MAU22C00 Assignment 4, Due Friday 20 October 2023
Solutions

1. State the following in the formal language of elementary arithmetic:
   (a) the commutative law for addition
   (b) the associative law for multiplication
   (c) the distributive law
   (d) that there are infinitely many odd numbers

   Solution:

   (a) $(\forall x.\{\forall y.[(x+y)=(y+x)]\})$
   (b) $\{\forall x.[\forall y.(\forall z.\{[(x\cdot y)\cdot z]=[x\cdot(y\cdot z)]\})]\}$
   (c) There are actually two distributive laws,
   $$\{\forall x.[\forall y.(\forall z.\{[x\cdot(y+z)]=[(x\cdot y)+(x\cdot z)]\})]\},$$
   and
   $$\{\forall x.[\forall y.(\forall z.\{[(x+y)\cdot z)]=[(x\cdot z)+(y\cdot z)]\})]\}.$$
   (d) We have to state this in the equivalent for any number x there is an odd number y which is greater than x. Of the various ways to say that y is an odd natural number we'll use the one that y is $2\cdot z+1$ for some z. This gives
   $$[\forall x.(\exists y.\{[\neg(y\leq x)]\wedge(\exists z.\{y=[(0''\cdot z)+1]\})\})].$$
   There are few different but equally correct ways to state this though.

2. Give an informal proof that the union of two arithmetic sets or the intersection of two arithmetic sets is arithmetic.
   Solution: A set A of natural numbers is called arithmetic if there is some Boolean expression P, involving x, such that x belongs to A if and only if P is true. If A and B are arithmetic then there are Boolean expressions P and Q such that x belongs to A if and only if P is true and x belongs to B if and only if Q is true. x belongs to $A\bigcup B$ if and only if it belongs to A or to B, if and only if P or Q is true, which happens if and only the expression $(P\vee Q)$ is true. So $A\bigcup B$ is arithmetic. The argument for $A\bigcap B$ is the same, except with "and" in place of "or", $\wedge$ in place of $\vee$, and $\bigcap$ in place of $\bigcup$.

3. Which axiom or rule of inference is being used on each line of the following

proof of $[(0'' \cdot 0'') = 0'''']$?

1. $\{\forall x.[(x \cdot 0) = 0]\}$
2. $[(0'' \cdot 0) = 0]$
3. $[(0'' \cdot 0)' = 0']$
4. $[(0'' \cdot 0)'' = 0'']$
5. $[\forall x.(\forall y.\{(x \cdot y') = [(x \cdot y) + x]\})]$
6. $(\forall y.\{(0'' \cdot y') = [(0'' \cdot y) + 0'']\})$
7. $\{(0'' \cdot 0') = [(0'' \cdot 0) + 0'']\}$
8. $(\forall x.\{\forall y.[(x + y') = (x + y)']\})$
9. $(\forall y.\{[(0'' \cdot 0) + y'] = [(0'' \cdot 0) + y]'\})$
10. $\{[(0'' \cdot 0) + 0''] = [(0'' \cdot 0) + 0']'\})$
11. $\{(0'' \cdot 0') = [(0'' \cdot 0) + 0']'\}$
12. $\{[(0'' \cdot 0) + 0'] = [(0'' \cdot 0) + 0]'\}$
13. $(\forall x.\{[x + 0] = x\})$
14. $\{[(0'' \cdot 0) + 0] = (0'' \cdot 0)\}$
15. $\{[(0'' \cdot 0) + 0] = 0\}$
16. $\{[(0'' \cdot 0) + 0]' = 0'\}$
17. $\{[(0'' \cdot 0) + 0'] = 0'\}$
18. $\{[(0'' \cdot 0) + 0']' = 0''\}$
19. $\{[(0'' \cdot 0) + 0''] = 0''\}$
20. $[(0'' \cdot 0') = 0'']$
21. $\{(0'' \cdot 0'') = [(0'' \cdot 0') + 0'']\}$
22. $(\forall y.\{[(0'' \cdot 0') + y'] = [(0'' \cdot 0') + y]'\})$
23. $\{[(0'' \cdot 0') + 0''] = [(0'' \cdot 0') + 0']'\}$
24. $\{(0'' \cdot 0'') = [(0'' \cdot 0') + 0']'\}$
25. $\{[(0'' \cdot 0') + 0'] = [(0'' \cdot 0') + 0]'\})$
26. $\{[(0'' \cdot 0') + 0] = (0'' \cdot 0')\}$
27. $\{[(0'' \cdot 0') + 0] = 0''\}$
28. $\{[(0'' \cdot 0') + 0]' = 0'''\}$
29. $\{[(0'' \cdot 0') + 0'] = 0'''\}$
30. $\{[(0'' \cdot 0') + 0']' = 0''''\}$
31. $[(0'' \cdot 0'') = 0'''']$

For reference here are the axioms and rules of inference for arithmetic.

1 $\{\forall x.[\neg(x' = 0)]\}$

2 $\{\forall x.[(x + 0) = x]\}$

3 $(\forall x.\{\forall y.[(x + y') = (x + y)']\})$

4 $\{\forall x.[(x \cdot 0) = 0]\}$

5 $[\forall x.(\forall y.\{(x \cdot y') = [(x \cdot y) + x]\})]$

1 From a statement of the form $(X = Y)$ we can derive $(Y = X)$.

2 From statements of the form $(X = Y)$ and $(Y = Z)$ we can derive $(X = Z)$.

3 From a statement of the form $(X = Y)$ we can derive $(X' = Y')$.

4 From a statement of the form $(X' = Y')$ we can derive $(X = Y)$.

5 Suppose Q is the Boolean expression P with all free occurences of v replaced by 0 and R is P with all free occurences of v replaced by v'. From Q and $[\forall v.(P \supset R)]$ we can derive $(\forall v.P)$. The same holds with any other variable in place of v.

You can also use the quantifier rules from first order logic.
Solution:

1 The fourth axiom.

2 The first quantifier rule from first order logic, substituting the numerical expression $0''$ for the variable x in the previous line

3 The third rule of inference for arithmetic, applied to the previous line.

4 The third rule of inference for arithmetic, applied to the previous line.

5 The fifth axiom.

6 The first quantifier rule from first order logic, substituting $0''$ for x in the previous line.

7 The first quantifier rule from first order logic, substituting $0'$ for y in the previous line.

8 The third axiom.

9 The first quantifier rule from first order logic, substituting $(0'' \cdot 0)$ for x in the previous line.

10 The first quantifier rule from first order logic, substituting $0'$ for y in the previous line.

11 The second rule of inference for arithmetic, applied to lines 7 and 10.

12 The first quantifier rule from first order logic, substituting 0 for y in line 9.

13 The second axiom.

14 The first quantifier rule from first order logic, substituting $(0'' \cdot 0)$ for x.

15 The second rule of inference for arithmetic, applied to lines 2 and 14.

16 The third rule of inference for arithmetic, applied to the previous line.

17 The second rule of inference for arithmetic, applied to lines 12 and 16.

18 The third rule of inference for arithmetic, applied to the previous line.

19 The second rule of inference for arithmetic, applied to lines 12 and 16.

20 The second rule of inference for arithmetic, applied to lines 7 and 19.

21 The first quantifier rule from first order logic, substituting $(0'' \cdot 0')$ for x in line 6.

22 The first quantifier rule from first order logic, substituting $(0'' \cdot 0')$ for x in line 8.

23 The first quantifier rule from first order logic, substituting $0'$ for y in the previous line.

24 The second rule of inference for arithmetic, applied to lines 21 and 23.

25 The first quantifier rule from first order logic, substituting 0 for y in line 22.

26 The first quantifier rule from first order logic, substituting $(0'' \cdot 0')$ for x in line 13.

27 The second rule of inference for arithmetic, applied to lines 20 and 26.

28 The third rule of inference for arithmetic, applied to the previous line.

29 The second rule of inference for arithmetic, applied to lines 25 and 26.

30 The third rule of inference for arithmetic, applied to the previous line.

31 The second rule of inference for arithmetic, applied to lines 24 and 30.