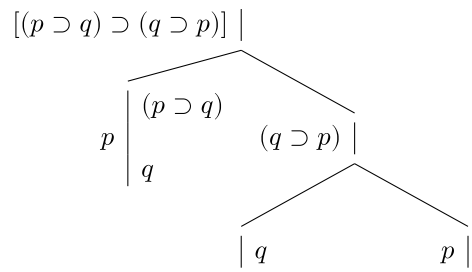
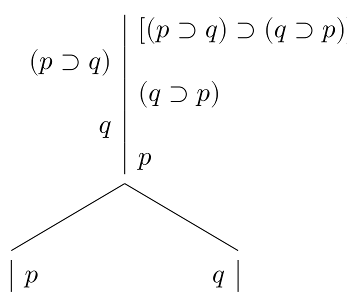
MAU22C00 Assignment 2, Due Friday 4 October 2023
Solutions

1. Write down a tableau to show that the statement $[(p \supset q) \supset (q \supset p)]$ is satisfiable and another to show that it is not a tautology.
Solution: To show that it's satisfiable we start a tableau with it on the left.



There's nothing further to be done and at least one branch has failed to close–in fact none of them have, but we only need one–so the statement is satisfiable.

To show that it is not a tautology we start a tableau with the same statement on the right.



Once again there's nothing further to be done and we have a branch which hasn't closed. In fact there's more than one but again we only need one. Since we started with our statement on the right and the tableau didn't close it must not have been a tautology.

2. Take the following proof in the natural deduction system and indicate

which rule of inference is being used in each line.

$$
\begin{array}{lll}
1 & . & [p \wedge (\neg p)] \\
2 & . & p \\
3 & . & (\neg p) \\
4 & . & . \quad (\neg q) \\
5 & . & . \quad [\neg(\neg p)] \\
6 & . & \{(\neg q) \supset [\neg(\neg p)]\} \\
7 & . & [(\neg p) \supset q] \\
8 & . & q \\
9 & \multicolumn{2}{l}{\{[p \wedge (\neg p)] \supset q\}}
\end{array}
$$

The line numbering isn't technically part of the proof; I've just added it to make it easier to refer to individual lines.

Solution: The first line is the introduction of a hypothesis, also known as the rule of fantasy. The hypothesis is $[p \wedge (\neg p)]$.

The second and third lines are from our first rule of inference, "From statements P and Q we can deduce the statement $(P \wedge Q)$. Also, from any statement of the form $(P \wedge Q)$ we can deduce the statment P and the statement Q." This is second part of it, applied to the first line.

The fourth line introduces the hypothesis $(\neg q)$.

The fifth line uses the third rule from the lecture notes, which is the fourth rule from lecture: "The expressions $[\neg(\neg P)]$ and P are freely interchangeable. In other words, anywhere an expression of one of these forms appears in a statement we may deduce the statement where it has been replaced by the other." This is applied to the second line.

The sixth line discharges the hypothesis $(\neg q)$ from the fourth line.

The seventh line uses the fifth rule of inference: "The expressions $(P \supset Q)$ and $[(\neg Q) \supset (\neg P)]$ are freely interchangeable." Here P is the expression $(\neg p)$ and Q is the expression q.

The eighth line uses the fourth rule of inference from the lecture notes, which is the third from lecture: "From P and $(P \supset Q)$ we can deduce Q." This is the rule known as modus ponens. It's applied to the third and seventh lines.

The ninth and final line discharges the hypothesis $[p \wedge (\neg p)]$.

3. The following proof of $\{[(p \vee r) \wedge (\neg(p \vee r))] \supset (r \supset q)\}$ is the same as the

one above, except for the last two lines, which use the rule of substitution.

$$
\begin{array}{ll}
1 & .\ \ [p \wedge (\neg p)] \\
2 & .\ \ p \\
3 & .\ \ (\neg p) \\
4 & .\ \ .\ \ (\neg q) \\
5 & .\ \ .\ \ [\neg(\neg p)] \\
6 & .\ \ \{(\neg q) \supset [\neg(\neg p)]\} \\
7 & .\ \ [(\neg p) \supset q] \\
8 & .\ \ q \\
9 & \{[p \wedge (\neg p)] \supset q\} \\
10 & \{[p \wedge (\neg p)] \supset (r \supset q)\} \\
11 & \{[(p \vee r) \wedge (\neg(p \vee r))] \supset (r \supset q)\}
\end{array}
$$

Give an alternate proof without substitution. What property of the natural deduction system makes this possible? Can the same be done with all proofs which use the rule of substitution?
Hint: Instead of substituting at the end you can run the same argument but with all the substitutions done from the start.
Solution:

$$
\begin{array}{ll}
1 & .\ \ [(p \vee r) \wedge (\neg(p \vee r))] \\
2 & .\ \ p \\
3 & .\ \ (\neg(p \vee r)) \\
4 & .\ \ .\ \ (\neg(r \supset q)) \\
5 & .\ \ .\ \ [\neg(\neg(p \vee r))] \\
6 & .\ \ \{(\neg(r \supset q)) \supset [\neg(\neg(p \vee r))]\} \\
7 & .\ \ [(\neg(p \vee r)) \supset (r \supset q)] \\
8 & .\ \ q \\
9 & \{[(p \vee r) \wedge (\neg(p \vee r))] \supset (r \supset q)\}
\end{array}
$$

This is just the proof given previously for $\{[p \wedge (\neg p)] \supset q\}$, except with $(p \vee r)$ replacing p everywhere and $(r \supset q)$ replacing q.

We could check that this is a valid proof but we don't really need to. The rules of inference used are the same as in the previous problem. We know this works because those rules only ever referred to statements and expressions, never to variables. So replacing variables by statements doesn't affect the validity of any inferences. We could have done the same thing with any tautology and any substitution.