = V(t,s)W(t,r)$. We just saw that $V'(t,r) = -V(t,r)A(t)$ and $W'(t,r) = A(t)W(t,r)$ so

$$\begin{aligned} U'(t,s,r) &= V'(t,s)W(t,r) + V(t,s)W'(t,r) \\ &= -V(t,s)A(t)W(t,r) + V(t,s)A(t)W(t,r) = O. \end{aligned}$$

So $U(t,s,r)$ is independent of t. A useful consequence is

$$\begin{aligned} V(r,s) &= V(r,s)I = V(r,s)W(r,r) = U(r,s,r) \\ &= U(s,s,r) = V(s,s)W(s,r) = IW(s,r) = W(s,r). \end{aligned}$$

Another is

$$V(t,s)W(t,s) = U(t,s,s) = U(s,s,s) = V(s,s)W(s,s) = I^2 = I.$$

So $V(t,s) = W(t,s)^{-1}$.

Properties of V and W (3/3)

A further consequence is

$$\begin{aligned} W(s,r) &= IW(s,r) = V(s,s)W(s,r) = U(s,s,r) \\ &= U(t,s,r) = V(t,s)W(t,r) = W(t,s)^{-1}W(t,r) \end{aligned}$$

Multiplying from the left by $W(t,s)$,

$$W(t,s)W(s,r) = W(t,r).$$

In view of the equation $W(t,s) = V(s,t)$ we can rewrite

$$\frac{\partial V}{\partial s}(s,t) = -V(s,t)A(s)$$

as

$$\frac{\partial W}{\partial s}(t,s) = -W(t,s)A(s).$$

The theorem on fundamental solutions (1/2)

If A is a continuous square matrix valued function on the closed interval J then there is a unique square matrix valued function W on $J \times J$ such that

1. $\frac{\partial W}{\partial t}(t,s) = A(t)W(t,s)$,
2. $W(t,s)$ is invertible, with $W(t,s)^{-1} = W(s,t)$.
3. $W(t,t) = I$,
4. $\frac{\partial W}{\partial s}(t,s) = -W(t,s)A(s)$,
5. $W(t,s)W(s,r) = W(t,r)$, and

This W is called *the fundamental solution* corresponding to the coefficient matrix A and the interval J. We just showed the existence of such a W. To see the uniqueness, suppose $\tilde{W}$ has the same properties.

The theorem on fundamental solutions (2/2)

Let $K(t,s) = W(t,s)\tilde{W}(s,t)$. Then

$$\begin{aligned}\frac{\partial K}{\partial s}(t,s) &= \frac{\partial W}{\partial s}(t,s)\tilde{W}(s,t) + W(t,s)\frac{\partial \tilde{W}}{\partial s}(s,t)\\ &= (-W(t,s)A(s))\tilde{W}(s,t) + W(t,s)(A(s)\tilde{W}(s,t)) = O.\end{aligned}$$

So $K(t,s)$ is independent of s.
$K(t,t) = W(t,t)\tilde{W}(t,t) = I^2 = I$ so $W(t,s)\tilde{W}(s,t) = I$ for all t and s. Then $W(t,s)\tilde{W}(t,s)^{-1} = I$. Multiplying from the right by $\tilde{W}(t,s)$ gives $W(t,s) = \tilde{W}(t,s)$.
An invertible square matrix valued function Y on J is called *a fundamental solution* corresponding to the coefficient matrix A and the interval J if $Y'(t) = A(t)Y(t)$. The first two properties above tell us that the fundamental solution is a fundamental solution for any fixed value of its second argument.

Constructing the fundamental solution from a fundamental solution (1/2)

Getting a fundamental solution from the fundamental solution is easy. Just fix the second argument. Getting the fundamental solution from a fundamental solution is also possible.
Suppose Y is a fundamental solution and set

$$W(t,s) = Y(t)Y(s)^{-1}.$$

$$\frac{\partial W}{\partial t}(t,s) = Y'(t)Y(s)^{-1} = A(t)Y(t)Y(s)^{-1} = A(t)W(t,s).$$

$$W(t,s)^{-1} = \left(Y(t)Y(s)^{-1}\right)^{-1} = Y(s)Y(t)^{-1} = W(s,t).$$

$$W(t,t) = Y(t)Y(t)^{-1} = I.$$

Constructing the fundamental solution from a fundamental solution (2/2)

$$\begin{aligned}\frac{\partial W}{\partial s}(t,s) &= Y(t)\frac{\partial Y^{-1}}{\partial s}(t,s) = -Y(t)Y(s)^{-1}Y'(s)Y(s)^{-1}\\ &= -Y(t)Y(s)^{-1}A(s)Y(s)Y(s)^{-1} = -W(t,s)A(s).\end{aligned}$$

$$\begin{aligned}W(t,r) &= Y(t)Y(r)^{-1} = Y(t)IY(r)^{-1}\\ &= Y(t)Y(s)^{-1}Y(s)Y(r)^{-1} = W(t,s)W(s,r).\end{aligned}$$

Of course this is only useful if you have some other way of finding a fundamental solution besides fixing the second argument of the fundamental solution.