# MAU22C00 Lecture 17

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# Infinite sets

We have various things which are definitely not finite sets.

- The natural numbers: We know that
  - any partial order on a finite set has a maximal member
  - $\leq$ is a partial order on the natural numbers, if they're a set
  - the natural numbers have no maximal member with respect to $\leq$ so they can't form a finite set.
- The list with items chosen from a non-empty set $A$: We know that
  - any injective function from a finite set to itself is surjective
  - if it's a set then the function which takes a list and appends an $x \in A$ to it is an injective function
  - it is not a surjective function so they can't form a finite set either.

On the other hand, our axioms aren't sufficient to prove the existence of any finite set.

Extensionality doesn't give us any sets. Elementary Sets only gives us finite sets. Separation, Union and Power Sets when applied to finite sets give finite sets.

If we want the natural numbers or the lists with items in a given set to be sets then we need a new axiom.

# Axiom of Infinity

Simply assuming that there is an infinite set isn't enough to get either the set of natural numbers or the set of lists with items in $A$ if $A$ is non-empty.

Assuming that we have sets of lists is enough to get the set of natural numbers.

Assuming we have the set of natural numbers is enough to get sets of lists, but we'd need a further axiom, the Axiom of Replacement.

I'll take the following as an axiom: For every set $A$ there is a set $B$ such that $C \in B$ if and only if $C$ is a list and every item in $C$ is a member of $A$.

A more common approach is take the existence of the natural numbers as an axiom, or rather to take the following statement, from which one can construct the set of natural numbers: There is a set $A$ such that $\emptyset \in A$ and if $B \in A$ then $B \bigcup \{B\} \in A$.

# The von Neumann ordinals

If you assume the axiom just described then there is a set whose members are $\emptyset$, $\{\emptyset, \{\emptyset\}\}$, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, etc.

These are called the von Neumann ordinals. The set from the axiom might be larger but we can use Separation to select only those members which are finite and satisfy

$$\forall B \in C : \forall C \in B : C \in A \vee \forall D \in C : D \in B.$$

We can define $0 = \emptyset$ and define an increment operator $'$ by $B' = B \bigcup \{B\}$. By a much more complicated procedure we can define an addition operator $+$ and a multiplication operator $\cdot$ and show that these satisfy the Peano axioms.

This is an implementation of the natural numbers within set theory.

# Natural numbers as lists

An alternate implementation is to pick some $x$ and declare that natural numbers are lists where each item is an $x$.

The idea is that all items are the same so the only distinguishing feature of the lists is their length, which is a natural number. We can identify the list of length $n$ with the natural number $n$.

We define $0 = ()$, the empty list and define $A'$ to be the list $A$ with an $x$ appended. Addition is just concatenation. Multiplication is harder to describe.

If you think of lists as stacks then this is the stack representation of the natural numbers from the discussion of pushdown automata in the introduction.

As usual, it doesn't really matter which implementation we choose but we need to pick one. I'll use the list version, with $x = \emptyset$.

From now on the set of natural numbers will be denoted by $N$.

## Comparing the size of sets

I've mostly avoided talking about sizes of sets until now.

We can define a certain notion of size, called cardinality, based on injective functions.

$A$ is said to be no larger than $B$ if there is an injective function from $A$ to $B$.

If $A \subseteq B$ then there is such a function, the set of $(x, x)$ with $x \in A$, so $A$ is no larger than $B$.

For infinite sets it can happen that $B$ is a proper subset of $A$ but $A$ is no larger than $B$.

This happens when $A$ is the set of natural numbers and $B$ is the set of even natural numbers. The injective function is the set of ordered pairs $(n, 2 \cdot n)$.

Nothing like this happens for finite sets.

$A$ and $B$ are said to be of the same size if $A$ is no larger than $B$ and vice versa.

The set of natural numbers and the set of even natural numbers are of the same size.

# Schröder-Bernstein

Suppose there is a bijective, i.e. injective and surjective, function $F$ from $A$ to $B$. Then $F$ is an injective function from $A$ to $B$ and $F^{-1}$ is an injective function from $B$ to $A$. So $A$ is of the same size as $B$.

Conversely, if $A$ is of the same size as $B$ then there is a bijective function $F$ from $A$ to $B$.

This is not obvious. We have an injective function from $A$ to $B$ and another from $B$ to $A$ but these don't have to be inverses of each other, or even have inverses.

We just saw that with the natural numbers and even natural numbers.

There is a theorem though, called the Schröder-Bernstein theorem, which guarantees the existence of a bijective function between sets of the same size.

## Properties of cardinality

"Is no larger than" behaves like a reflexive relation in that $A$ is no larger than $A$.

This is true because the identity function is injective.

"Is no larger than" behaves like a transitive relation in that if $A$ is no larger than $B$ and $B$ is no larger than $C$ then $A$ is no larger than $C$.

This is true because the composition of injective functions is injective

"Is no larger than" behaves sort of like an antisymmetric relation in that if $A$ is no larger than $B$ and $B$ is no larger than $A$ then and $A$ is of the same size as $B$. So it behaves sort of like a partial order on sets.

"Is no larger than" is not a relation. Relations are defined on sets and there is no set of all sets!

Whether it behaves sort of like a total order, i.e. whether for all $A$ and $B$ it's true that $A$ is no larger than $B$ or $B$ is no larger than $A$ is a complicated question.

## Power sets

For any set $A$ it's true that $A$ is no larger than $PA$.

For no set $A$ is it true that $PA$ is no larger than $A$.

The proof is via the Cantor diagonalisation argument, which you can find in the notes. I recommend reading it. The idea is one which occurs in many places.

So $A$ and $PA$ cannot be of the same size.

We say that $A$ is strictly smaller than $B$ if $A$ is no larger than $B$ and $A$ and $B$ are not of the same size. So $A$ is always strictly smaller than $PA$

In particular $N$ is strictly smaller than $PN$, which is strictly smaller than $PPN$, which is strictly smaller than $PPPN$, etc.

There are different levels of infinity. We'd like to identify a class of sets which are not necessarily finite but aren't *too* infinite.

# Countable sets

Sets which are no larger than $N$ are called countable.

Unfortunately there are two different conventions. One is the convention above. The other says that sets of the same size as $N$ are countable. Both agree that uncountable means strictly larger than $N$. In the convention used here finite sets are countable. In the other convention they are neither countable nor uncountable.

Subsets, intersections, and relative complements of countable sets are countable. This is easy.

The union of two countable sets is countable. This is a bit harder.

The Cartesian product of countable sets is countable. This one is tricky.

If $A$ is a countable set then the set of lists all of whose items are members of $A$ is countable. This one's quite tricky.

The power set of a countable set needn't be countable. Indeed $PN$ is not countable.

# Uncountable sets

$PN$ is uncountable.

The set of tokens in our language for arithmetic is finite, hence countable.

The set of lists of such tokens is countable, so the set of lists in our language must be countable.

Suppose every element of $PN$ were characterised by a statement in our language. By associating to each set in $PN$ the first natural number which is the encoding of such a statement we'd get an injective function from $PN$ to $N$.

There is no injective function from $PN$ to $N$, so not every element $PN$ is characterised by a statement in our language.

In other words, there are sets of natural numbers which are not arithmetic.

Tarski's theorem gives us a specific non-arithmetic set but it's much easier to show that there is a non-arithmetic set than to identify one.

## Uncountable sets, continued

Similarly, for any countable (including finite) set of tokens the set of phrase structure grammars for languages with those tokens is countable.

If the set of tokens is non-empty then the set of languages is uncountable.

So there are languages with no phrase structure grammar.

Arguments like the ones above are called counting arguments, even though the key step is identifying a set we can't count.