MAU22C00 Lecture 7

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The Łukasiewicz system

Three axioms:

- $[p \supset (q \supset p)]$
- $\{[p \supset (q \supset r)] \supset [(p \supset q) \supset (p \supset r)]\}$
- $\{[(\neg p) \supset (\neg q)] \supset (q \supset p)\}$

Two rules of inference:

- Substitution: From any statement we can derive the result of substituting an expression for all occurences of a particular variable in that statement.
- Modus ponens: From P and $(P \supset Q)$ we can derive Q.

A proof

Here is a formal proof of $(p \supset p)$:

$$
\begin{array}{ll}
1 & (p \supset (q \supset p)) \\
2 & ((p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))) \\
3 & (p \supset ((q \supset p) \supset p)) \\
4 & ((p \supset ((q \supset p) \supset p)) \supset ((p \supset (q \supset p)) \supset (p \supset p))) \\
5 & (p \supset (q \supset p)) \supset (p \supset p)) \\
6 & (p \supset p)
\end{array}
$$

Each line is either an axiom or is derived from previous lines using one of the two rules of inference. Which are which?

The first two are axioms, the next two are substitution and the last two are modus ponens.

It's obvious that $(p \supset p)$ is a tautology. What wasn't obvious is that we could prove it in this formal system.

Disadvantages of traditional axiomatic systems

- Proofs are long. We took six lines to prove $(p \supset p)$. Proving more complicated statements is harder.
- It's hard to show completeness of the system. It's not too hard to show that every theorem is a tautology but how do we show that *every* tautology is a theorem?
- Proofs are unenlightening. Do you feel you got any insight into *why* $(p \supset p)$ from the proof above?

If you're unimpressed by traditional axiomatic systems you're not alone. Even Łukasiewicz, who created this system, did not like traditional axiomatic systems, for the reasons above.

He and others developed natural deduction systems as a reaction against traditional axiomatic systems.

Natural deduction systems

- Fewer (or no) axioms
- More rules of inference
- Proofs much more similar to typical mathematical reasoning.
- Easier to turn an informal proof, e.g. a tableau, into a proof.
- Disadvantage: usually harder to reason *about* a natural deduction system than an axiomatic one.

A natural deduction system

The system from the notes has *no axioms* and ten rules of inference:

- From statements P and Q we can deduce the statement $(P \wedge Q)$. Also, from any statement of the form $(P \wedge Q)$ we can deduce the statment P and the statement Q.
- From the statement P we can deduce the statement $(P \vee Q)$, where Q is any expression.
- From the statements P and $(P \supset Q)$ we can deduce the statement Q.
- The expressions $[\neg(\neg P)]$ and P are freely interchangeable. In other words, anywhere an expression of one of these forms appears in a statement we may deduce the statement where it has been replaced by the other.
- The expressions $(P \supset Q)$ and $[(\neg Q) \supset (\neg P)]$ are freely interchangeable.
- The expressions $[(\neg P) \wedge (\neg Q)]$ and $[\neg(P \vee Q)]$ are freely interchangeable.

A natural deduction system, continued

- The expressions $[(\neg P) \vee (\neg Q)]$ and $[\neg(P \wedge Q)]$ are freely interchangeable.
- The expressions $(P \vee Q)$ and $[(\neg P) \supset Q]$ are freely interchangeable.
- The "Rule of Fantasy", to be described below.
- The "Rule of Substitution", subject to restrictions to be discussed below.

The first three rules operate on statements as a whole. The next five operate on expressions within statements, which could be the whole statement but don't have to be.

The "rule of fantasy" allows us to introduce a hypothesis, reason for a while on the assumption that that hypothesis is true, reach a conclusion, and then say that that conclusion holds if the hypothesis is true.

In more detail, we can introduce as a hypothesis any expression P. Until we discharge that hypothesis every statement we make, including that one, is said to be within the *scope* of the hypothesis.

Discharging hypotheses

At any point we can discharge the most recent undischarged hypothesis. If that hypothesis was P and the last statement we derived was Q then we can derive $(P \supset Q)$ *outside the scope of the hypothesis* P.

Once we have discharged a hypothesis we no longer have access to any statements derived within its scope.

You should think of everything within the scope as being (implicitly) conditional on the validity of the hypothesis.

We can only finish a proof outside the scope of all hypotheses, i.e. after all hypotheses we've introduced have been discharged.

Two common reasons to introduce a hypothesis are for case by case analysis and for proof by contradiction. Often when reading a proof you don't find out why a hypothesis has been introduced until it's discharged.

An example proof

Here's a formal proof of $\{[(p \supset q) \wedge (q \supset r)] \supset (p \supset r)\}$:

$$
\begin{array}{l}
.\quad [(p \supset q) \wedge (q \supset r)] \\
.\quad .\quad p \\
.\quad .\quad (p \supset q) \\
.\quad .\quad (q \supset r) \\
.\quad .\quad q \\
.\quad .\quad r \\
.\quad (p \supset r) \\
\{[(p \supset q) \wedge (q \supset r)] \supset (p \supset r)\}
\end{array}
$$

Here I've followed the common notational convention of using indentation to indicate scope. Everything indented once is in the scope of the hypothesis $[(p \supset q) \wedge (q \supset r)]$. Everything indented twice is in the scope of that hypothesis and the hypothesis p.

A translation

Here's a translation of

$$
\begin{array}{l}
.\quad [(p \supset q) \wedge (q \supset r)] \\
.\quad .\quad p \\
.\quad .\quad (p \supset q) \\
.\quad .\quad (q \supset r) \\
.\quad .\quad q \\
.\quad .\quad r \\
.\quad (p \supset r) \\
\{[(p \supset q) \wedge (q \supset r)] \supset (p \supset r)\}
\end{array}
$$

into an informal proof.

Suppose $[(p \supset q) \wedge (q \supset r)]$ is true. Then $(p \supset q)$ and $(q \supset r)$ are both true. If p is true then q is true and therefore r is true. So $(p \supset r)$. This holds under our assumption that $[(p \supset q) \wedge (q \supset r)]$ is true, so $\{[(p \supset q) \wedge (q \supset r)] \supset (p \supset r)\}$.

The rule of substitution

In a traditional axiomatic system the following rule of inference is sound: from any statement we can derive another statement by replacing all occurences of one of its variables by any expression.

In our natural deduction system this would not be sound. Consider the following "proof" of $[p \supset (\neg p)]$:

$$
\begin{array}{l}
\cdot \quad p \\
\cdot \quad q \\
(p \supset q) \\
(p \supset (\neg p))
\end{array}
$$

The first step is to introduce p as a hypothesis. The next step is to take the statement p and substitute the expression q for the variable p in it. Then we discharge the hypothesis p, obtaining $(p \supset q)$. Then we substitute the expression $(\neg p)$ for the variable q in that.

Of course $(p \supset (\neg p))$ is false in any interpretation!

Fixing substitution

There are three ways to fix this problem.

- Ban substitution! It's not actually needed. We can prove all tautologies of zeroeth order logic without it.
- Only allow substitution outside the scope of any hypothesis.
- Allow substitution within the scope of one or more hypothesis, but only for variables which don't appear in any of those hypotheses.

The first option is the simplest, but leads to very long proofs.

The second option gives shorter proofs, but they have a weird structure. First you prove every statement you want to substitute into and do all your substitutions. Then you give the main proof, introducing hypotheses and using those substitution instances as needed.

The third option leads to the simplest proofs, but it's harder to show that it's sound.

Lessons for reading informal proofs

One reason to look at formal proofs is the insight they give into reading and writing informal proofs.

Mathematicians introduce and discharge hypotheses all the time. We don't use indentation, just phrases like "Suppose ..." or "Assuming ..." or "If ..." buried in the text. If you see a statement in the middle of a long proof it can be difficult to identify the hypotheses under which it's true.

Using statements without identifying their hypotheses is dangerous!

There may be some clues why a hypothesis has been introduced.

Conditional statements like "if A *were* finite" often signal a proof by contradiction.

If a number of cases are listed and then one is introduced as a hypothesis you're looking at case by case analysis, with each case introduced in turn and then discharged.

Often you won't find out why a hypothesis was introduced until after it's discharged. Proofs are sometimes easier to read out of sequence!