# MAU22C00 Lecture 15

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## Announcements

- I'm not here on Thursday. I will post a recorded lecture to Blackboard and to the module web page.
- I've posted more lecture notes. There will be still more in the next few days.

## Lots of identities

Last time I described how to get all of these identities from zeroeth order logic.
There's more detail in the notes.

$$
\begin{array}{ll}
[[A \bigcap B] \subseteq A] & [[A \setminus [A \setminus B]] = [A \bigcap B]] \\
[[A \bigcap B] \subseteq B] & [[[A \bigcap B] \bigcap C] = [A \bigcap [B \bigcap C]]] \\
[A \subseteq [A \bigcup B]] & [[[A \bigcup B] \bigcup C] = [A \bigcup [B \bigcup C]]] \\
[B \subseteq [A \bigcup B]] & [[[A \setminus B] \setminus C] = [A \setminus [B \bigcup C]]] \\
[[A \setminus B] \subseteq A] & [[[A \setminus B] \bigcap C] = [A \bigcap [C \setminus B]]] \\
[[A \bigcap A] = A] & [[A \setminus [B \setminus C]] = [[A \bigcap C] \bigcup [B \setminus C] \\
[[A \bigcup A] = A] & [[[A \bigcap [B \bigcup C]] = [[A \bigcup C] \bigcap [B \bigcup C]]] \\
[[A \bigcap B] = [B \bigcap A]] & [[[A \bigcup [B \bigcap C]] = [[A \bigcap C] \bigcup [B \bigcap C]]] \\
[[A \bigcup B] = [B \bigcup A]] & [[C \setminus [A \bigcap B]] = [[C \setminus B] \bigcup [C \setminus A]]] \\
[[A \bigcap [A \bigcup B]] = A] & [[C \setminus [A \bigcup B]] = [[C \setminus B] \bigcap [C \setminus A]]] \\
[[A \bigcup [A \bigcap B]] = A] &
\end{array}
$$

The first few are obviously true. That's not really the point though. The point is that they are theorems. Not every true statement is a theorem.

# Finite sets

Can we express finiteness in the language of set theory?

The answer had better be yes or we need a different language.

It's not immediately obvious that we have a way to talk about sizes of sets.

Even if we did, we don't have a notation for natural numbers within the language.

One option is an inductive definition. $\emptyset$ is finite and if $A$ is finite then $A \bigcup \{x\}$ is finite for all $x$.

It's not clear how you could prove a set is infinite using this definition, but we will return to this idea.

There are a number of properties finite sets have which infinite sets don't. That's why we need to define finiteness but it also gives us a way to define it.

We just need to pick one such property which makes it (relatively) easy to prove the others.

# Tarski finiteness

There are a few definitions in common use. Tarski's not the most intuitive but it is one of the most efficient.

A member $C$ of a set $A$ of sets is said to minimal if it is a member of $A$ and no proper subset of $C$ is a member of $A$. $C$ is said to maximal if no proper superset of $C$ is a member of $A$.

This is something we can express in our language.

As examples, here are some sets of subsets of $\{x, y\}$ and their minimal an maximal members. Note that the subsets of $\{x, y\}$ are $\emptyset$ and $\{x\}$, $\{y\}$, and $\{x, y\}$.

| set | minimal member(s) | maximal member(s) |
|---|---|---|
| $\{\emptyset, \{x\}, \{y\}, \{x, y\}\}$ | $\emptyset$ | $\{x, y\}$ |
| $\{\{x\}, \{y\}, \{x, y\}\}$ | $\{x\}, \{y\}$ | $\{x, y\}$ |
| $\{\emptyset, \{x\}, \{y\}\}$ | $\emptyset$ | $\{x\}, \{y\}$ |
| $\{\{x\}, \{y\}\}$ | $\{x\}, \{y\}$ | $\{x\}, \{y\}$ |

# Tarski finiteness

There are 12 more sets of subsets of $\{x, y\}$. 11 of those are non-empty and each has at least one minimal and at least one maximal member.

The empty set of subsets has no members, so clearly no minimal or maximal member.

A set $E$ is said to be finite if every non-empty set of subsets of $E$ has both a minimal and a maximal member.

This also is something we can express in our language.

A set is said to be infinite if it is not finite.

$\{x, y\}$ is a finite set by this definition.

It's fairly clear that sets we think of as finite are finite according to this definition.

It's a bit less clear that sets we think of as infinite, e.g. the natural numbers, are infinite according to this definition.

# Related criteria for finiteness

If every non-empty set of subsets of $E$ has a minimal member then every non-empty set of subsets has a maximal member and vice-versa.

This is true because taking complements relative to $E$ reverses inclusions. In other words, if $B \subseteq C$ then $E \setminus C \subseteq E \setminus B$.

I'm dropping brackets where it's "obvious" where they should go.

We could use either condition to define finiteness, or use both.

I use both, for symmetry, but to prove a set is finite it suffices to show either.

# Properties

Proving by $\{x,y,z\}$ is finite by listing non-empty sets of subsets and finding minimal and maximal member would need a table with 255 rows.

It's better just to prove that if $A$ and $B$ are finite then so is $A \bigcup B$.

The proof is fairly ugly, but better than listing maximal and minimal members of 255 sets. You can find it in the notes.

It's very easy to see that subsets of finite sets are finite, so $\emptyset$ is finite and $\{x\}$ is finite.

So $\emptyset$ is finite and if $A$ is finite then so is $A \bigcup \{x\}$ for any $x$.

That's not our definition; that's a theorem.

Intersections and relative complements of finite sets are also finite. In fact for $A \setminus B$ to be finite it's sufficient for $A$ to be finite and for $A \bigcap B$ to be finite it's sufficient for $A$ or $B$ to be finite.

# Induction on sets

If the union of two finite sets is finite then the union of finitely many sets should also be finite.

If we were able to define finiteness in terms of the number of members we would prove this by induction on the number of sets in our set of sets.

In other words, the union of 0 sets is finite. If it's true that the union of $n$ finite sets is always finite then it's true that the union of $n+1$ finite sets is finite, because we can write it as the union of a union of $n$ sets and another set.

We don't have numbers and didn't define finiteness that way but you can do induction on sets!

Suppose $A$ is a finite set and $B$ is a set of sets such that $\emptyset \in B$ and for all $C \in B$ and $x \in A$ we have $C \bigcup \{x\} \in B$. Then $A \in B$.

# Induction on sets

*Suppose $A$ is a finite set and $B$ is a set of sets such that $\emptyset \in B$ and for all $C \in B$ and $x \in A$ we have $C \bigcup \{x\} \in B$. Then $A \in B$.*

You can find the proof in the notes. It's relatively short.

Note that in Peano Arithmetic the principle of mathematical induction was an axiom. The corresponding principle for set theory is a theorem.

Application: Suppose the members of $A$ are all finite sets. Let $B$ be the set of subsets of $A$ such that $\bigcup B$ is finite. Then $\emptyset \in B$ and for all $C \in B$ and and $x \in A$ we have $C \bigcup \{x\} \in B$. So $A \in B$.

In other words, $\bigcup A$ is finite, so the union of finitely many finite sets is finite.

If you're tempted to prove a result about finite sets by (arithmetic) induction on the number of members then you can actually prove it by set induction.

A slight repackaging of the set induction principle is

*Suppose $A$ is a member of every set of sets $B$ such that $\emptyset \in B$ and for all $C \in B$ and $x \in A$ we have $C \bigcup \{x\} \in B$. Then $A$ is finite.*

# Power sets

Another application of set induction is to prove that power sets of finite sets are finite.

$P\emptyset = \{\emptyset\}$, which is finite.

Note that $P\emptyset \neq \emptyset$!

If $PA$ is finite then so is $P[A \bigcup \{x\}] \setminus PA$.

There are two cases. If $x \in A$ then $P[A \bigcup \{x\}] \setminus PA$ is empty. Otherwise each member of $P[A \bigcup \{x\}] \setminus PA$ is a member of $PA$ with an extra $x$.

Now $P[A \bigcup \{x\}] = [PA] \bigcup [P[A \bigcup \{x\}] \setminus PA]$ so $P[A \bigcup \{x\}]$ is finite.

Therefore $PA$ is finite whenever $A$ is finite.

## Kuratowski pairs

Sets of the form $\{\{x\},\{x,y\}\}$ are called Kuratowski pairs.

They are one way to represent the concept of an ordered pair in set theory.

Note that we're not assuming $x \neq y$. If $x = y$ then $\{x,y\} = \{x\}$ and $\{\{x\},\{x,y\}\} = \{\{x\}\}$.

Questions: Given a set $A$, constructed in some way,

- Can you identify whether $A$ is a Kuratowski pair?
- Can you identify $x$ and $y$?
- Can you *distinguish* $x$ from $y$.

Answers:

- Yes, but it's somewhat painful.
- Yes, $x$ and $y$ are the members of the set $\bigcup A$.
- Yes, $x$ is the unique member of $\bigcap A$. $y$ is the unique element of $[\bigcup A] \setminus [\bigcap A]$ if it is non-empty, or the same as $x$ if it's empty.

# Kuratowski pairs

The ability to distinguish $x$ from $y$, if they're different is what makes Kuratowski pairs suitable as an implementation of ordered pairs, unlike the set $\{x, y\}$.

A consequence–really just a rephrasing of this–is that $\{\{v\}, \{v, w\}\} = \{\{x\}, \{x, y\}\}$ if and only if $v = x$ and $w = y$.

Ordered pairs are very useful. We'll define functions, order relations and various other things in terms of them.

Ordered triples are also useful. Unfortunately the obvious generalisation of Kuratowski pairs does not work!

It's not true that

$$\{\{u\}, \{u, v\}, \{u, v, w\}\} = \{\{x\}, \{x, y\}, \{x, y, z\}\}$$

only if $u = x$, $v = y$ and $w = z$.

There's a counter-example in the notes.