MAU 23205 Lecture 2

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Terminology: single equations vs systems

A differential equation is an equation, possibly with some parameters, relating derivatives of one or more functions with respect to one or more independent variables, like

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \nu(\nu+1)y = 0.$$

A system of differential equations is just a finite set of at least two such equations, like

$$x'(t) + x(t) + y(t) = 0, \quad y'(t) - x(t) + y(t) = 0.$$

A solution to a system must solve all of these equations.

Terminology: variables and parameters

Three types of variables appear in differential equations: dependent variables, independent variables and parameters.

- Dependent variables represent the function or functions we differentiate. They're what we're meant to be solving for.
- Independent variables are the ones with respect to which those derivatives are taken.
- Parameters are variables which don't appear in any derivatives.

Example: The Legendre differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \nu(\nu + 1)y = 0$$

has one dependent variable, y, one independent variable, x, and one parameter, ν.

Terminology: ordinary vs partial

A differential equation (or system) is called an ordinary differential equation (or system of ordinary differential equations) if there is only one independent variable and is called a partial differential equation (or system) if there is more than one. All the examples I've given so far are ordinary differential equations. An example of a partial differential equation is the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This has one dependent variable, u and two independent variables, x and y, so it is a partial differential equation. In this module we only consider ordinary differential equations. There is a separate module on partial differential equations.

Terminology: scalar vs vector

An equation (or system) is called scalar if there is only one dependent variable and is called vector if there is more than one. The Legendre equation is a scalar equation, while

$$x'(t) + x(t) + y(t) = 0 \quad y'(t) - x(t) + y(t) = 0,$$

is a vector system. The term "vector" makes more sense if you rewrite it as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ +1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

The system

$$x'(t) = y(t)z(t) \quad y'(t) = -x(t)z(t) \quad z'(t) = -k^2 x(t)y(t)$$

is also a vector system though, even though it can't be written conveniently in vector notation.

Terminology: order

The order of a differential equation or system is the order of the highest derivative appearing in it.

$$x'(t) = y(t)z(t) \quad y'(t) = -x(t)z(t) \quad z'(t) = -k^2 x(t)y(t)$$

is a first order system.

$$(1-x^2)y''(x) - 2xy'(x) + \nu(\nu+1)y(x) = 0$$

is a second order equation.

$$2\frac{\tau'''(j)}{\tau'(j)} - 3\frac{\tau''(j)^2}{\tau'(j)^2} = \frac{8}{9}\frac{1}{j^2} + \frac{3}{4}\frac{1}{(1-j)^2} + \frac{23}{36}\frac{1}{j(1-j)}$$

is a third order equation.

Terminology: Linear vs Non-linear

An equation (or system) is called linear if it is a linear equation (or system) in the independent variable(s) and its/their derivatives, with coefficients which are functions, possibly constant functions, of the independent variable(s) and any parameters.

$$\frac{d^2y}{dx^2} + y = 0$$

is linear, while

$$\left(\frac{dy}{dx}\right)^2 + y^2 = I$$

is not.

More about linearity

Note that coefficients are allowed to depend on the independent variables and the parameters, but are not allowed to depend on the dependent variables or their derivatives. So

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \nu(\nu+1)y = 0$$

is linear, while

$$(1-x^2)\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} + \nu(\nu+1)y = 0$$

is not.

Yet about more linearity

$$x'(t) + x(t) + y(t) = 0, \quad y'(t) - x(t) + y(t) = 0.$$

is a linear system while

$$x'(t) = y(t)z(t) \quad y'(t) = -x(t)z(t) \quad z'(t) = -k^2 x(t)y(t).$$

is not.
In Linear Algebra we distinguish between homogeneous linear equations or systems and inhomogeneous equations or systems. For example, $x + 2y = 0$ is homogeneous while $x + 2y = 3$ is inhomogeneous. It's similar for linear differential equations. $y''(x) + y(x) = 0$ is homogeneous; $y''(x) + y(x) = A\cos(x - \theta)$ is inhomogeneous. Note that the "constant term" in this linear equation depends on the independent variable and parameters.

Unfortunate terminology

There is nothing interesting you can say about systems which doesn't also apply to single equations. There is nothing interesting you can say about vector systems which doesn't also apply to scalar systems. There is nothing interesting you can say about non-linear systems which doesn't also apply to linear systems. There is nothing interesting you can say about linear inhomogeneous systems which doesn't also apply to linear homogeneous systems. When discussing the general theory, those terms, as defined above, just aren't very useful, and so we often talk about vector systems of linear inhomogeneous equations, for example, without meaning to exclude single equations, scalar systems or homogeneous equations. We only use the strict definitions for describing individual examples.

More unfortunate terminology

In most equations or systems that arise in applications it happens that the number of equations is equal to the number of dependent variables.

When that happens the equation/system distinction coincides with the scalar/vector distinction. Because of this, people often use the terms interchangeably.

Unfortunately the number of equations is not *always* the same as the number of dependent variables. Common counterexamples in Physics are Maxwell's Equations in Electrodynamics and Euler's Equations in Hydrodynamics. Those examples are Partial Differential Equations and this situation won't ever come up in this module, but it's good to be aware of it.