MAU 23205 Lecture 3

John Stalker

Trinity College Dublin

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Reduction of Order

Recall that the order of an equation or system is the order of the highest derivative appearing. There are few different contexts in which we can reduce higher order equations or systems to lower order equations or systems. For example, the third order equation

$$2\frac{\tau'''(j)}{\tau'(j)} - 3\frac{\tau''(j)^2}{\tau'(j)^2} = \frac{8}{9}\frac{1}{j^2} + \frac{3}{4}\frac{1}{(1-j)^2} + \frac{23}{36}\frac{1}{j(1-j)}$$

is equivalent to the first order system

$$\tau'(j) = \theta(j) \qquad \theta'(j) = \varphi(j)$$
$$2\frac{\varphi'(j)}{\theta(j)} - 3\frac{\varphi(j)^2}{\theta(j)^2} = \frac{8}{9}\frac{1}{j^2} + \frac{3}{4}\frac{1}{(1-j)^2} + \frac{23}{36}\frac{1}{j(1-j)}$$

What are we doing and why?

Equivalent here means that we can get a solution to the third order equation by taking a solution to the first order system and just ignoring θ and φ and that we can get a solution to the first order system by taking a solution to the third order equation and defining θ and φ via $\theta(j) = \tau'(j)$ and $\varphi(j) = \theta'(j)$.
We clearly *can* do this, and do something similar for any other differential equation, but *should* we? Is there a point?

- To get an explicit solution to this equation, no.
- For writing general purpose numerical code, yes, probably.
- For proving general theorems, definitely.

For general theorems or algorithms we only need to consider first order systems.

Another example

There is generally more than one way to reduce an equation or system to first order. Consider $x''(t) - 3x'(t) + 2x(t) = 0$. Following what we did above, this is equivalent to the first order system

$$x'(t) = y(t) \qquad y'(t) = -2x(t) + 3y(t).$$

A more useful reduction is

$$x'(t) = x(t) + y(t) \qquad y'(t) = 2y(t).$$

The advantage is that x doesn't appear in the second equation, which makes it easier to find the solution $y(t) = A\exp(2t)$. The solution to the first equation is then $x(t) = A\exp(2t) + B\exp(t)$.

Yet another example

Consider the equation $\frac{d^2y}{dx^2} + y = 0$. Set $I = \left(\frac{dy}{dx}\right)^2 + y^2$. Differentiating,

$$\frac{dI}{dx} = 2\frac{dy}{dx}\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} = 2\frac{dy}{dx}\left(\frac{d^2y}{dx^2} + y\right) = 0.$$

So is the second order equation $I = \left(\frac{dy}{dx}\right)^2 + y^2$ equivalent to the first order system

$$\left(\frac{dy}{dx}\right)^2 + y^2 = I \qquad \frac{dI}{dx} = 0 \ ?$$

That would be quite convenient, since the solutions to the second equation are obviously constant.

Yet another example, continued

Life is complicated.
If y solves the second order equation $\frac{d^2y}{dx^2} + y = 0$ and I is defined by $I = \left(\frac{dy}{dx}\right)^2 + y^2$ then y and I satisfy the first order system

$$\left(\frac{dy}{dx}\right)^2 + y^2 = I \qquad \frac{dI}{dx} = 0.$$

The converse isn't true though.

$$y(x) = A \qquad I(x) = A^2$$

solves the first order system for any A, but doesn't solve the second order equation unless $A = 0$.

What went wrong?

In going from the second order equation to the first order system we introduced extraneous solutions. How? If $I(x) = y'(x)^2 + y(x)^2$

$$I'(x) = 2y'(x)y''(x) + 2y(x)y'(x) = 2y'(x)(y''(x) + y(x)) = 0.$$

That's fine. If y solves the second order equation then $I'(x) = 0$. But $I'(x) = 0$ if $y'(x) = 0$, even if y doesn't solve the second order equation.
Just as with algebraic equations, you have to be careful when manipulating differential equations not to introduce extraneous solutions. If you want to say that two systems are equivalent then be sure to check that your argument is reversible.

Comments

I cheated slightly when I said that the solutions of $I'(x) = 0$ are constant. They are if the domain of I is an interval. If it's not an interval then we could get things like

$$I(x) = \begin{cases} -1 & \text{if } -2 < x < -1, \\ +1 & \text{if } +1 < x < +2. \end{cases}$$

This isn't a very interesting solution. In this module solutions are *always* defined on intervals. That ensures that anything with derivative 0 is constant. A quantity whose derivative is 0 for every solution is called an *invariant*.

Are there still more solutions of $y'(x)^2 + y(x)^2 = I$ besides the solutions of $y''(x) + y(x) = 0$ and $y(x) = A$? Yes!

Uses of invariants

Returning to our example, if y solves the second order equation $y''(x) + y(x) = 0$ and I is defined by $I(x) = y'(x)^2 + y(x)^2$ then y and I satisfy the first order system

$$y'(x)^2 + y(x)^2 = I(x) \qquad I'(x) = 0$$

and I is constant. Then

$$|y(x)| = \sqrt{I - y'(x)^2} \leq \sqrt{I},$$

so y is bounded. I've assumed here that we consider only real valued solutions. That's what I'll generally do in this module.

More about invariants

In this case all the solutions to $y'' + y = 0$ happen to be of the form $y(x) = a\cos x + b\sin x$, from which it's clear that they're bounded. But this method of showing boundedness doesn't require an explicit solution. It works equally well for the system

$$x'(t) = y(t)z(t) \qquad y'(t) = -x(t)z(t) \quad z'(t) = -k^2 x(t)y(t)$$

with invariants $x(t)^2 + y(t)^2$ and $k^2 x(t)^2 + z(t)^2$. Here it's not possible to find an explicit solution in terms of elementary functions, but we can still conclude that solutions are bounded.

$x' = yz$, $y' = -xz$, $z' = -k^2xy$ in more detail

Here's that last example in more detail. Suppose x, y and z are solutions to

$$x'(t) = y(t)z(t), \qquad y'(t) = -x(t)z(t), \quad z'(t) = -k^2x(t)y(t).$$

Set $I = x^2 + y^2$ and $J = k^2x^2 + z^2$. Then

$$\begin{aligned} I'(t) &= 2x(t)x'(t) + 2y(t)y'(t) \\ &= 2x(t)\left[y(t)z(t)\right] + 2y(t)\left[-x(t)z(t)\right] = 0 \end{aligned}$$

and similarly for J. So I and J are invariants. It follows that

$$|x(t)| \leq \sqrt{I}, \qquad |y(t)| \leq \sqrt{I}, \qquad |z(t)| \leq \sqrt{J}$$

for all t.

Not quite invariants

Sometimes we have a quantity whose derivative isn't zero, but is of constant sign. This can still be useful.
Consider the system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ +1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

If $V = x^2 + y^2$ then $dV/dt = -2V$. V isn't invariant, but it is monotone decreasing, and strictly monotone decreasing unless $V = 0$. It follows that solutions remain bounded as $t \to \infty$. In fact, $V(t) = V(0)\exp(-2t)$, so all solutions tend to 0 as $t \to \infty$.
We'll see a lot more of this type of argument when we discuss stability later in the module.

A question to think about

For which 2×2 systems

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

can we apply this trick? The not quite invariant might not be $x^2 + y^2$; It could be $V(x, y) = px^2 + qxy + ry^2$. Under what conditions on a, b, c and d are there p, q and r such that

- $V(x, y) \geq 0$, and $V(x, y) = 0$ only if $(x, y) = 0$, and
- $V(x(t), y(t))$ is strictly monotone decreasing if (x, y) is a non-zero solution of the system above?

This is a Linear Algebra problem, but not an easy one. We'll learn how to answer it, in the $n \times n$ case, later in the module.