MAU 23205 Lecture 4

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Existence and uniqueness

The main existence and uniqueness theorem on ODEs is a *local* theorem about the *initial value problem*.
Suppose $t_* \in \mathbf{R}$ and $\mathbf{x}_*$ in $\mathbf{R}^n$ Suppose that
$\mathbf{F}\colon B(t_*, S) \times B(\mathbf{x}_*, R) \to \mathbf{R}^n$ is continuous for some $R, S > 0$.
$B(\mathbf{x}_*, R)$ is the ball of radius R about $\mathbf{x}_*$ in $\mathbf{R}^n$. Similarly $B(t_*, S)$ is the ball of radius S about t_*, i.e. the interval $(t_* - S, t_* + S)$.
Then there are $r, s > 0$ such that if $t_0 \in B(t_*, s)$ and $\mathbf{x}_0 \in B(\mathbf{x}_*, r)$ then the initial value problem

$$\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}(t)) \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a continuously differentiable solution $\mathbf{x}\colon B(t_0, s) \to B(\mathbf{x}_0, r)$.
If $\mathbf{F}(t, \mathbf{x})$ is a continuously differentiable function of $\mathbf{x}$ then there is only one solution to the initial value problem. This solution depends continuously on t_0 and $\mathbf{x}_0$ as well as t.

Why these restrictions?

Any theorem this general can only give local existence. Consider $x'(t) = 1 + x(t)^2$. The solution with $x(t_0) = x_0$ is

$$x(t) = \frac{x_0 + \tan(t - t_0)}{1 - x_0 \tan(t - t_0)}.$$

This makes sense only for t sufficiently close to t_0, specifically within an interval of length π. At the ends of the interval the solution tends to $\pm\infty$.

Continuity of $\mathbf{F}$ is enough for existence, but not uniqueness. Consider $x'(t) = x(t)^{1/3}$, $x(0) = 0$. $x(t) = 0$ is a continuously differentiable solution. But so is

$$x(t) = \begin{cases} (2t/3)^{3/2} & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases}$$

We'll find out later where these solutions come from.

Illusory restrictions

Some of the restrictions are more apparent than real though. The theorem covers first order systems, but reduction of order can make anything into a first order system.

The theorem is for an equation without parameters. We could apply it to each value of any parameter separately, but then it wouldn't tell us how the solution depends on the parameters. There is a better way.

Consider the initial value problem $y(x_0) = y_0$, $y'(x_0) = v_0$ for the Legendre equation

$$(1 - x^2)y''(x) - 2xy'(x) + \nu(\nu + 1)y(x) = 0.$$

If $z_1 = y$, $z_2 = y'$ and $z_3 = \nu$ then $\mathbf{z}$ satisfies the initial value problem

$$(z_1, z_2, z_3)(x_0) = (y_0, v_0, \nu)$$

$$z_1'(x) = z_2(x) \quad z_2'(x) = \frac{2xz_2(x) - z_3(x)(z_3(x) + 1)z_1(x)}{1 - x^2} \quad z_3'(x) = 0.$$

Illusory restrictions (continued)

Conversely, if $\mathbf{z}$ satisfies

$$(z_1, z_2, z_3)(x_0) = (y_0, v_0, \nu)$$

$$z_1'(x) = z_2(x) \quad z_2'(x) = \frac{2xz_2(x) - z_3(x)(z_3(x)+1)z_1(x)}{1-x^2} \quad z_3'(x) = 0$$

and we set $y = z_1$ then y satisfies

$$y(x_0) = y_0, \quad y'(x_0) = v_0$$

$$(1-x^2)y''(x) - 2xy'(x) + \nu(\nu+1)y(x) = 0.$$

We can apply the existence and uniqueness theorem to

$$\mathbf{F}(x, z_1, z_2, z_3) = \left(z_2, \frac{2xz_2 - z_3(z_3+1)z_1}{1-x^2}, 0 \right)$$

to get existence and uniqueness of solutions to

$$y(x_0) = y_0, \quad y'(x_0) = v_0$$

$$(1-x^2)y''(x) - 2xy'(x) + \nu(\nu+1)y(x) = 0.$$

We need to restrict to an interval not containing $(-1, 1)$.

Illusory restrictions (conclusion)

The theorem only gives us existence and uniqueness in a still smaller interval. But it gives us continuous dependence not just on x, x_0, y_0 and v_0 but also on ν.

You wouldn't gain any real generality by including higher order differential equations or equations with parameters in the statement of the theorem.

You could even restrict it to autonomous systems, i.e those of the form $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$.

$$(z_1, z_2, z_3, z_4)(x_0) = (y_0, v_0, \nu, x_0)$$

$$z_1'(x) = z_2(x) \quad z_2'(x) = \frac{2z_4(t)z_2(x) - z_3(x)(z_3(x)+1)z_1(x)}{1 - z_4(t)^2}$$

$$z_3'(x) = 0 \quad z_4'(x) = 1$$

is equivalent to the original initial value problem for the Legendre equation.

Explicit versus implicit

Not every equation is, or can be, naturally written as $\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}(t))$. Sometimes the form $\mathbf{G}(t, \mathbf{x}(t), \mathbf{x}'(t)) = \mathbf{0}$ is more natural.

Consider $x(t)^2 + x'(t)^2 = I$, for example. You could try to rewrite this as

$$x'(t) = \pm\sqrt{I - x(t)^2}$$

but this isn't single valued. Near the initial values you can make it single valued though by choosing the correct branch, i.e. sign.

We can fix this problem in the general case with an appeal to the Implicit Function Theorem. If $\mathbf{G}$ is continuously differentiable near $(t_*, \mathbf{x}_*, \mathbf{v}_*)$, $\mathbf{G}(t_*, \mathbf{x}_*, \mathbf{v}_*) = \mathbf{0}$ and $\partial\mathbf{G}/\partial\mathbf{v}(t_*, \mathbf{x}_*, \mathbf{v}_*)$ is invertible then there is a unique continuously differentiable function $\mathbf{F}$ defined near $(t_*, \mathbf{x}_*)$ such that $\mathbf{G}(t, \mathbf{x}, \mathbf{F}(t, \mathbf{x})) = \mathbf{0}$ and $F(t_*, \mathbf{x}_*) = \mathbf{v}_*$. You can make all the references to "near" precise by using balls, as in the existence and uniqueness theorem. You can also use open sets, if you know what they are.

Approximations

The statement of the existence and uniqueness theorem gives no hint of how to find the solution it claims exists. There are two main methods, each with advantages and disadvantages.

- Picard's "method of successive approximations", via a sequence of integrals.
- Peano's method, based on minimisation in a function space.

Both give, or can give, a sequence of approximations converging to the solution. In the case where $\mathbf{F}$ is merely continuous, where we can't expect uniqueness, they give a sequence such that some subsequence converges to a solution, but different subsequences might converge to different solutions.

Picard and integral equations

The Fundamental Theorem of Calculus applies to vector valued functions as well as to real valued functions. So

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{x}'(s)\, ds.$$

If $\mathbf{x}$ satisfies the initial value problem

$$\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}(t)) \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

then

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{F}(s, \mathbf{x}(s))\, ds.$$

Conversely, if $\mathbf{x}$ satisfies the integral equation above then it satisfies the initial value problem.

Picard and integral equations (continued)

Note that

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{F}(s, \mathbf{x}(s))\, ds.$$

isn't really a solution formula. To get the function $\mathbf{x}$ from the integral we would already need to know it, since it appears in the integrand.

To get around this, define a sequence inductively by

$$\mathbf{x}_0(t) = \mathbf{x}_0 \quad \mathbf{x}_{k+1}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{F}(s, \mathbf{x}_k(s))\, ds.$$

You could hope that this sequence converges and that its limit satisfies the integral equation and therefore the initial value problem. This largely works, at least in a small enough interval containing the initial data. Either the sequence converges or it has a convergent subsequence, depending on the differentiability of $\mathbf{F}$.