MAU22C00 Assignment 2, Due Friday 13 October 2023
Solutions

1. In the following expressions, which variables are free and which are bound?

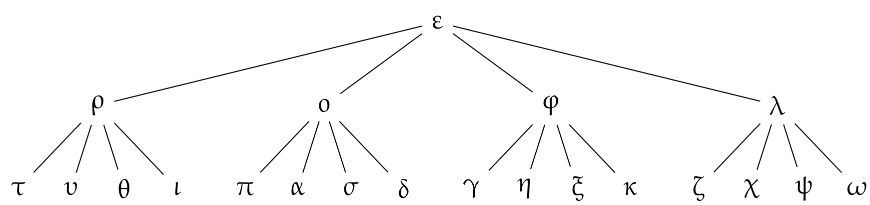
   (a) $\{[\exists x.(fx)] \supset [\exists y.(fy)]\}$
   (b) $\{[\forall x.(fx)] \supset (fy)\}$
   (c) $([(fx) \supset (gx)] \supset \{[(gx) \supset (hx)] \supset [(fx) \supset (hx)]\})$

   Solution:

   (a) x and y are both bound.
   (b) x is bound and y is free.
   (c) x is free.

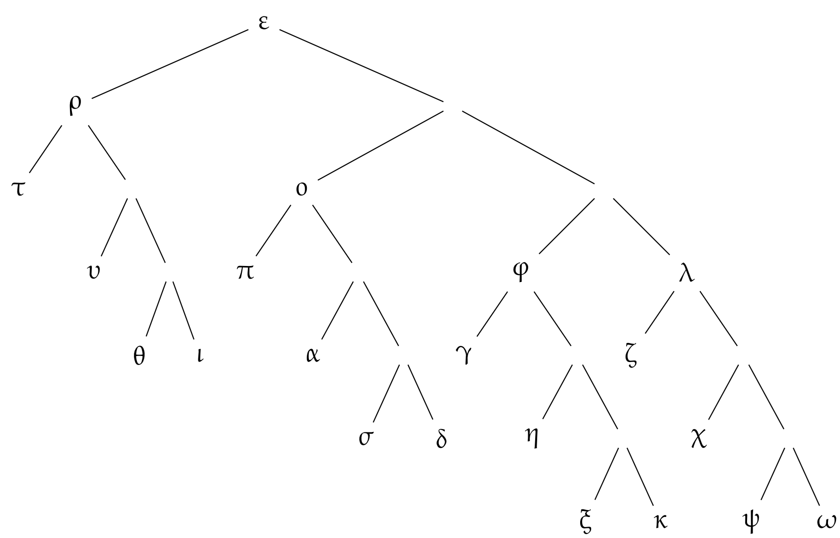
2. For the following tree, indicate the order in which the nodes are traversed

   (a) in a depth first traversal?
   (b) in a breadth first traversal?
   (c) in the hybrid traversal discussed in lecture?



   Solution:

   (a) $\varepsilon\ \rho\ \tau\ \upsilon\ \theta\ \iota\ o\ \pi\ \alpha\ \sigma\ \delta\ \varphi\ \gamma\ \eta\ \xi\ \kappa\ \lambda\ \zeta\ \chi\ \psi\ \omega$
   (b) $\varepsilon\ \rho\ o\ \varphi\ \lambda\ \tau\ \upsilon\ \theta\ \iota\ \pi\ \alpha\ \sigma\ \delta\ \gamma\ \eta\ \xi\ \kappa\ \zeta\ \chi\ \psi\ \omega$
   (c) $\varepsilon\ \rho\ \tau\ o\ \upsilon\ \pi\ \varphi\ \lambda\ \theta\ \iota\ \alpha\ \gamma\ \zeta\ \sigma\ \delta\ \eta\ \chi\ \xi\ \kappa\ \psi\ \omega$
   The first two can be read off from the tree above. For the last one it's helpful to construct the corresponding binary tree. The added nodes



are not labelled.

3. Give a tableau to show that the statement $\{[\exists x.(fx)] \supset [\exists y.(fy)]\}$ is valid.

   Solution:

	$\{[\exists x.(fx)] \supset [\exists y.(fy)]\}$
$[\exists x.(fx)]$	
	$[\exists y.(fy)]$
(fa)	
	$\underline{(fa)}$

   Note that we needed to process the two existentially quantified statements in the order shown above. Had we processed the one on the right of the line first then we we came to the one on the left we would have had to replace y by a new parameter, not a, since the tableau rule for existential quantifiers to the left of the line is one of the restricted ones.

4. Give a formal proof of the statement $\{[\exists x.(fx)] \supset [\exists y.(fy)]\}$.
   Hint: The statement is an implication. All the proofs of implications I've given in lecture start and end in the same way and this one is no exception. You have a rule of inference which eliminates existential quantifiers and another which introduces them. You want to use those rules in that order. If your proof is more than half a dozen lines long then it is unnecessarily complicated.
   Solution: The following is a proof.

   $$\begin{array}{l} .\quad [\exists x.(fx)] \\ .\quad .\quad (fa) \\ .\quad .\quad [\exists y.(fy)] \\ .\quad [\exists y.(fy)] \\ \{[\exists x.(fx)] \supset [\exists y.(fy)]\} \end{array}$$

   As always with an implication we start by introducing the premise as a hypothesis, in a new scope. One of our rules of inference, the last of the four rules for quantifiers, allows us to take the statement $[\exists x.(fx)]$ and introduce the same statement with the quantifier removed and a parameter substituted for free occurences of the variable as a hypothesis in a new scope. We replace the variable x by the parameter a. Note that "free" in this context means free in the expression after the dot, i.e. (fx), not the larger expression $[\exists x.(fx)]$. Our other rule of inference for existential quantifiers allows us to derive $[\exists y.(fy)]$ from (fa). We now discharge the hypothesis (fa). In keeping with the rule of inference we used to introduce it we are allowed to bring the statement $[\exists y.(fy)]$ outside the scope, since the parameter a does no appear in it. Finally we discharge the hypothesis $[\exists x.(fx)]$.