MAU22C00 Assignment 5, Due Friday 3 November 2023
Solutions

1. Show that if A and B are finite sets then so is $A \times B$.
Hint: Use induction on sets, twice. Let C be the set of subsets D of A such that $D \times B$ is finite. Show that $\emptyset \in C$ and that if $D \in C$ and $x \in A$ then $D \bigcup \{x\} \in C$. For the second of these it's helpful to prove that $\{x\} \times B$ is finite. For that, let E be the set of subsets F of B such that $\{x\} \times F$ is finite. Show that $\emptyset \in E$ and that if $F \in E$ and $y \in B$ then $F \bigcup \{y\} \in E$.
Solution: If $C \in \emptyset \times B$ then $C = (x, y)$ for some $x \in \emptyset$ and $y \in B$, but there is no $x \in \emptyset$ and hence no $C \in \emptyset \times B$. Therefore $\emptyset \times B = \emptyset$. $\emptyset$ is finite so $\emptyset \times B$ is finite. Therefore $\emptyset \in C$.

   Similarly $\{y\} \times \emptyset = \emptyset$ so $\emptyset \in E$. If $F \in E$ then $\{x\} \times F$ is finite. Now

$$\{x\} \times (F \bigcup \{y\}) = (\{x\} \times F) \bigcup (\{x\} \times \{y\})$$

   and

$$\{x\} \times \{y\} = \{(x, y)\}$$

   is finite. The union of two finite sets is finite so $\{x\} \times (F \bigcup \{y\})$ is finite. In other words, $F \bigcup \{y\} \in E$. E is a set of subsets of the finite set B and we've shown that $\emptyset \in E$ and that if $F \in E$ and $y \in B$ then $F \bigcup \{y\} \in E$. It follows from the induction theorem for sets that $B \in E$, i.e. that $\{x\} \times B$ is finite.

   Suppose that $D \in C$, i.e. that $D \times B$ is finite, and $x \in A$. Then

$$(D \bigcup \{x\}) \times B = (D \times B) \bigcup (\{x\} \times B)$$

   and we've just seen that $\{x\} \times B$ is finite and the union of two finite sets is finite so $(D \bigcup \{x\}) \times B$ is finite. In other words $(D \bigcup \{x\}) \in C$. So C is a set of subsets of the finite set A, $\emptyset \in C$ and if $D \in C$ and $x \in A$ then $D \bigcup \{x\} \in C$. By the induction theorem for sets then $A \in C$, so $A \times B$ is finite.

2. Suppose A is a set and $B = PA$ is its power set. Let C be the set of ordered pairs (D, E) of members of B such that $D \subseteq E$. Is the binary relation C on B

   (a) left total?
   (b) right total?
   (c) left unique?
   (d) right unique?

   Some of the answers could depend on A.
Solution:

(a) Yes. For every $D \in B$ we have $(D, E) \in C$ with $D = A$, so there is an $E \in B$ such that $(D, E) \in C$ and therefore C is left total. Note that $E = D$ would also have worked.

(b) Yes. For every $E \in B$ we have $(D, E) \in C$ with $E = \emptyset$, so there is a $D \in B$ such that $(D, E) \in C$ and therefore C is right total. Note that $E = D$ would also have worked.

(c) Not unless A is empty. Suppose C is left unique, i.e. that for every $E \in B$ there is at most one $D \in B$ such that $(D, E) \in C$. This applies in particular to $E = A$, so there is at most one subset of A. $\emptyset$ and A are subsets of A so they must be equal. Conversely, if A is empty then $C = \{(\emptyset, \emptyset)\}$ and this is left unique.

(d) Not unless A is empty. Suppose C is right unique, i.e. that for every $D \in B$ there is at most one $E \in B$ such that $(D, E) \in C$. This applies in particular to $D = \emptyset$, so there is at most one subset of A which is a superset of $\emptyset$. All sets are supersets of $\emptyset$ so there is at most one subset of A. $\emptyset$ and A are subsets of A so they must be equal. Conversely, if A is empty then $C = \{(\emptyset, \emptyset)\}$ and this is right unique.

3. There is a proof in the notes that N^3 is countable, but unlike the proof given there for N^2 it doesn't give an actual injective function to N. Given an example of an injective function from N^3 to N.
Hint: You can use the same idea as for N^2, although the picture given there is harder to draw in three dimensions.
Solution: One way to describe the construction given for N^2 is that we list the pairs (i, j) by increasing value of $i + j$ and by increasing value of j within those with the same value of $i + j$. So for N^3 we want to list the triples (i, j, k) by increasing value of $i + j + k$, by increasing value of $j + k$ within those with the same value of $i + j + k$, and by increasing value of k within those with the same values of $i + j + k$ and of $j + k$. In other words we'll visit the triples in the following order: $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(2, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, $(0, 2, 0)$, $(0, 1, 1)$, $(0, 0, 2)$, ... The function which accomplishes this is

$$\frac{(i+j+k)(i+j+k+1)(i+j+k+2)}{6} + \frac{(j+k)(j+k+1)}{2} + k.$$

The first summand is the number of triples with a lower value of $i + j + k$ than the current one, the second summand is the number with the same value of $i + j + k$ but a lower value of $j + k$ and the last summand is the number of triples with the same value of $i + j + k$ and $j + k$ but a lower value of k.

There are other ways to do this problem, but this is the one which implements the suggestion in the hint.