MAU23205 Lecture 1

John Stalker

Trinity College Dublin

15 September 2021

Module information

- Module: MAU23205 Ordinary Differential Equations
- Instructor: John Stalker (`stalker@maths.tcd.ie`)
- Webpage: `https://www.maths.tcd.ie/~stalker/23205/`
- I will use try to post everything both to the webpage and to Blackboard.
- Marking will be 80% exam, 20% continuous assessment.
- There will also be practice problems, unmarked, but with solutions posted.
- There are 3 lectures per week, posted online, and a weekly Q&A session.
- There is a discussion board on Blackboard.

Module content

- Definitions, terminology, notation
- Numerical solution methods
- Integral equations
- General existence and uniqueness theorems
- Review of analysis and linear algebra
- First order linear equations and systems
- Linear constant coefficient equations and systems
- Separable and integrable equations
- Stability of autonomous systems
- General existence and uniqueness theorems

Examples of ordinary differential equations

More precise definitions will follow, but here are some examples:

$$y''(x) + y(x) = 0$$

$$y'(x)^2 + y(x)^2 = I$$

$$x''(t) + 2x'(t) + 2x(t) = 0$$

$$(1 - x^2)y''(x) - 2xy'(x) + \nu(\nu + 1)y(x) = 0$$

$$2\frac{\tau'''(j)}{\tau'(j)} - 3\frac{\tau''(j)^2}{\tau'(j)^2} = \frac{8}{9}\frac{1}{j^2} + \frac{3}{4}\frac{1}{(1-j)^2} + \frac{23}{36}\frac{1}{j(1-j)}$$

Each of these equations has one independent variable, one dependent variable and various derivatives of the dependent variable with respect to the independent variable.

Alternate notations

There's an alternate notation that's often used:

$$\frac{d^2y}{dx^2} + y = 0 \qquad \left(\frac{dy}{dx}\right)^2 + y^2 = I$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 0$$

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \nu(\nu + 1)y = 0$$

Also, one often abbreviates the previous notation, e.g. writing

$$(1 - x^2)y'' - 2xy' + \nu(\nu + 1)y = 0$$

instead of $(1 - x^2)y''(x) - 2xy'(x) + \nu(\nu + 1)y(x) = 0$.

Examples of systems

We're also interested in systems of ordinary differential equations, like

$$\begin{aligned} x'(t) + x(t) + y(t) &= 0 \\ y'(t) - x(t) + y(t) &= 0 \end{aligned}$$

or

$$\begin{aligned} x'(t) &= y(t)z(t) \\ y'(t) &= -x(t)z(t) \\ z'(t) &= -k^2x(t)y(t) \end{aligned}$$

These examples still have a single independent variable but have multiple dependent variables.

Vector/matrix notation

The system

$$x'(t) + x(t) + y(t) = 0 \qquad y'(t) - x(t) + y(t) = 0$$

can also be written as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ +1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The system

$$x'(t) = y(t)z(t) \qquad y'(t) = -x(t)z(t) \qquad z'(t) = -k^2 x(t)y(t)$$

cannot be written in matrix notation.

Important questions about an equation or system

- Is there a solution?
- Can we find one?
- Can we find all of them?
- Can we find one with particular initial or boundary values?
- Is this solution unique?
- Does this solution depend continuously on those values and the values of any parameters?
- How differentiable is that solution?
- Does the solution exist for all "time"?
- Does it remain bounded?
- Is it stable?
- Can we find solutions numerically?

$y''(x) + y(x) = 0$ as an example

- Is there a solution? Yes. $y(x) = \sin x$ and $y(x) = \cos(x)$ are solutions.
- Can we find one? Yes, we just did!
- Can we find all of them? Yes. Every solution on an interval must be of the form $y(x) = a\cos x + b\sin x$ for some constants a and b. It's easy to check that all such functions are solutions, but it's not obvious that every solution is of that form.

$y'' + y = 0$ continued

- Can we find one with particular initial or boundary values? We can specify the value of y and y' at an arbitrary point. For example,

  $$y(x) = y_0 \cos(x - x_0) + z_0 \sin(x - x_0)$$

  is a solution with $y(x_0) = y_0$ and $y'(x_0) = z_0$. We may or may not be able to find solutions with specified boundary values.

  $$y(x) = \frac{y_1 \sin(x - x_0) + y_0 \sin(x_1 - x)}{\sin(x_1 - x_0)}$$

  is a solution with $y(x_0) = y_0$ and $y(x_1) = y_1$, provided that $x_1 - x_0$ is not an integer multiple of π. If it is an integer multiple of π then there needn't be a solution with $y(x_0) = y_0$ and $y(x_1) = y_1$.

$y'' + y = 0$ continued

- Is this solution unique? If $x_1 - x_0$ is not an integer multiple of π then yes. This follows from the general form of solutions and some trigonometric identities. If $x_1 - x_0$ is an integer multiple of π then no, since we can add $\sin(x - x_0)$ to any solution to obtain a different solution.
- Does this solution depend continuously on those values and the values of any parameters? Yes.
- How differentiable is that solution? It's infinitely differentiable.
- Does the solution exist for all "time"? Yes.
- Does it remain bounded? Yes.
- Is it stable? That depends on what you mean by stable.
- Can we find solutions numerically? Yes, but the numerical solutions may not be bounded.

Good news and bad news

Unfortunately the example $y''(x) + y(x) = 0$ is not typical. We usually can't find solutions explicitly in terms of familiar functions. We can usually answer all or most of the questions asked above though, even without an explicit solution. Learning how to do that is the main content of this module.