MAU22200 Lecture notes

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1 Limits

1.1 Levels of generality

Topology began as an attempt to generalise various notions from real and complex analysis. Generalisation has a number of purposes, but one is to avoid giving very similar proofs over and over again in slightly different settings. There are, for example, a number of elementary results in analysis saying that the limit of a finite sum is the sum of the limits. This can be proved for functions or sequences, with limits at finite points or infinity. There are other theorems which are not usually stated in terms of limits of sums, but which could be. The integral of a finite sum is the sum of the integrals, for example. We'd like to think of the Riemann integral as a limit of Riemann sums and think of the theorem about integrals of sums as a special case of a theorem about limits of sums. In what sense though is the Riemann integral a limit of Riemann sums?

It's helpful to work simultaneously at several different levels of generality. For most theorems there is an optimal level of generality, one where the definitions reflect all the properties needed in the proof, but none of the extraneous detail associated with special cases, even if those special cases are what we're ultimately most interested in. The most efficient way to prove everything we need is to start at the most general level and then progressively specialise, proving at each stage those results which are true at that level of generality but not the previous ones. The historical development is usually the opposite of this, with the results in special cases coming first and then gradually being generalised further and further. At each level of generality some results which held at previous levels may fail to generalise, either because their statements no longer make sense or because they're no longer true. A presentation along those lines is more intuitive, because it starts with familiar properties of familiar objects, but it necessarily involves repeatedly reproving the same results at various levels of generality, which defeats the main purpose of generalisation. A less obvious but equally important problem is that we need to redefine a number of terms each time we generalise, and this requires showing that the new definition when restricted to the old setting agrees with the old definition. Mostly I will follow the efficient but ahistorical and somewhat unintuitive approach, but for the particular case of the theorem about limits of sums I will start with a special case and gradually generalise it. Hopefully this will give some idea of why various definitions look the way they do.

1.2 Limits of real valued functions of a real variable

The following definition and the two theorems which follow it should be familiar:

Definition 1.2.1. If $f\colon \mathbf{R} \to \mathbf{R}$ is a function, $w \in \mathbf{R}$ and $z \in \mathbf{R}$ then we say that z is the *limit* of f at w, written

$$\lim_{x\to w} f(x) = z,$$

if for all $\epsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x - w| < \delta$ we have $|f(x) - z| < \epsilon$.

The word "the" requires a justification:

Theorem 1.2.2. *There is at most one z such that $\lim_{x\to w} f(x) = z$.*

Proof. Suppose on the contrary that

$$\lim_{x\to w} f(x) = z_1,$$

$$\lim_{x\to w} f(x) = z_2$$

and

$$z_1 \neq z_2.$$

Then set

$$\epsilon = |z_1 - z_2|/2.$$

Clearly $\epsilon > 0$, so by the definition of the limit there are $\delta_1 > 0$ and $\delta_2 > 0$ such that whenever

$$0 < |x - w| < \delta_1$$

and

$$0 < |x - w| < \delta_2$$

we have

$$|f(x) - z_1| < \epsilon$$

and

$$|f(x) - z_2| < \epsilon.$$

There is an x which satisfies both

$$0 < |x - w| < \delta_1$$

and

$$0 < |x - w| < \delta_2.$$

For example,

$$x = w + \min(\delta_1, \delta_2)/2$$

works. We therefore have

$$|f(x) - z_1| < \epsilon$$

and

$$|f(x) - z_2| < \epsilon.$$

An elementary property of the absolute value function is that if $a \in \mathbf{R}$ and $b \in \mathbf{R}$ then

$$|a + b| \leq |a| + |b|.$$

Taking $a = f(x) - z_1$ and $b = z_2 - f(x)$ we find that

$$|z_2 - z_1| \leq |f(x) - z_1| + |(-1)(f(x) - z_1)|.$$

Further properties of the absolute value function are that

$$|\alpha b| = |\alpha||b|$$

for all $\alpha \in \mathbf{R}$ and $b \in \mathbf{R}$ and

$$|-1| = 1.$$

Using these,

$$|(-1)(f(x) - z_1)| = |f(x) - z_1|.$$

So

$$|z_2 - z_1| \leq |f(x) - z_1| + |(f(x) - z_1)| < \epsilon + \epsilon.$$

But ϵ was chosen such that

$$\epsilon + \epsilon = |z_2 - z_1|.$$

There is no real number which is less than itself, so we have a contradiction. Our assumption that there are distinct z_1 and z_2 such that $\lim_{x\to w} f(x) = z_1$ and $\lim_{x\to w} f(x) = z_2$ must therefore be false. □

As promised, the limit of a finite sum is the sum of the limits.

Theorem 1.2.3. *Suppose that $f_1, \ldots, f_k$ are functions from $\mathbf{R}$ to $\mathbf{R}$ such that*

$$\lim_{x\to w} f_j(x) = z_j$$

for $j = 1, \ldots, k$. Then

$$\lim_{x\to w} \sum_{j=1}^{k} f_j(x) = \sum_{j=1}^{k} z_j.$$

Proof. Let $\epsilon > 0$. Then $\epsilon/k > 0$. Since

$$\lim_{x\to w} f_j(x) = z_j$$

there is a $\delta_j > 0$ such that if

$$0 < |x - w| < \delta_j$$

then

$$|f_j(x) - z_j| < \frac{\epsilon}{k}.$$

Let

$$\delta = \min(\delta_1, \ldots, \delta_k).$$

Then $\delta > 0$. Also, if

$$0 < |x - w| < \delta$$

then

$$0 < |x - w| < \delta_j$$

for $1 \le j \le k$ and so

$$|f_j(x) - z_j| < \frac{\epsilon}{k}.$$

Starting from

$$|a + b| \le |a| + |b|$$

we can easily show by induction that

$$\left|\sum_{j=1}^{k} a_j\right| \le \sum_{j=1}^{k} |a_j|.$$

So

$$\left|\sum_{j=1}^{k} (f_j(x) - z_j)\right| < \sum_{j=1}^{k} \frac{\epsilon}{k} = \epsilon.$$

By the associativity and commutativity of addition this is equivalent to

$$\left|\sum_{j=1}^{k} f_j(x) - \sum_{j=1}^{k} z_j\right| < \epsilon.$$

We've just constructed a $\delta > 0$ such that if

$$0 < |x - w| < \delta$$

then

$$\left|\sum_{j=1}^{k} f_j(x) - \sum_{j=1}^{k} z_j\right| < \epsilon.$$

This shows that

$$\lim_{x\to w} \sum_{j=1}^{k} f_j(x) = \sum_{j=1}^{k} z_j.$$

□

I won't generally give this level of detail in proofs. This could be abbreviated considerably, with gaps left for you to fill in. The reason that I've given this level of detail in this case is to make it clear which properties of the real numbers and the absolute value function are being used, and therefore which are not. That's what has to guide us in generalising both the definitions and the theorems.

1.3 Limits of vector valued functions of a vector variable

The easiest generalisation is to higher dimensions.

Definition 1.3.1. If $\mathbf{f}\colon \mathbf{R}^m \to \mathbf{R}^n$ is a function, $\mathbf{w} \in \mathbf{R}^m$ and $\mathbf{z} \in \mathbf{R}^n$ then we say that $\mathbf{z}$ is the *limit* of $\mathbf{f}$ at $\mathbf{w}$, written

$$\lim_{\mathbf{x}\to\mathbf{w}} \mathbf{f}(\mathbf{x}) = \mathbf{z},$$

if for all $\epsilon > 0$ there is a $\delta > 0$ such that whenever $0 < \|\mathbf{x} - \mathbf{w}\| < \delta$ we have $\|\mathbf{f}(\mathbf{x}) - \mathbf{z}\| < \epsilon$.

Theorem 1.3.2. *There is at most one* $\mathbf{z}$ *such that* $\lim_{\mathbf{x}\to\mathbf{w}} \mathbf{f}(\mathbf{x}) = \mathbf{z}$*.*

Proof. Suppose on the contrary that

$$\lim_{\mathbf{x}\to\mathbf{w}} \mathbf{f}(\mathbf{x}) = \mathbf{z}_1,$$

$$\lim_{\mathbf{x}\to\mathbf{w}} \mathbf{f}(\mathbf{x}) = \mathbf{z}_2$$

and

$$\mathbf{z}_1 \neq \mathbf{z}_2.$$

Then set

$$\epsilon = \|\mathbf{z}_1 - \mathbf{z}_2\|/2.$$

Clearly $\epsilon > 0$, so by the definition of the limit there are $\delta_1 > 0$ and $\delta_2 > 0$ such that whenever

$$0 < \|\mathbf{x} - \mathbf{w}\| < \delta_1$$

and

$$0 < \|\mathbf{x} - \mathbf{w}\| < \delta_2$$

we have

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{z}_1\| < \epsilon$$

and

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{z}_2\| < \epsilon.$$

There is an $\mathbf{x}$ which satisfies both

$$0 < \|\mathbf{x} - \mathbf{w}\| < \delta_1$$

and

$$0 < \|\mathbf{x} - \mathbf{w}\| < \delta_2.$$

For example,

$$\mathbf{x} = \mathbf{w} + \frac{\min(\delta_1, \delta_2)}{2}\mathbf{u}$$

works, where $\mathbf{u} = (1, 0, \ldots, 0)$. We therefore have

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{z}_1\| < \epsilon$$

and

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{z}_2\| < \epsilon.$$

An elementary property of the length function on vectors is that if $\mathbf{a} \in \mathbf{R}^n$ and $\mathbf{b} \in \mathbf{R}^n$ then

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|.$$

Taking $\mathbf{a} = \mathbf{f}(\mathbf{x}) - \mathbf{z}_1$ and $\mathbf{b} = \mathbf{z}_2 - \mathbf{f}(\mathbf{x})$ we find that

$$\|\mathbf{z}_2 - \mathbf{z}_1| \leq \|\mathbf{f}(\mathbf{x}) - \mathbf{z}_1\| + \|(-1)(\mathbf{f}(\mathbf{x}) - \mathbf{z}_1)\|.$$

A further property of the length function is that

$$\|\alpha\mathbf{b}\| = |\alpha|\|\mathbf{b}\|$$

for all $\alpha \in \mathbf{R}$ and $\mathbf{b} \in \mathbf{R}^n$. Using this,

$$\|(-1)(\mathbf{f}(\mathbf{x}) - \mathbf{z}_1)\| = \|\mathbf{f}(\mathbf{x}) - \mathbf{z}_1\|.$$

So

$$\|\mathbf{z}_2 - \mathbf{z}_1\| \leq \|\mathbf{f}(\mathbf{x}) - \mathbf{z}_1\| + \|(\mathbf{f}(\mathbf{x}) - \mathbf{z}_1)\| < \epsilon + \epsilon.$$

But ϵ was chosen such that

$$\epsilon + \epsilon = \|\mathbf{z}_2 - \mathbf{z}_1\|.$$

There is no real number which is less than itself, so we have a contradiction. Our assumption that there are distinct $\mathbf{z}_1$ and $\mathbf{z}_2$ such that $\lim_{\mathbf{x}\to\mathbf{w}} \mathbf{f}(\mathbf{x}) = \mathbf{z}_1$ and $\lim_{x\to w} \mathbf{f}(\mathbf{x}) = \mathbf{z}_2$ must therefore be false. $\square$

Theorem 1.3.3. *Suppose that* $\mathbf{f}_1, \ldots, \mathbf{f}_k$ *are functions from* $\mathbf{R}^m$ *to* $\mathbf{R}^n$ *such that*

$$\lim_{\mathbf{x}\to\mathbf{w}} \mathbf{f}_j(\mathbf{x}) = \mathbf{z}_j$$

for $j = 1, \ldots, k$*. Then*

$$\lim_{\mathbf{x}\to\mathbf{w}} \sum_{j=1}^{k} \mathbf{f}_j(\mathbf{x}) = \sum_{j=1}^{k} \mathbf{z}_j.$$

Proof. Let $\epsilon > 0$. Then $\epsilon/k > 0$. Since

$$\lim_{\mathbf{z}\to\mathbf{w}} \mathbf{f}_j(\mathbf{z}) = \mathbf{z}_j$$

there is a $\delta_j > 0$ such that if

$$0 < \|\mathbf{x} - \mathbf{w}\| < \delta_j$$

then

$$\|\mathbf{f}_j(\mathbf{x}) - \mathbf{z}_j\| < \frac{\epsilon}{k}.$$

Let

$$\delta = \min(\delta_1, \ldots, \delta_k).$$

Then $\delta > 0$. Also, if

$$0 < \|\mathbf{x} - \mathbf{w}\| < \delta$$

then

$$0 < \|\mathbf{x} - \mathbf{w}\| < \delta_j$$

and so

$$\|\mathbf{f}_j(x) - \mathbf{z}_j\| < \frac{\epsilon}{k}.$$

Starting from

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$$

we can easily show by induction that

$$\left\|\sum_{j=1}^{k} \mathbf{a}_j\right\| \leq \sum_{j=1}^{k} \|\mathbf{a}_j\|.$$

So

$$\left\|\sum_{j=1}^{k} (\mathbf{f}_j(\mathbf{x}) - \mathbf{z}_j)\right\| < \sum_{j=1}^{k} \frac{\epsilon}{k} = \epsilon.$$

By the associativity and commutativity of addition this is equivalent to

$$\left\|\sum_{j=1}^{k} \mathbf{f}_j(\mathbf{x}) - \sum_{j=1}^{k} \mathbf{z}_j\right\| < \epsilon.$$

We've just constructed a $\delta > 0$ such that if

$$0 < \|\mathbf{x} - \mathbf{w}\| < \delta$$

then

$$\left\|\sum_{j=1}^{k} \mathbf{f}_j(\mathbf{x}) - \sum_{j=1}^{k} \mathbf{z}_j\right\| < \epsilon.$$

This shows that

$$\lim_{\mathbf{x} \to \mathbf{w}} \sum_{j=1}^{k} \mathbf{f}_j(\mathbf{x}) = \sum_{j=1}^{k} \mathbf{z}_j.$$

□

The notation here is that boldface is used for vectors and for vector valued functions. The length of a vector is denoted with double bars, e.g. $\|\mathbf{a}\|$. It's defined by

$$\|\mathbf{a}\| = \sqrt{\sum_{j=1}^{n} a_j^2}$$

for $\mathbf{a} \in \mathbf{R}^n$. Not all of our vectors are in $\mathbf{R}^n$ though. For vectors in $\mathbf{R}^m$ we need to replace the n by an m in the definition above. Since we aren't assuming that $m = n$ we have in fact two different length functions, although they're so closely related that the usual notation doesn't bother to distinguish between them.

The definition and proofs above are of course obtained from the definition and proofs in the case of real valued functions of a real variable by replacing real numbers by vectors and absolute values by lengths. This isn't quite a mechanical substitution though. Some of the real numbers remain real numbers and a small number of absolute values remain absolute values. We also need to distinguish between vectors in $\mathbf{R}^m$ and vectors in $\mathbf{R}^n$. Furthermore, not every property of the real numbers has a vector analogue. For example, the real numbers have a natural order relation. Vectors don't. There are quite a few inequalities above, so it's rather fortunate that all of them involve real quantities and none involve vector quantities and that the properties which we need to apply to vectors, like associativity, do apply to them. There are one or two places where the proofs require minor modifications to avoid objects which are not well defined. As an example, in the original proof we construct an x such that $0 < |x - w| < \delta$ by adding $\min(\delta_1, \delta_2)/2$ to w. In the vector context we can't add $\min(\delta_1, \delta_2)/2$ to $\mathbf{w}$, since there's no notion of adding scalars to vectors, so we multiply the scalar quantity by a vector of length 1 to get an addition which is well defined and accomplishes the same thing. The typographic convention of writing scalars in italics and vectors in bold helps to highlight instances where mechanical substitution yields nonsensical expressions. I will eventually drop this practice, but not yet.

It's rather tedious to check that proofs generalise and it's very tempting to skip over details, but it's also rather dangerous. It's particularly easy to miss problems which arise in trivial cases. Theorem 1.3.2 provides an example. There is something wrong with it. You might want to take a moment to try to figure out what the problem is before reading on. In case you don't see the problem I'll explain it after the next paragraph.

Theorems 1.3.2 and 1.3.3 are generalisations of Theorems 1.2.2 and 1.2.3, not just analogues of them. By this I mean two things. First of all, the real results are special cases of the vector ones, specifically the special cases where $m = n = 1$. The length in this case is the absolute value and so the statements of both the definition and the theorems in the vector case imply those in the real case. Secondly, the vector result covers cases that the real case does not. That's rather obvious here since there are integers greater than 1, but for some later generalisations it will be less clear whether a seemingly more general result is genuinely more general.

Did you spot the problem with Theorem 1.3.2? Nothing is specified about the dimensions m and n, so the natural assumption is that any meaningful values are allowed. $\mathbf{R}^m$ and $\mathbf{R}^n$ make sense for all non-negative integer values of m and n. There's no problem with $n = 0$, although the theorem isn't very interesting in that case. For $m = 0$ we have a problem though. The vector $\mathbf{u}$ was defined to have a 1 as its first entry and zeroes as all the other entries. If $m = 0$ then there is no first entry though. If you take $\mathbf{u}$ to be the zero vector, and there aren't any other vectors in $\mathbf{R}^0$ to choose, then you won't have $0 < \|\mathbf{x} - \mathbf{w}\| < \delta_1$ and $0 < \|\mathbf{x} - \mathbf{w}\| < \delta_2$. In fact there's no $\mathbf{x} \in \mathbf{R}^0$ satisfying those inequalities. And it's not just the proof which has a problem if $m = 0$. The statement of the theorem is false as well, as long as $n > 0$. To see this, choose any distinct $\mathbf{z}_1$ and $\mathbf{z}_2$ in $\mathbf{R}^n$. Then

$$\lim_{\mathbf{x} \to \mathbf{w}} \mathbf{f}(\mathbf{x}) = \mathbf{z}_1$$

and

$$\lim_{\mathbf{x} \to \mathbf{w}} \mathbf{f}(\mathbf{x}) = \mathbf{z}_2.$$

This is a simple, if somewhat surprising, consequence of Definition 1.3.1. To show it we need to find a δ for each ϵ. $\delta = 1$ works fine for each epsilon. For $j = 1$ or $j = 2$ we have

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{z}_j\| < \epsilon$$

for all $\mathbf{x} \in \mathbf{R}^0$ such that

$$0 < \|\mathbf{x} - \mathbf{w}\| < \delta.$$

This is true trivially, because there are no such $\mathbf{x}$ and so every statement holds for all such $\mathbf{x}$.

This problem arose from carelessness, or at least simulated carelessness, but also from a somewhat badly chosen definition. The restriction $0 < \|\mathbf{x} - \mathbf{w}\|$ in Definition 1.3.1 mirrors the restriction $0 < |x - w|$ in Definition 1.2.1. The theory of limits, even in the special case of real valued functions of a real variable, would be simpler without it. There are a number of theorems, like the one on limits of compositions of functions, which require extra hypotheses because this restriction in some sense allows too many functions to have limits. There are historical reasons why this is the standard definition. The main one is that in the definition of the derivative as a limit of difference quotients we need to evaluate the limit of a function at a point where it's not defined. A simple solution to that problem was to phrase the definition of the limit in a way which ignored any possible value of the function at the point where the limit was to be evaluated. But simple solutions aren't always good solutions. There are better, if slightly more complicated, ways of coping with limits of functions at points where they're not defined, which we will see soon. It's too late to change this particular definition but it's useful as a cautionary tale about the dangers of not choosing one's definitions carefully so as not to cause unnecessary problems later on.

To salvage Theorem 1.3.2 we need to add the hypothesis that $m > 0$ if $n > 0$. There will be other similar hypotheses needed in later theorems to avoid problems with trivial cases. In some theorems, like this one, the problem could have been avoided by a better choice of definitions but in others it simply reflects the fact that trivial cases are sometimes just different from non-trivial cases. In the remainder of these notes such hypotheses will be written in explicitly though.

1.4 Norms and normed vector spaces

The properties of $\mathbf{R}^m$ and $\mathbf{R}^n$ that we used above were related to addition and scalar multiplication and also certain properties of the length function. We can use that observation to generalise a bit further. For the moment I'll defer the question of why one would want to.

We start by defining a norm.

Definition 1.4.1. If V is a vector space then we say that $p\colon V \to \mathbf{R}$ is a *norm* on V if it has the following three properties:

(a) For all $\mathbf{v} \in V$, $p(\mathbf{x}) \geq 0$ and $p(\mathbf{v}) > 0$ unless $\mathbf{v} = \mathbf{0}$.

(b) For all $\alpha \in \mathbf{R}$ and $\mathbf{v} \in V$, $p(\alpha\mathbf{v}) = |\alpha| p(\mathbf{v})$.

(c) For all $\mathbf{v}, \mathbf{w} \in V$, $p(\mathbf{v} + \mathbf{w}) \leq p(\mathbf{v}) + p(\mathbf{w})$.

A pair (V, p) consisting of a vector space V and a norm p on V is called a *normed vector space*.

Of course the function which assigns to the vector $\mathbf{v}$ in $\mathbf{R}^n$ its length $\|\mathbf{v}\|$ is a norm. It's also possible, and useful, to define norms for complex vector spaces, but that won't concern us in these notes.

Some easy consequences of the definition are given in the following lemma.

Lemma 1.4.2. *Suppose p is a norm on V.*

(a) $p(\mathbf{0}) = 0$.

(b) If $\mathbf{v} \in V$ then $p(-\mathbf{v}) = p(\mathbf{v})$.

(c) If $\mathbf{v}, \mathbf{w} \in V$ then $p(\mathbf{v} - \mathbf{w}) = p(\mathbf{w} - \mathbf{v})$.

(d) If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ then

$$p(\mathbf{u} - \mathbf{w}) \leq p(\mathbf{u} - \mathbf{v}) + p(\mathbf{v} - \mathbf{w}).$$

(e) If $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ then

$$p\left(\sum_{j=1}^{k} \mathbf{v}_j\right) \leq \sum_{j=1}^{k} p(\mathbf{v}_j).$$

Proof. 1.4.2a is just 1.4.1b with $\alpha = 0$. Similarly, 1.4.2b is just 1.4.1b with $\alpha = -1$. 1.4.2c is 1.4.2b applied with $\mathbf{v} - \mathbf{w}$ in place of $\mathbf{v}$. 1.4.2d is 1.4.1c with $\mathbf{u} - \mathbf{v}$ and $\mathbf{v} - \mathbf{w}$ in place of $\mathbf{v}$ and $\mathbf{w}$. 1.4.2e is proved by induction on k. The base case, with $k = 0$, is just 1.4.2a, since we follow the usual convention that a sum with no terms is equal to zero. For the inductive step we assume

$$p\left(\sum_{j=1}^{k} \mathbf{v}_j\right) \leq \sum_{j=1}^{k} p(\mathbf{v}_j).$$

and apply 1.4.1c with $\sum_{j=1}^{k} \mathbf{v}_j$ and $\mathbf{v}_{k+1}$ in place of $\mathbf{v}$ and $\mathbf{w}$ to get

$$p\left(\sum_{j=1}^{k+1} \mathbf{v}_j\right) \leq p\left(\sum_{j=1}^{k} \mathbf{v}_j\right) + p\left(\mathbf{v}_{k+1}\right)$$

It follows that

$$p\left(\sum_{j=1}^{k+1} \mathbf{v}_j\right) \leq \sum_{j=1}^{k} p\left(\mathbf{v}_j\right) + p\left(\mathbf{v}_{k+1}\right) = \sum_{j=1}^{k+1} p\left(\mathbf{v}_j\right).$$

This is the same statement we assumed, but with $k + 1$ in place of k, which completes the inductive step. □

The reason for starting the induction at $k = 0$ rather than $k = 2$ is just to make sure that the result holds for $k = 0$ and $k = 1$, without needing to supply the admittedly trivial proofs in those special cases. In general it's always best to start inductions from the most trivial case which makes sense.

Some examples of norms are

$$q(\mathbf{x}) = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}$$

on $\mathbf{R}^n$ for $p \geq 1$ and

$$q(f) = \left(\int_a^b |f(t)|^p \, dt\right)^{1/p}$$

on the vector space of continuous real valued functions on the interval $[a, b]$.

1.5 Limits of functions between normed vector spaces

Definition 1.5.1. Suppose that (X,p) and (Y,q) are normed vector spaces. If $\mathbf{f}\colon X\to Y$ is a function, $\mathbf{w}\in X$ and $\mathbf{z}\in Y$ then we say that $\mathbf{z}$ is the *limit* of $\mathbf{f}$ at $\mathbf{w}$, written

$$\lim_{\mathbf{x}\to\mathbf{w}}\mathbf{f}(\mathbf{x})=\mathbf{z},$$

if for all $\epsilon>0$ there is a $\delta>0$ such that whenever $0<p\left(\mathbf{x}-\mathbf{w}\right)<\delta$ we have $q\left(\mathbf{f}(\mathbf{x})-\mathbf{z}\right)<\epsilon$.

Theorem 1.5.2. *Suppose that (X,p) and (Y,q) are normed vector spaces and that X is not the zero vector space. If $\mathbf{f}\colon X\to Y$ is a function and $\mathbf{w}\in X$ then there is at most one $\mathbf{z}\in Y$ such that $\lim_{\mathbf{x}\to\mathbf{w}}\mathbf{f}(\mathbf{x})=\mathbf{z}$.*

Proof. Suppose on the contrary that

$$\lim_{\mathbf{x}\to\mathbf{w}}\mathbf{f}(\mathbf{x})=\mathbf{z}_1,$$

$$\lim_{\mathbf{x}\to\mathbf{w}}\mathbf{f}(\mathbf{x})=\mathbf{z}_2$$

and

$$\mathbf{z}_1\neq\mathbf{z}_2.$$

Then set

$$\epsilon=q\left(\mathbf{z}_1-\mathbf{z}_2\right)/2.$$

Clearly $\epsilon>0$, so by the definition of the limit there are $\delta_1>0$ and $\delta_2>0$ such that whenever

$$0<p\left(\mathbf{x}-\mathbf{w}\right)<\delta_1$$

and

$$0<p\left(\mathbf{x}-\mathbf{w}\right)<\delta_2$$

we have

$$q\left(\mathbf{f}(\mathbf{x})-\mathbf{z}_1\right)<\epsilon$$

and

$$q\left(\mathbf{f}(\mathbf{x})-\mathbf{z}_2\right)<\epsilon.$$

There is an $\mathbf{x}$ which satisfies both

$$0<p\left(\mathbf{x}-\mathbf{w}\right)<\delta_1$$

and

$$0<p\left(\mathbf{x}-\mathbf{w}\right)<\delta_2.$$

To find such an $\mathbf{x}$ first note that X is not the zero vector space so there is a $\mathbf{v}\neq\mathbf{0}$ in X and set

$$\mathbf{u}=\frac{1}{p(\mathbf{v})}\mathbf{v}.$$

It follows from 1.4.1a that this is well defined and it follows from 1.4.1b that $p(\mathbf{u})=1$. Then set

$$\mathbf{x}=\mathbf{w}+\frac{\min(\delta_1,\delta_2)}{2}\mathbf{u}.$$

Another application of 1.4.1b shows that $q(\mathbf{x}-\mathbf{w})=\frac{\min(\delta_1,\delta_2)}{2}$. We therefore have

$$q\left(\mathbf{f}(\mathbf{x})-\mathbf{z}_1\right)<\epsilon$$

and

$$q\left(\mathbf{f}(\mathbf{x})-\mathbf{z}_2\right)<\epsilon.$$

By 1.4.2d,

$$q\left(\mathbf{z}_2-\mathbf{z}_1\right)\leq q\left(\mathbf{f}(\mathbf{x})-\mathbf{z}_1\right)+q\left(\mathbf{f}(\mathbf{x})-\mathbf{z}_1\right).$$

But ϵ was chosen such that

$$\epsilon+\epsilon=q\left(\mathbf{z}_2-\mathbf{z}_1\right).$$

There is no real number which is less than itself, so we have a contradiction. Our assumption that there are distinct $\mathbf{z}_1$ and $\mathbf{z}_2$ such that $\lim_{\mathbf{x}\to\mathbf{w}}\mathbf{f}(\mathbf{x})=\mathbf{z}_1$ and $\lim_{\mathbf{x}\to\mathbf{w}}\mathbf{f}(x)=\mathbf{z}_2$ must therefore be false. □

Theorem 1.5.3. *Suppose that $\mathbf{f}_1,\ldots,\mathbf{f}_k$ are functions from X to Y such that*

$$\lim_{\mathbf{x}\to\mathbf{w}}\mathbf{f}_j(\mathbf{x})=\mathbf{z}_j$$

for $j=1,\ldots,k$. Then

$$\lim_{\mathbf{x}\to\mathbf{w}}\sum_{j=1}^{k}\mathbf{f}_j(\mathbf{x})=\sum_{j=1}^{k}\mathbf{z}_j.$$

Proof. Let $\epsilon>0$. Then $\epsilon/k>0$. Since $\lim_{\mathbf{z}\to\mathbf{w}}\mathbf{f}_j(\mathbf{z})=\mathbf{z}_j$ there is a $\delta_j>0$ such that if

$$0<p\left(\mathbf{x}-\mathbf{w}\right)<\delta_j$$

then

$$q\left(\mathbf{f}_j(\mathbf{x})-\mathbf{z}_j\right)<\frac{\epsilon}{k}.$$

Let

$$\delta = \min(\delta_1, \ldots, \delta_k).$$

Then $\delta > 0$. Also, if

$$0 < p\,(\mathbf{x} - \mathbf{w}) < \delta$$

then

$$0 < p\,(\mathbf{x} - \mathbf{w}) < \delta_j$$

and so

$$q\,(\mathbf{f}_j(x) - \mathbf{z}_j) < \frac{\epsilon}{k}.$$

By 1.4.2e,

$$q\left(\sum_{j=1}^{k} (\mathbf{f}_j(\mathbf{x}) - \mathbf{z}_j)\right) \leq \sum_{j=1}^{k} q\,(\mathbf{f}_j(\mathbf{x}) - \mathbf{z}_j) < \sum_{j=1}^{k} \frac{\epsilon}{k}.$$

By the associativity and commutativity of addition this is equivalent to

$$q\left(\sum_{j=1}^{k} \mathbf{f}_j(\mathbf{x}) - \sum_{j=1}^{k} \mathbf{z}_j\right) < \sum_{j=1}^{k} \frac{\epsilon}{k} = \epsilon.$$

We've just constructed a $\delta > 0$ such that if

$$0 < p(\mathbf{x} - \mathbf{w}) < \delta$$

then

$$q\left(\sum_{j=1}^{k} \mathbf{f}_j(\mathbf{x}) - \sum_{j=1}^{k} \mathbf{z}_j\right) < \epsilon.$$

This shows that

$$\lim_{\mathbf{x} \to \mathbf{w}} \sum_{j=1}^{k} \mathbf{f}_j(\mathbf{x}) = \sum_{j=1}^{k} \mathbf{z}_j.$$

□

These proofs were, of course, largely constructed by taking the proofs from the previous subsection and replacing the length function everywhere either by p or by q as appropriate. Parts of the proofs from that section have moved to the proof of Lemma 1.4.2 however. In addition, the construction of $\mathbf{u}$ has been modified slightly, since there's no longer a particular vector whose norm we know to be 1. This adds a small complication to the proof, but it makes it harder not to notice the restriction that X should be non-trivial, since we'd otherwise be dividing by zero.

There's a problem with the proof of Theorem 1.5.3 given above, which I'll explain after the next paragraph. It's not a new problem. The same problem affects Theorems 1.2.3 and 1.3.3. The theorems are all correct as stated, but their proofs need modification. You might want to take a moment to see if you can identify the problem and find an appropriate modification.

Theorems 1.5.2 and 1.5.3 are generalisations of Theorems 1.3.2 and 1.3.3, not just analogues of them. Again, this means two things. First of all, the results for vector spaces where the norm is the Euclidean norm, i.e. the length function are a special cases of the one ones for general norms, since the length function is a norm. Secondly, the results for normed vector spaces cover cases other than just the Euclidean case. This is less obvious than when we moved from the real valued case to the vector valued case, since it's not immediately obvious that there are norms on a vector space other than the Euclidean norm. We will see later that there are, but we will also see that for finite dimensional vector spaces the increase in generality is more apparent than real.

Did you identify the problem with the proof of Theorem 1.5.3? We've learned to be careful of trivial cases, but it can be quite difficult sometimes to spot when an argument fails in a trivial case. The proof of Theorem 1.5.3 given above fails in the case $k = 0$ for the very simple reason that we divided ϵ by k. The proofs of Theorem 1.2.3 and Theorem 1.3.3 had the same problem, of course, although I didn't mention it in either of the two previous sections. Fortunately the theorems remain true if $k = 0$, although they only say that the limit at any point of the function which is zero everywhere is zero. The proofs require modification however if we want to include this case. There are two ways of accomplishing this. One is to treat the trivial case separately. It's not difficult, but it is distracting. Also, treating trivial cases separately is a bad habit to get into. The reason is that in more complicated theorems the number of trivial cases can be prohibitively large. If, for example, a theorem refers to a set of functions $\mathbf{f}_1, \ldots, \mathbf{f}_m$ from a set S to a vector space V then we might have $m = 0$ or $S = \varnothing$ or $\dim(V) = 0$ or various combinations of these. And this is a relatively simple example. It's usually better if possible to modify proofs slightly so that the trivial cases no longer cause difficulties. In this case the simplest fix is to replace the words

> Let $\epsilon > 0$. Then $\epsilon/k > 0$. Since $\lim_{\mathbf{z}\to\mathbf{w}} \mathbf{f}_j(\mathbf{z}) = \mathbf{z}_j$ there is a $\delta_j > 0$ such that if
> $$0 < p\left(\mathbf{x} - \mathbf{w}\right) < \delta_j$$
> then
> $$q\left(\mathbf{f}_j(\mathbf{x}) - \mathbf{z}_j\right) < \frac{\epsilon}{k}.$$

with

> Let $\epsilon > 0$. Then $\epsilon/(k+1) > 0$. Since $\lim_{\mathbf{z}\to\mathbf{w}} \mathbf{f}_j(\mathbf{z}) = \mathbf{z}_j$ there is a $\delta_j > 0$ such that if
> $$0 < p\left(\mathbf{x} - \mathbf{w}\right) < \delta_j$$
> then
> $$q\left(\mathbf{f}_j(\mathbf{x}) - \mathbf{z}_j\right) < \frac{\epsilon}{k+1}.$$

There's no longer a division by zero. Subsequent references to $\frac{\epsilon}{k}$ must of course also be replaced by $\frac{\epsilon}{k+1}$ and the "$\sum_{j=1}^{k} \frac{\epsilon}{k} = \epsilon$" becomes "$\sum_{j=1}^{k} \frac{\epsilon}{k+1} < \epsilon$", but everything works. The only disadvantage to this approach is that the problem we're carefully avoiding becomes so invisible that the reader may not even realise it's there.

1.6 Metrics and metric spaces

Our next generalisation is based on the observation that we used the length function, and then more generally a norm, to measure distances between points. We don't really need a vector space. Any set with a suitable notion of distance will work just as well, at least as the domain of our function. For the theorem about the limit of a sum of functions we still need to be able to sum functions, which means we will still need some sort of additive structure. The properties we need for a suitable notion of distance are captured in the definition of a metric.

Definition 1.6.1. If S is a set then we say that $d\colon S \times S \to \mathbf{R}$ is a *metric* on S if it has the following three properties.

(a) If $a, b \in S$ then $d(a, b) \geq 0$ and $d(a, b) = 0$ if and only if $a = b$.

(b) If $a, b \in S$ then $d(a, b) = d(b, a)$.

(c) If $a, b, c \in S$ then $d(a, c) \leq d(a, b) + d(b, c)$.

A pair (S, d) consisting of a set S and a metric d on S is called a *metric space*.

One source of metrics is norms on vector spaces.

Theorem 1.6.2. *If p is a norm on a vector space V then $d\colon V \times V$, defined by*

$$d(\mathbf{u}, \mathbf{v}) = p(\mathbf{u} - \mathbf{v}),$$

is a metric on V.

Proof. If $\mathbf{u}, \mathbf{v} \in V$ then $d(\mathbf{u}, \mathbf{v}) = p(\mathbf{u} - \mathbf{v}) \geq 0$ by 1.4.1a. Also $d(\mathbf{u}, \mathbf{v}) > 0$ unless $p(\mathbf{u} - \mathbf{v}) > 0$, which happens only if $\mathbf{u} - \mathbf{v} = 0$, i.e. only if $\mathbf{u} = \mathbf{v}$, again by 1.4.1a. This establishes 1.6.1a. If $\mathbf{u}, \mathbf{v} \in V$ then

$$d(\mathbf{u}, \mathbf{v}) = p(\mathbf{u} - \mathbf{v}) = p(\mathbf{v} - \mathbf{u}) = d(\mathbf{v}, \mathbf{u})$$

The middle equation is 1.4.2c. This establishes 1.6.1b. Finally, if $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ then

$$\begin{aligned} d(\mathbf{u}, \mathbf{w}) &= p(\mathbf{u} - \mathbf{w}) \\ &\leq p(\mathbf{u} - \mathbf{v}) + p(\mathbf{v} - \mathbf{w}) \\ &= d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w}). \end{aligned}$$

The middle inequality is 1.4.2d. This establishes 1.6.1c. □

Another way to get a metric space is to take a subset of a metric space we already have.

Lemma 1.6.3. *If $S \subseteq T$ and d is a metric on T then the restriction of d from $T \times T$ to $S \times S$ is a metric on S.*

Proof. For simplicity I'll use d to refer both to the original function from $T \times T$ to $\mathbf{R}$ and the restricted function from $S \times S$ to $\mathbf{R}$. This can't really cause any trouble. When either of the arguments of d is not an element of S the original function must be meant. When both arguments of d are elements of S we can interpret it either as the original function or the restriction. The value of the function at that point is the same under either interpretation though, so no ambiguity arises.

If $a, b \in S$ then $a, b \in T$. d is a metric on T and so, by 1.6.1a, $d(a, b) \geq 0$ and $d(a, b) > 0$ unless $a = b$. This establishes the property 1.6.1a for d as a metric on S. Similarly, if $a, b \in S$ then $a, b \in T$. d is a metric on T and so, by 1.6.1b, $d(a, b) = d(b, a)$. This establishes the property 1.6.1b for d on S. If $a, b, c \in S$ then $a, b, c \in T$. d is a metric on T and so, by 1.6.1c, $d(a, c) \leq d(a, b) + d(b, c)$. This establishes the property 1.6.1c for d on S. $\square$

We now have a variety of metric spaces. Any subset of the plane taken together with the Euclidean distance function is a metric space. This follows from the preceding theorem and lemma, together with the fact that the Euclidean norm is a norm.

There are other ways to get metric spaces though.

Example 1.6.4. If F is a finite set. We'll denote the *power set* of F, i.e. the set of all its subsets, by $\wp(F)$. Define $d\colon \wp(F) \times \wp(F) \to \mathbf{R}$ by

$$d(A, B) = \#(A \triangle B)$$

where $\#$ denotes *cardinality*, i.e. the number of elements, and $\triangle$ denotes *symmetric difference*, i.e.

$$A \triangle B = A \cup B \setminus A \cap B.$$

Then d is a metric on $\wp(F)$. It is called the *Hamming distance*.

This is fairly straightforward to verify. Cardinalities are always non-negative. $d(A, B) = 0$ if and only if $A \triangle B$ has cardinality 0, i.e. if and only if $A \triangle B$ is empty, which happens if and only if $A \cup B = A \cap B$ and hence if and only if $A = B$. This establishes 1.6.1a. 1.6.1b is easier. $A \cup B = B \cup A$ and $A \cap B = B \cap A$ so $A \triangle B = B \triangle A$ and hence $d(A, B) = d(B, A)$. For 1.6.1c we note that

$$A \triangle C = (A \triangle B) \triangle (B \triangle C)$$

It then follows from the definition of $\triangle$ that

$$A \triangle C \subseteq (A \triangle B) \cup (B \triangle C).$$

The cardinality of a subset is at most the cardinality of the containing set and the cardinality of a union is at most the sum of the cardinalities, so

$$d(A, C) \leq d(A, B) + d(B, C).$$

Another interesting class of examples comes from number theory.

Example 1.6.5. Suppose p is a prime number. As usual, denote the set of integers by $\mathbf{Z}$. For any integers m and n, set $d_p(m, n) = 0$ if $m = n$ and otherwise set $d_p(m, n) = p^{-k}$ where p^k is the highest power of p which divides $m - n$. Then d_p is a metric on $\mathbf{Z}$. It is called the *p-adic metric*.

Again the verification that this is a metric is fairly straightforward. $p^k > 0$ for all p and k so $d_p(m, n) \geq 0$ and $d_p(m, n) > 0$ unless $m = n$. That establishes 1.6.1a. $m = n$ if and only if $n = m$ and the highest power of p which divides $m - n$ is the same as the highest power which divides $n - m$. It follows that $d_p(m, n) = d_p(n, m)$, establishing 1.6.1b. As usual, 1.6.1c requires more work. Let p^{k_1} be the highest power of p dividing $l - n$, p^{k_2} be the highest power dividing $l - m$ and p^{k_3} be the highest dividing $m - n$. If p^k divides both $l - m$ and $m - n$ then it also divides

$$l - n = (l - m) + (m - n).$$

Applying this to $k = \min(k_2, k_3)$ gives the relation

$$k_1 \geq \min(k_2, k_3).$$

From this it follows that

$$p^{-k_1} \leq \max\left(p^{-k_2}, p^{-k_3}\right).$$

In other words,

$$d_p(l, n) \leq \max\left(d_p(l, m), d_p(m, n)\right).$$

The sum of two non-negative numbers is always greater than or equal to their maximum, so

$$d_p(l, n) \leq d_p(l, m) + d_p(m, n),$$

establishing 1.6.1c.

Of course $\mathbf{Z}$ already has an entirely different metric, namely $d(x,y) = |x-y|$. This is sometimes referred to as the *usual metric* when it's necessary to distinguish it from the p-adic metrics, but more often one just refers to it as "the metric" on $\mathbf{Z}$, when there's no reason to believe that any other metric is meant. As we'll see, the p-adic metrics behave quite differently from the usual metric.

As with norms there are a number of useful properties of metrics which don't form part of the definition but which follow easily from it.

Lemma 1.6.6. *Suppose d is a metric on S.*

(a) If $a \in S$ then $d(a,a) = 0$.

(b) If $a, b, c \in S$ then

$$d(a,c) \geq |d(a,b) - d(b,c)|.$$

(c) If $a_0, a_1, \ldots, a_k \in S$ then

$$d(a_0, a_k) \leq \sum_{j=1}^{k} d(a_{k-1}, a_k).$$

Proof. By 1.6.1c,

$$d(a,b) \leq d(a,c) + d(c,b).$$

By 1.6.1b,

$$d(c,b) = d(b,c).$$

Combining those,

$$d(a,b) \leq d(a,c) + d(b,c),$$

and hence

$$d(a,c) \geq d(a,b) - d(b,c).$$

Similarly,

$$d(b,c) \leq d(b,a) + d(a,c)$$

and

$$d(b,a) = d(a,b)$$

so

$$d(a,c) \geq d(b,c) - d(a,b).$$

Therefore

$$d(a,c) \geq \max(d(a,b) - d(b,c), d(b,c) - d(b,a)).$$

The right hand side is just $|d(a,b) - d(b,c)|$, establishing 1.6.6b. 1.6.6c is then proved by induction on k, using $d(a,a) = 0$ for the base case, $k = 0$, and 1.6.6b for the inductive step. □

1.7 Limits of functions between metric spaces

We're now able to state the metric space analogues of Definition 1.5.1 and Theorems 1.5.2 and 1.5.3. As was mentioned earlier, we will eventually need to evaluate limits at points which are not in the domain of definition of our function, and this seems as good a time as any to make the necessary changes in order to do that.

Definition 1.7.1. Suppose that (X, d_X) and (Y, d_Y) are metric spaces. If $f \colon U \to Y$ is a function defined on a subset $U \subseteq X$, $w \in X$ and $z \in Y$ then we say that z is the *limit* of f at w, written

$$\lim_{x \to w} f(x) = z,$$

if for all $x \in U$ and $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < d_X(x, w) < \delta$ then $d_Y(f(x), z) < \epsilon$.

Note that we don't require $w \in U$, but neither do we prohibit it. Similarly, U is allowed to be a proper subset of X, but not required to be. The case $U = X$ is in fact the case we're usually interested in.

Also, none of the variables are in boldface any more. They might belong to a vector space, since subsets of normed vector spaces are metric spaces, but they don't have to be and the notation shouldn't imply that they are.

We've already seen that some further hypothesis will be required to obtain uniqueness, since $\mathbf{R}^0$ would otherwise be a counter-example. The problem there was that there were no $\mathbf{x}$ such that $0 < p(\mathbf{x} - \mathbf{w}) < \delta$. Now that we've generalised from normed vector spaces to metric spaces there are many more ways for this to happen. Consider the Hamming distance on the power set of a finite set, from Example 1.6.4, for example. $d(A,B)$ is the cardinality of the finite set $A \triangle B$ and so is necessarily a non-negative integer. If $\delta \leq 1$ then there will never be a $B \in \wp(F)$ such that $0 < d(A,B) < \delta$. The same thing can happen if U is a proper subset of a normed vector space of positive dimension, rather than the whole normed vector space. The simplest option is to take the condition we need to make the proof work and turn it into a definition.

Definition 1.7.2. A point w in a metric space (X,d) is called a *limit point* of $U \subseteq X$ if for every $\delta > 0$ there is a $x \in U$ with

$$0 < d(w,x) < \delta.$$

Theorem 1.7.3. *Suppose that (X, d_X) and (Y, d_Y) are metric spaces and that $U \subseteq X$. If $f \colon U \to Y$ is a function and $w \in X$ is a limit point of U then there is at most one $z \in Y$ such that $\lim_{x \to w} f(x) = z$.*

Proof. Suppose on the contrary that

$$\lim_{x \to w} f(x) = z_1,$$

$$\lim_{x \to w} f(x) = z_2$$

and

$$z_1 \neq z_2.$$

Then set

$$\epsilon = d_Y\left(z_1, z_2\right)/2.$$

Clearly $\epsilon > 0$, so by the definition of the limit there are $\delta_1 > 0$ and $\delta_2 > 0$ such that whenever $x \in U$ and

$$0 < d_X\left(x, w\right) < \delta_1$$

and

$$0 < d_X\left(x, w\right) < \delta_2$$

we have

$$d_Y\left(f(x), z_1\right) < \epsilon$$

and

$$d_Y\left(f(x), z_2\right) < \epsilon.$$

By assumption w is a limit point of U so there are $x_1 \in U$ and $x_2 \in U$ such that

$$0 < d_Y\left(x_1, w\right) < \delta_1$$

and

$$0 < d_Y\left(x_2, w\right) < \delta_2.$$

Take $\delta = \min(\delta_1, \delta_2)$ and take x to be either x_1 or x_2 according to whether δ_1 or δ_2 is smaller. If $\delta_1 = \delta_2$ then either choice is fine. Then $x \in U$, $0 < d_X(x,w) < \delta_1$ and $0 < d_X(x,w) < \delta_2$ and so

$$d_Y\left(f(x), z_1\right) < \epsilon$$

and

$$d_Y\left(f(x), z_2\right) < \epsilon.$$

By 1.6.1c,

$$d_Y\left(z_2, z_1\right) \leq d_Y\left(f(x), z_1\right) + d_Y\left(f(x), z_1\right).$$

But ϵ was chosen such that

$$\epsilon + \epsilon = d_Y\left(z_2, z_1\right).$$

There is no real number which is less than itself, so we have a contradiction. Our assumption that there are distinct z_1 and z_2 such that $\lim_{x \to w} f(x) = z_1$ and $\lim_{x \to w} f(x) = z_2$ must therefore be false. $\square$

Theorem 1.7.4. *Suppose that (X, d_X) is a metric space and (Y, q) is a normed vector space and that $U \subseteq X$. Let $d_Y(\mathbf{a}, \mathbf{b}) = q(\mathbf{a} - \mathbf{b})$, which, by Theorem 1.6.2, is a metric on Y. Suppose that $\mathbf{f}_1, \ldots, \mathbf{f}_k$ are functions from U to Y such that*

$$\lim_{x \to w} \mathbf{f}_j(x) = \mathbf{z}_j$$

for $j = 1, \ldots, k$. Then

$$\lim_{x \to w} \sum_{j=1}^{k} \mathbf{f}_j(x) = \sum_{j=1}^{k} \mathbf{z}_j.$$

Proof. Let $\epsilon > 0$. Then $\epsilon/(k+1) > 0$. Since

$$\lim_{x \to w} \mathbf{f}_j(x) = \mathbf{z}_j$$

there is a $\delta_j > 0$ such that if

$$0 < d_X\left(x, w\right) < \delta_j$$

then

$$d_Y\left(\mathbf{f}_j(x), \mathbf{z}_j\right) < \frac{\epsilon}{k+1}.$$

Let

$$\delta = \min(\delta_1, \dots, \delta_k).$$

Then $\delta > 0$. Also, if

$$0 < d_X(x, w) < \delta$$

then

$$0 < d_X(x, w) < \delta_j$$

and so

$$d_Y(\mathbf{f}_j(x), \mathbf{z}_j) < \frac{\epsilon}{k+1}.$$

In view of the definition of d_Y,

$$q(\mathbf{f}_j(x) - \mathbf{z}_j) < \frac{\epsilon}{k+1}.$$

By 1.4.2e,

$$\begin{aligned} q\left(\sum_{j=1}^{k}(\mathbf{f}_j(x) - \mathbf{z}_j)\right) &\le \sum_{j=1}^{k} q(\mathbf{f}_j(x) - \mathbf{z}_j) \\ &< \sum_{j=1}^{k} \frac{\epsilon}{k+1}. \end{aligned}$$

By the associativity and commutativity of addition this is equivalent to

$$q\left(\sum_{j=1}^{k}\mathbf{f}_j(x) - \sum_{j=1}^{k}\mathbf{z}_j\right) < \sum_{j=1}^{k} \frac{\epsilon}{k+1} < \epsilon.$$

Using the definition of d_Y again,

$$d_Y\left(\sum_{j=1}^{k}\mathbf{f}_j(x), \sum_{j=1}^{k}\mathbf{z}_j\right) < \epsilon.$$

We've just constructed a $\delta > 0$ such that if $x \in U$ and

$$0 < d_X(x, w) < \delta$$

then

$$d_Y\left(\sum_{j=1}^{k}\mathbf{f}_j(x), \sum_{j=1}^{k}\mathbf{z}_j\right) < \epsilon.$$

so

$$\lim_{x \to w} \sum_{j=1}^{k}\mathbf{f}_j(x) = \sum_{j=1}^{k}\mathbf{z}_j.$$

□

Note that Y was still required to have a norm, not just a metric, and to be a vector space, not just a set. This was done because we need to add and substract elements of Y, which doesn't make sense in an arbitrary set. We could still have generalised a bit further, since scalar multiplication is not needed, and allowed Y to be an abelian group and d_Y to be a translation-invariant metric. That's sometimes a useful generalisation, but among the various generalisations we could make it's a low priority.

Theorems 1.7.3 and 1.7.4 are generalisations rather than analogues of Theorems 1.5.2 and 1.5.3, in a sense which should by now be familiar. The former include the latter as special cases, but also cover cases which the latter do not.

1.8 Open and closed balls

Still staying within the context of metric spaces, there are various ways we can usefully reformulate the results of the previous section. For these we'll need some more definitions.

Definition 1.8.1. If (X, d) is a metric space, $r > 0$ and $w \in X$ then the *open ball* of radius r about w is the set

$$B(w, r) = \{x \in X : d(x, w) < r\}.$$

The *closed ball* of radius r about w is the set

$$\bar{B}(w, r) = \{x \in X : d(x, w) \le r\}.$$

I've deliberately not defined balls of zero or negative radius. The word "ball" therefore means the same thing as "ball of positive radius." Note that the notation specifies the centre w and the radius r but not the ambient space X or the metric d. These must be understood from context.

We can describe limit points in terms of open balls.

Proposition 1.8.2. *A point w in a metric space (X, d) is a limit point of $U \subseteq X$ if for every $\delta > 0$ there is an $x \in U$ such that $U \cap B(w, \delta) \setminus \{w\}$ is non-empty.*

Proof.

$$x \in U \cap B(w, \delta) \setminus \{w\}$$

if and only if $x \in U$, $d(w,x) < \delta$ and $x \neq w$. The last of these conditions is equivalent to $d(w,x) > 0$. So

$$x \in U \cap B(w,\delta) \setminus \{w\}$$

if and only $x \in U$ and $0 < d(w,x) < \delta$ and so $U \cap B(w,\delta) \setminus \{w\}$ is non-empty if and only if there is an $x \in U$ such that $0 < d(w,x) < \delta$. □

Similarly, we can describe limits in terms of open balls.

Proposition 1.8.3. *Suppose that (X, d_X) and (Y, d_Y) are metric spaces. If $f \colon U \to Y$ is a function defined on a subset $U \subseteq X$, $w \in X$ and $z \in Y$ then*

$$\lim_{x \to w} f(x) = z$$

if and only if for all $\epsilon > 0$ there is a $\delta > 0$ such that if $x \in U \cap B(w,\delta) \setminus \{w\}$ then $f(x) \in B(z,\epsilon)$.

Proof. This is just Definition 1.7.1 with the statement "$x \in U$ and $0 < d_X(x,w) < \delta$" replaced by "$x \in U \cap B(w,\delta) \setminus \{w\}$" and the statement "$d_Y(f(x),z) < \epsilon$" replaced by $f(x) \in B(z,\epsilon)$, which are clearly equivalent in view of the definitions of $B(w,\delta)$ and $B(z,\epsilon)$. □

The balls in $\mathbf{R}^m$ are the usual balls, so

$$B(\mathbf{x}, r) = \{\mathbf{y} \in \mathbf{R}^m \colon \|\mathbf{x} - \mathbf{y}\| < r\}$$

and

$$\bar{B}(\mathbf{x}, r) = \{\mathbf{y} \in \mathbf{R}^m \colon \|\mathbf{x} - \mathbf{y}\| \leq r\}.$$

Note that mathematical usage, unlike ordinary usage, maintains a clear distinction between balls and spheres. The sphere of radius r would be the set

$$\{\mathbf{y} \in \mathbf{R}^m \colon \|\mathbf{x} - \mathbf{y}\| = r\}.$$

In $\mathbf{R}^m$ the open and closed balls are always distinct and balls of strictly smaller radius are proper subsets of balls of larger radius, but there are metric spaces where that's not true. For the Hamming metric both the open and closed balls of any radius in the interval $(0,1)$ about a subset A are just $\{A\}$. The open ball of radius 1 is also just $\{A\}$ while the closed ball of radius 1 also includes sets which differ from A by inserting or removing a single element.

1.9 Images and preimages

We now need the notions of image a preimage of a set under a function, which is best stated in terms of power sets, which we already met in the context of the Hamming distance.

Definition 1.9.1. The *power set* of a set X is the set of all subsets of X. It is denoted $\wp(X)$.

Note that all subsets means all subsets. They needn't be proper and could be empty.

Definition 1.9.2. If $\varphi \colon X \to Y$ is a function then $\varphi^* \colon \wp(Y) \to \wp(X)$, defined by

$$\varphi^*(V) = \{x \in X \colon \varphi(x) \in V\},$$

is called the *preimage* function of φ. The set $\varphi^*(V)$ is often called the preimage of V under φ.

Definition 1.9.3. If $\varphi \colon X \to Y$ is a function then $\varphi_* \colon \wp(X) \to \wp(Y)$, defined by

$$\varphi_*(U) = \{y \in Y \colon \exists x \in U \colon \varphi(x) = y\},$$

is called the *image* function of φ. The set $\varphi_*(V)$ is often called the image of U under φ.

A more common notation is $\varphi(U)$ for the image and $\varphi^{-1}(V)$ for the preimage. There are a few problems with that notation though. One is that $\varphi^{-1}(V)$ suggests the image of V under the inverse function φ. In one sense that's okay since it is equal to the image of V under the inverse function φ if φ has an inverse, but the preimage makes sense even when φ is not a bijection. So there's no ambiguity, but it is easy to see φ^{-1} and think, incorrectly, that φ must have an inverse. A more subtle problem arises for functions between sets of sets. Consider, for example the infimum and supremum functions on the set of subsets of $[0,1]$. In other words, $\inf(A)$ for $A \subseteq [0,1]$ is the largest element of $[0,1]$ which is a lower bound for A and $\sup(A)$ is the smallest element of $[0,1]$ which is an upper bound for it. With these definitions

$$\inf(\varnothing) = 1,$$

$$\inf_*(\varnothing) = \varnothing,$$

$$\sup(\varnothing) = 0$$

and

$$\sup_*(\varnothing) = \varnothing.$$

In the more common notation $\inf_*(\varnothing)$ would be denoted $\inf(\varnothing)$ and $\sup_*(\varnothing)$ would be denoted $\sup(\varnothing)$. It would then appear, from the transitivity of the $=$ sign, that $0 = 1$. Of course $0 \neq 1$ and it is simply ambiguous notation which makes it appear otherwise.

The basic properties of the preimage and image are summarised in the following lemma.

Lemma 1.9.4. *If $\varphi\colon X \to Y$ and $\psi\colon T \to X$ are functions, $A, B \in \wp(X)$, $C, D \in \wp(Y)$, $\mathcal{E} \in \wp(\wp(X))$ and $\mathcal{F} \in \wp(\wp(Y))$ then*

(a) $\varphi_*(\varnothing) = \varnothing$,

(b) $\varphi^*(\varnothing) = \varnothing$,

(c) $\varphi_*(X) \subseteq Y$,

(d) $\varphi^*(Y) = X$,

(e) $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$,

(f) $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$,

(g) *if* $A \subseteq B$ *then* $\varphi_*(A) \subseteq \varphi_*(B)$,

(h) *if* $C \subseteq D$ *then* $\varphi^*(A) \subseteq \varphi^*(B)$,

(i) $\varphi_*(A \cup B) = \varphi_*(A) \cup \varphi_*(B)$,

(j) $\varphi^*(C \cup D) = \varphi^*(C) \cup \varphi^*(D)$,

(k) $\varphi_*(A \cap B) \subseteq \varphi_*(A) \cap \varphi_*(B)$,

(l) $\varphi^*(C \cap D) = \varphi^*(C) \cap \varphi^*(D)$,

(m) $\varphi_*(A \setminus B) \supseteq \varphi_*(A) \setminus \varphi_*(B)$,

(n) $\varphi^*(C \setminus D) = \varphi^*(C) \setminus \varphi^*(D)$,

(o) $\varphi_*\left(\bigcup_{V \in \mathcal{E}} V\right) = \bigcup_{V \in \mathcal{E}} \varphi_*(V)$,

(p) $\varphi^*\left(\bigcup_{W \in \mathcal{F}} W\right) = \bigcup_{W \in \mathcal{F}} \varphi^*(W)$,

(q) $\varphi_*\left(\bigcap_{V \in \mathcal{E}} V\right) \subseteq \bigcap_{V \in \mathcal{E}} \varphi_*(V)$,

(r) $\varphi^*\left(\bigcap_{W \in \mathcal{F}} W\right) = \bigcap_{W \in \mathcal{F}} \varphi^*(W)$.

Note that the properties of the preimage are generally better than the properties of the image, in that we often have equations in place of inclusions.

Proof. Although there are a lot of statements to be checked each is relatively straightforward.

(a) There no $x \in \varnothing$ and hence there is no $y \in Y$ which is $\varphi(x)$ for such an x.

(b) There are no $x \in X$ such that $\varphi(x) \in \varnothing$.

(c) $\varphi(x) \in Y$ for all $x \in X$.

(d) $\varphi(x) \in Y$ for all $x \in X$.

(e) If $y \in (\varphi \circ \psi)_*(U)$ then there is a $t \in U$ such that

$$(\varphi \circ \psi)(t) = \varphi(\psi(t)) = y.$$

Clearly $\psi(t) \in \psi_*(U)$. Also, there is an $x \in \psi_*(U)$ such that $\varphi(x) = y$, namely $x = \psi(t)$, so $y \in \varphi_*(\psi_*(U))$. Conversely, suppose

$$y \in (\varphi_* \circ \psi_*)(U) = \varphi_*(\psi_*(U)).$$

Then there is an $x \in \psi_*(U)$ such that $y = \varphi(x)$. Because $x \in \psi_*(U)$ there must be a $t \in U$ such that $x = \psi(t)$. Then $y = (\varphi \circ \psi)(t)$, so $y \in (\varphi \circ \psi)_*(U)$. In other words, $y \in (\varphi \circ \psi)_*(U)$ if and only if $y \in \varphi_*(\psi_*(U))$. It follows that

$$(\varphi \circ \psi)_*(U) = (\varphi_* \circ \psi_*)(U).$$

This holds for any $U \in \wp(T)$, so

$$(\varphi \circ \psi)_* = \varphi_* \circ \psi_*.$$

(f)

$$t \in (\varphi \circ \psi)^*(W)$$

if and only if

$$\varphi(\psi(t)) = (\varphi \circ \psi)(t) \in W,$$

which happens if and only if

$$\psi(t) \in \varphi^*(W),$$

i.e. if and only if

$$t \in \psi^*(\varphi^*(W)).$$

In other words, $t \in (\varphi \circ \psi)^*(W)$ if and only if $t \in (\psi^* \circ \varphi^*)(W)$. It follows that

$$(\varphi \circ \psi)^*(W) = (\psi^* \circ \varphi^*)(W)$$

This holds for any $W \in \wp(Y)$, so

$$(\varphi \circ \psi)^* = \psi^* \circ \varphi^*.$$

(g) If $A \subseteq B$ and $y \in \varphi_*(A)$ then there's an $x \in A$ such that $\varphi(x) = y$. But then $x \in B$ and so $y \in \varphi_*(B)$. Every element of $\varphi_*(A)$ is therefore an element of $\varphi_*(B)$. This implies $\varphi_*(A) \subseteq \varphi_*(B)$.

(h) If $C \subseteq D$ and $x \in \varphi^*(C)$ then $\varphi(x) \in C$, so $\varphi(x) \in D$ and hence $x \in \varphi^*(D)$. Every element of $\varphi^*(C)$ is therefore an element of $\varphi^*(D)$. This implies $\varphi^*(C) \subseteq \varphi^*(D)$.

(i) This will follow from 1.9.4o with $\mathcal{E} = \{A, B\}$.

(j) This will follow from 1.9.4p with $\mathcal{F} = \{C, D\}$.

(k) This will follow from 1.9.4q with $\mathcal{E} = \{A, B\}$.

(l) This will follow from 1.9.4r with $\mathcal{F} = \{C, D\}$.

(m) Suppose $y \in \varphi_*(A) \setminus \varphi_*(B)$. Then there is an $x \in A$ such that $\varphi(x) = y$. This x is not in B because otherwise we would have $y \in \varphi_*(B)$. So $x \in A \setminus B$ and hence $y \in \varphi_*(A \setminus B)$.

(n) $x \in \varphi^*(C \setminus D)$ if and only if $\varphi(x) \in C \setminus D$, i.e. if and only $\varphi(x) \in C$ and $\varphi(x) \notin D$. $\varphi(x) \in C$ if and only if $x \in \varphi^*(C)$. $\varphi(x) \notin D$ if and only if $x \notin \varphi^*(D)$. Together, those show that $x \in \varphi^*(C \setminus D)$ if and only if $x \in \varphi^*(C)$ and $x \notin \varphi^*(D)$, i.e. if and only if $x \in \varphi^*(C) \setminus \varphi^*(D)$.

(o) $y \in \varphi_*\left(\bigcup_{V \in \mathcal{E}} V\right)$ if and only if there is an $x \in \bigcup_{V \in \mathcal{E}} V$ such that $\varphi(x) = y$, which happens if and only if there is a $V \in \mathcal{E}$ and an $x \in V$ such that $\varphi(x) = y$. If so then $y \in \varphi_*(V)$ for this V and hence $y \in \bigcup_{V \in \mathcal{E}} \varphi_*(V)$. Conversely, if $y \in \bigcup_{V \in \mathcal{E}} \varphi_*(V)$ then $y \in \varphi_*(V)$ for some $V \in \mathcal{E}$, i.e. there is a $V \in \mathcal{E}$ and an $x \in V$ such that $\varphi(x) = y$.

(p) $x \in \varphi^*\left(\bigcup_{W \in \mathcal{F}} W\right)$ if and only if $\varphi(x) \in \bigcup_{W \in \mathcal{F}} W$, i.e. if and only if there is a $W \in \mathcal{F}$ such that $\varphi \in W$, which happens if and only if there is a $W \in \mathcal{F}$ such that $x \in \varphi^*(W)$, i.e. if and only if $x \in \bigcup_{W \in \mathcal{F}} \varphi^*(W)$.

(q) If $y \in \varphi_*\left(\bigcap_{V \in \mathcal{E}} V\right)$ then there an $x \in \bigcap_{V \in \mathcal{E}} V$ such that $\varphi(x) = y$. Then $x \in V$ for each $V \in \mathcal{E}$, so $y \in \varphi_*(V)$ for each $V \in \mathcal{E}$. But this implies $y \in \bigcap_{V \in \mathcal{E}} \varphi_*(V)$.

(r) $x \in \varphi^*\left(\bigcap_{W \in \mathcal{F}} W\right)$ if and only if $\varphi(x) \in \bigcap_{W \in \mathcal{F}} W$, i.e. if and only if $\varphi(x) \in W$ for each $W \in \mathcal{F}$. This happens if and only if $x \in \varphi^*(W)$ for each $W \in \mathcal{F}$, i.e. if and only if $\bigcap_{W \in \mathcal{F}} \varphi^*(W)$.

□

We can use preimages to give an alternate characterisation of limits.

Proposition 1.9.5. *Suppose that (X, d_X) and (Y, d_Y) are metric spaces. If $f \colon U \to Y$ is a function defined on a subset $U \subseteq X$, $w \in X$ and $z \in Y$ then*

$$\lim_{x \to w} f(x) = z$$

if and only if for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$U \cap B(w, \delta) \setminus \{w\} \subseteq f^*(B(z, \epsilon)).$$

Proof. In view of the definition of the preimage

$$x \in f^*(B(z, \epsilon))$$

if and only if $f(x) \in B(z, \epsilon)$, so this is equivalent to our previous characterisation of limits. □

1.10 Open and closed sets

Definition 1.10.1. A subset S of a metric space (X, d) is called *open* if whenever $x \in X$ there is an $r > 0$ such that $B(x, r) \subseteq S$. A subset is said to be *closed* if $X \setminus S$ is open.

Note that "closed" is not the opposite of "open", as one might naively expect. A subset can be both open and closed. In fact $\varnothing$ and X are always both open and closed subsets of X. There may or may not be non-empty proper subsets which are both open and closed. Also, subsets can be neither open nor closed. $[0,1) \subseteq \mathbf{R}$ is an example. It is not open because $0 \in [0,1)$ but there is no open ball about 0 which is a subset of $[0,1)$. It is not closed because 1 is in $\mathbf{R} \setminus [0,1)$ but there is no open ball about 1 which is a subset of $\mathbf{R} \setminus [0,1)$.

Also note that the definition of closed sets is not obtained by taking the definition of open sets and replacing open balls by closed balls. In fact, if you take the definition of open sets and replace open balls by closed balls what you get is just an alternate characterisation of open balls.

Proposition 1.10.2. *A subset S of a metric space (X,d) is open if and only if whenever $x \in X$ there is a closed ball centred at x which is contained in S.*

Proof. Suppose $S \subseteq X$ satisfies the condition above and $x \in X$. Then there is some $r > 0$ such that $\bar{B}(x,r) \subseteq S$. But $B(x,r) \subseteq \bar{B}(x,r)$ so $B(x,r) \subseteq S$. So S is open. Suppose, conversely that S is open. Then there is some $r > 0$ such that $B(x,r) \subseteq S$. But $\bar{B}(x,r/2) \subseteq B(x,r)$ so $\bar{B}(x,r/2) \subseteq S$. Thus S satisfies the condition above. □

Open balls will appear much more often in definitions than closed balls. Sometimes there are reasons why open balls are easier to work with but often it's just a matter of convention, as in the choice to define open sets in terms of open balls rather than closed balls.

The most important properties of open sets are the following.

Theorem 1.10.3. *Suppose (X,d) is a metric space.*

(a) $\varnothing$ and X are open subsets of X

(b) If V and W are open subsets of X then $V \cap W$ is an open subset of X.

(c) If $\mathcal{E}$ is a set of open subsets of X then $\bigcup_{V\in\mathcal{E}} V$ is an open subset of X.

(d) If $x \in X$, $y \in X$ and $x \neq y$ then there are open subsets V and W of X such that $x \in V$, $y \in W$ and $V \cap W = \varnothing$.

Proof. We check each statement in turn.

(a) There are no points in $\varnothing$ so the statement that each point in $\varnothing$ is contained in an open ball is vacuously true. So $\varnothing$ is an open subset. Every open ball about every point in X is contained in X. In particular, the ball of radius 1 about any point is in X, so there is at least one such ball about every point in X and hence X is open.

(b) If $x \in V \cap W$ then $x \in V$. Since V is an open subset there is an $s > 0$ such that $B(x,s) \subseteq V$. Similarly there is a $t > 0$ such that $B(x,t) \subseteq W$. Let $r = \min(s,t)$. Then $r > 0$, $B(x,r) \subseteq B(x,s) \subseteq V$, and $B(x,r) \subseteq B(x,t) \subseteq W$, so $B(x,r) \subseteq V \cap W$. So for any $x \in V \cap W$ there is an $r > 0$ such that $B(x,r) \subseteq V \cap W$. In other words, $V \cap W$ is an open subset.

(c) If $x \in \bigcup_{V\in\mathcal{E}} V$ then there is some $V \in \mathcal{E}$ such that $x \in V$. This V, like all elements of $\mathcal{E}$, is an open subset, so there is an $r > 0$ such that $B(x,r) \subseteq V$. But then $B(x,r) \subseteq \bigcup_{V\in\mathcal{E}} V$, so $\bigcup_{V\in\mathcal{E}} V$ is an open subset.

(d) Define
$$V = \{v \in X \colon d(x,v) < d(y,v)\}$$
and
$$W = \{w \in X \colon d(x,w) > d(y,w)\}.$$
If $z \in V \cap W$ then $d(x,z) < d(y,z)$ and $d(x,z) > d(y,z)$. This is impossible so $V \cap W = \varnothing$. If $v \in V$ then $B(v,r) \subseteq V$ where
$$r = \frac{d(y,v) - d(x,v)}{2}.$$
To see this, note that if $z \in B(v,r)$ then
$$d(v,z) < \frac{d(y,v) - d(x,v)}{2}.$$
By the definition of a metric
$$d(x,z) \leq d(x,v) + d(v,z).$$
By Lemma 1.6.6
$$d(y,z) \geq d(y,v) - d(v,z).$$

Combining these gives the inequality

$$d(x,z) < d(y,z).$$

So $z \in V$. This is true for every $z \in B(v,r)$ so $B(v,r) \subseteq V$. We've just seen that for every $z \in V$ there is an $r > 0$ such that $B(v,r) \subseteq V$. In other words, V is open. Similarly, if $z \in W$ then $B(z,r) \subseteq W$ with

$$r = \frac{d(x,v) - d(y,v)}{2}.$$

so W is open.

□

The corresponding statement for closed sets is that X and $\varnothing$ are closed sets, the union of two closed sets is closed, the intersection of arbitrarily many closed sets is closed, and given any two distinct points there is a pair closed sets whose union is X such that each point is not in one of the sets.

1.11 Topologies

In order to generalise limits beyond metric spaces we turn the previous theorem into a definition, or rather into two definitions.

Definition 1.11.1. A *topology* on a set X is a $\mathcal{T} \in \wp(\wp(X))$ such that

(a) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$,

(b) If $V \in \mathcal{T}$ and $W \in \mathcal{T}$ then $V \cap W \in \mathcal{T}$.

(c) If $\mathcal{E} \subseteq \mathcal{T}$ then $\bigcup_{V \in \mathcal{E}} V \in \mathcal{T}$.

A pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X is called a *topological space*.

Note that the second property refers to the intersection of a pair of elements of $\mathcal{T}$, but that we can then easily obtain by induction that the intersection of any finite collection of elements of $\mathcal{T}$ belongs to $\mathcal{T}$.

Definition 1.11.2. A topology $\mathcal{T}$ on X is said to be *Hausdorff* if for every $x, y \in X$ such that $x \neq y$ there are $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $V \cap W = \varnothing$.

The preceding theorem shows that the set of open sets in a metric space is a topology. In view of this we can, and will, refer to the elements of any topology as open sets and refer to their complements as closed sets. The theorem also shows that the topology of open sets in a metric space is a Hausdorff topology. We could, of course, have defined topological spaces in terms of their closed sets rather than their open sets. It's usually easier to work with open sets, but the Zariski topology, discussed below, is easier to describe in terms of closed sets.

There is a possible conflict of terminology since we've defined open and closed balls as subsets of a metric space and then defined open and closed subsets. If open balls weren't open or closed balls weren't closed then we would quickly run into trouble. Fortunately that doesn't happen.

Proposition 1.11.3. *Open balls are open and closed balls are closed.*

Proof. If $x \in B(w,r)$ then let $s = r - d(w,x)$. By the definition of an open ball $s > 0$. If $y \in B(x,s)$ then $d(x,y) < s$ and hence

$$d(w,y) \leq d(w,x) + d(x,y) < d(w,x) + s = r,$$

so $y \in B(x,r)$. In other words, $B(x,s) \subseteq B(w,r)$. For every $x \in B(w,r)$ there is therefore an $s > 0$ such that $B(x,s) \subseteq B(w,r)$. In other words, $B(w,r)$ is an open subset.

If $x \notin \bar{B}(w,r)$ then let $s = d(w,x) - r$. By the definition of a closed ball $s > 0$. If $y \in B(x,s)$ then $d(x,y) < s$ and hence

$$d(w,y) \geq d(w,x) - d(x,y) > d(w,x) - s = r,$$

so $y \notin \bar{B}(x,r)$. In other words,

$$B(x,s) \subseteq X \setminus \bar{B}(w,r).$$

For every

$$x \in X \setminus \bar{B}(w,r)$$

there is therefore an $s > 0$ such that

$$B(x,s) \subseteq X \setminus \bar{B}(w,r).$$

In other words, $X \setminus \bar{B}(w,r)$ is an open subset. Therefore $\bar{B}(w,r)$ is a closed subset. □

We'll also need a definition of limit points in general topological spaces.

Definition 1.11.4. A point w in a topological space $(X, \mathcal{T})$ is called a *limit point* of $U \subseteq X$ if for every $W \in \mathcal{T}$ such that $w \in W$ the set $U \cap W \setminus \{w\}$ is non-empty.

We need to check however that this is consistent with the definition given previously for metric spaces.

Proposition 1.11.5. *If $\mathcal{T}$ is the topology of open sets in the metric space (X, d) then w is a limit point of U in the sense of the definition above if and only if it is a limit point in the sense of Definition 1.7.2.*

Proof. If w satisfies the definition above then we can apply it with $W = B(w, \delta)$ for each $\delta > 0$ to see that for all such δ the set $U \cap B(w, \delta) \setminus \{w\}$ is non-empty. By Proposition 1.8.2 w is a limit point of U in the sense of Definition 1.7.2

Conversely, suppose w is a limit point of U in the sense of Definition 1.7.2 If $w \in W$ and $W \in \mathcal{T}$ then there is a $\delta > 0$ such that $B(w, \delta) \subseteq W$, since that's how the topology corresponding to a metric was defined. By Proposition 1.8.2 the set $U \cap B(w, \delta) \setminus \{w\}$ is non-empty. But this is a subset of $U \cap W \setminus \{w\}$, which therefore is also non-empty. □

Do all topologies arise from metrics? There are two different ways we could interpret this question, although the answer will be no under both interpretations.

We could, first of all, ask whether it is possible to describe a topology on a set without referring to a metric on it. That's easy enough.

Definition 1.11.6. The *discrete topology* on a set X is just $\wp(X)$. The *trivial topology* is $\mathcal{T} = \{\varnothing, X\}$.

It's straightforward to check that each of these satisfies all the requirements for a topology. Neither of these topologies was described in terms of a metric but the discrete topology could have been described in terms of one. If we set $d(x, x) = 0$ and $d(x, y) = 1$ when $x \neq y$ then it's straightforward to check that d is a metric and that the topology defined by its open sets is the discrete topology. It is called the *discrete metric*. It's not necessarily the only metric which gives the discrete topology though. If F is a finite set then the Hamming distance is a metric on $\wp(F)$. This metric is not equal to the discrete metric on $\wp(F)$ when $\#F > 1$, but both metrics give rise to the discrete topology. A topology which is the set of open sets for some metric, even if it wasn't initially defined using that metric, is called *metrisable*.

At this point we return to the question of norms on finite dimensional vector spaces, which was considered earlier. There are norms on such a space which are not equal to the Euclidean norm. These norms give rise to metrics which are not the Euclidean metric. We will see later though that they all give the same topology. For infinite dimension spaces, by contrast, it is possible to find norms which give rise to distinct topologies.

Returning now to the question whether all topologies come from metrics we could also ask whether there are topologies which are not the set of open sets for any choice of metric, i.e. topologies which are not metrisable. In fact the trivial topology on any set with more than one element is non-metrisable. At first sight this might seem difficult to prove. It would certainly be very painful to enumerate all possible metrics and then to check that none of them have the trivial topology as their set of open sets. Fortunately there's a much easier approach. We've already seen that metrics give Hausdorff topologies. The discrete topology on a set with at least two points fails to be Hausdorff, and so can't be metrisable. Indeed, if $x, y \in X$ and $x \neq y$ then the only $V \in \mathcal{T}$ such that $x \in V$ is $V = X$ and the only $W \in \mathcal{T}$ such that $y \in V$ is $W = X$. But then $V \cap W = X \neq \varnothing$.

The trivial topology is, of course, a rather uninteresting example. A much more interesting example is the Zariski topology on $\mathbf{C}^n$.

Example 1.11.7. Let $\mathcal{Z} \in \wp(\wp(\mathbf{C}^n))$ be the set of subsets of $\mathbf{C}^n$ of the form

$$V = \{\mathbf{z} \in \mathbf{C}^n : p_1(\mathbf{z}) = p_2(\mathbf{z}) = \cdots = p_j(\mathbf{z}) = 0\}$$

for some finite set of polynomials $p_1, \ldots, p_j$. In other words, a set belongs to $\mathcal{Z}$ if and only if it is zero set of a finite set of polynomials. Let $\mathcal{T}$ be the set of complements of elements of $\mathcal{Z}$, i.e.

$$\mathcal{T} = \{U \in \wp(\mathbf{C}^n) : \mathbf{C}^n \setminus U \in \mathcal{Z}\}.$$

Then $\mathcal{T}$ is a topology on $\mathbf{C}^n$, called the *Zariski topology*.

To check that this is a topology we need to check that $\mathbf{C}^n$ and $\varnothing$ belong to $\mathcal{Z}$, that the union of any two elements of $\mathcal{Z}$ belongs to $\mathcal{Z}$ sets is closed and that the intersection of an arbitrary set of elements of $\mathcal{Z}$ belongs to $\mathcal{Z}$. All but the last of these is straightforward. $\mathbf{C}^n$ is the zero set of $\{0\}$. $\varnothing$ is the zero set of $\{1\}$. If V is the zero set of $\{p_1, \ldots, p_j\}$ and W is the zero set of $\{q_1, \ldots, q_k\}$ then $V \cup W$ is the zero set of $\{p_1q_1, p_1q_2, \ldots, p_jq_k\}$. For the statement about intersections one needs, however, the fact that any set given as the common zero set of an arbitrary collection of polynomials is in fact the common zero set of a finite collection. This is a theorem of Hilbert but is somewhat difficult to prove for $n > 1$. For $n = 1$ it is however very easy to prove since a proper subset of $\mathbf{C}$ is the zero set of a polynomial if and only if it is finite.

1.12 Topologies and limits

We're now in a position to define limits of functions between topological spaces.

Definition 1.12.1. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces. If $f : U \to Y$ is a function defined on a subset $U \subseteq X$, $w \in X$ and $z \in Y$ then we say that z is the *limit* of f at w, written

$$\lim_{x \to w} f(x) = z,$$

if for all $Z \in \mathcal{T}_Y$ such that $z \in Z$ there is a $W \in \mathcal{T}_X$ such that $w \in W$ and

$$U \cap W \setminus \{w\} \subseteq f^*(Z).$$

We should check that this definition is consistent with the definition given previously for metric spaces.

Proposition 1.12.2. *If (X, d_X) and (Y, d_Y) are metric spaces and $\mathcal{T}_X$ and $\mathcal{T}_Y$ are the topologies of open sets on X and Y with respect to the metrics d_X and d_Y respectively then $\lim_{x\to w} f(x) = z$ in the sense of the definition above if and only if $\lim_{x\to w} f(x) = z$ in the sense of Definition 1.7.1.*

Proof. Instead of using Definition 1.7.1 directly we use the condition that for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$U \cap B(w, \delta) \setminus \{w\} \subseteq f^*(B(z, \epsilon)),$$

which is equivalent to it by Proposition 1.9.5.

Suppose the condition from Proposition 1.9.5 holds and that $Z \in \mathcal{T}_Y$ is such that $z \in Z$. By the definition of open sets in a metric space there is then an $\epsilon > 0$ such that

$$B(z, \epsilon) \subseteq Z,$$

and hence

$$f^*(B(z, \epsilon)) \subseteq f^*(Z).$$

Proposition 1.9.5 then gives us a $\delta > 0$ such that

$$U \cap B(w, \delta) \setminus \{w\} \subseteq f^*(B(z, \epsilon)).$$

Let $W = B(w, \delta)$. By Proposition 1.10.2 W is open and so $W \in \mathcal{T}_X$. Also, $w \in W$. So for every $Z \in \mathcal{T}_Y$ such that $z \in Z$ there is a $W \in \mathcal{T}_X$ such that $w \in W$ and

$$U \cap W \setminus \{w\} \subseteq f^*(Z).$$

In other words, $\lim_{x\to w} f(x) = z$ in the sense of Definition 1.12.1.

Suppose, conversely, that $\lim_{x\to w} f(x) = z$ in the sense of Definition 1.12.1, i.e. that for every $Z \in \mathcal{T}_Y$ such that $z \in Z$ there is a $W \in \mathcal{T}_X$ such that $w \in W$ and

$$U \cap W \setminus \{w\} \subseteq f^*(Z).$$

For any $\epsilon > 0$ we have $z \in B(z, \epsilon)$ and , by Proposition 1.10.2, $B(z, \epsilon) \in \mathcal{T}_Y$, so there is a $W \in \mathcal{T}_X$ such that $w \in W$ and

$$U \cap W \setminus \{w\} \subseteq f^*(B(z, \epsilon)).$$

From the definition of open sets there is a $\delta > 0$ such that

$$B(w, \delta) \subseteq W.$$

But then

$$U \cap B(w, \delta) \setminus \{w\} \subseteq U \cap W \setminus \{w\}$$

so

$$U \cap B(w,\delta) \setminus \{w\} \subseteq f^*(B(z,\epsilon)).$$

Thus the criterion from Proposition 1.9.5 is satisfied and so In other words, $\lim_{x\to w} f(x) = z$ in the sense of Definition 1.7.1. ☐

We can now state and prove our limit theorems in the context of topological spaces.

Theorem 1.12.3. *Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and that $U \subseteq X$. If $f\colon U \to Y$ is a function, $w \in X$ is a limit point of U, and $\mathcal{T}_Y$ is Hausdorff then there is at most one $z \in Y$ such that $\lim_{x\to w} f(x) = z$.*

Proof. Suppose on the contrary that

$$\lim_{x\to w} f(x) = z_1,$$

$$\lim_{x\to w} f(x) = z_2$$

and

$$z_1 \neq z_2.$$

Because $\mathcal{T}_Y$ is Hausdorff there are $Z_1, Z_2 \in \mathcal{T}_Y$ such that $z_1 \in Z_1$, $z_2 \in Z_2$ and $Z_1 \cap Z_2 = \varnothing$. By the definition of the limit there are $W_1, W_2 \in \mathcal{T}_X$ such that $w \in W_1$, $w \in W_2$,

$$U \cap W_1 \setminus \{w\} \subseteq f^*(Z_1)$$

and

$$U \cap W_2 \setminus \{w\} \subseteq f^*(Z_2).$$

Let $W = W_1 \cap W_2$. Then $w \in W$ and $W \in \mathcal{T}_Y$. By assumption w is a limit point of U so $x \in U \cap W \setminus \{w\}$ is non-empty. But

$$U \cap W \setminus \{w\} \subseteq U \cap W_1 \setminus \{w\} \subseteq f^*(Z_1)$$

and

$$U \cap W \setminus \{w\} \subseteq U \cap W_2 \setminus \{w\} \subseteq f^*(Z_2).$$

so

$$U \cap W \setminus \{w\} \subseteq f^*(Z_1) \cap f^*(Z_2) \subseteq f^*(Z_1 \cap Z_2) = \varnothing.$$

So we have a non-empty subset of an empty set and we have a contradiction. Our assumption that there are distinct z_1 and z_2 such that $\lim_{x\to w} f(x) = z_1$ and $\lim_{x\to w} f(x) = z_2$ must therefore be false. ☐

The ideas of the proof are the same as those in the proof of Theorem 1.7.3, but very little of the actual text has survived.

Note that the new theorem requires an extra hypothesis, that the topological space $(Y, \mathcal{T}_Y)$ is Hausdorff, but this hypothesis is automatically satisfied when $\mathcal{T}_Y$ is the topology of open sets in a metric space, so this is a genuine generalisation of our earlier theorem.

Theorem 1.12.4. *Suppose that $(X, \mathcal{T}_X)$ is a metric space and (Y, q) is a normed vector space and that $U \subseteq X$. Let $d_Y(\mathbf{a}, \mathbf{b}) = q(\mathbf{a} - \mathbf{b})$, which, by Theorem 1.6.2, is a metric on Y and let $\mathcal{T}_Y$ be the topology of open sets for the metric d_Y. Suppose that $\mathbf{f}_1$, ..., $\mathbf{f}_k$ are functions from U to Y such that*

$$\lim_{x\to w} \mathbf{f}_j(x) = \mathbf{z}_j$$

for $j = 1, \ldots, k$. Then

$$\lim_{x\to w} \sum_{j=1}^{k} \mathbf{f}_j(x) = \sum_{j=1}^{k} \mathbf{z}_j.$$

Proof. Suppose $Z \in \mathcal{T}_Y$ and $\sum_{j=1}^{k} \mathbf{z}_j \in Z$. Then there is an $\epsilon > 0$ such that

$$B\left(\sum_{j=1}^{k} \mathbf{z}_j, \epsilon\right) \subseteq Z.$$

Then $\epsilon/(k+1) > 0$. Since $\lim_{\mathbf{x}\to w} \mathbf{f}_j(\mathbf{z}) = \mathbf{z}_j$ there is a $W_j \in \mathcal{T}_X$ such that $w \in W_j$ and

$$\mathbf{f}_j^*(B(\mathbf{z}_j, \epsilon/(k+1)) \subseteq U \cap W_j \setminus \{w\}.$$

Let

$$W = \bigcap_{j=1}^{k} W_j.$$

Then $w \in W$ and $W \in \mathcal{T}_X$. If $x \in U \cap W \setminus \{w\}$ then $x \in U \cap W_j \setminus \{w\}$ for each j and hence $\mathbf{f}_j(x) \in B(\mathbf{z}_j, \epsilon/(k+1)$. In other words,

$$d_Y\left(\mathbf{f}_j(x), \mathbf{z}_j\right) < \frac{\epsilon}{k+1}.$$

In view of the definition of d_Y,

$$q\left(\mathbf{f}_j(x) - \mathbf{z}_j\right) < \frac{\epsilon}{k+1}$$

and so

$$q\left(\sum_{j=1}^{k}\left(\mathbf{f}_j(x)-\mathbf{z}_j\right)\right) \le \sum_{j=1}^{k} q\left(\mathbf{f}_j(x)-\mathbf{z}_j\right) < \sum_{j=1}^{k}\frac{\epsilon}{k+1}.$$

By the associativity and commutativity of addition this is equivalent to

$$q\left(\sum_{j=1}^{k}\mathbf{f}_j(x)-\sum_{j=1}^{k}\mathbf{z}_j\right) < \sum_{j=1}^{k}\frac{\epsilon}{k+1} < \epsilon.$$

Using the definition of d_Y again,

$$d_Y\left(\sum_{j=1}^{k}\mathbf{f}_j(x), \sum_{j=1}^{k}\mathbf{z}_j\right) < \epsilon.$$

In other words,

$$\sum_{j=1}^{k}\mathbf{f}_j(x) \in B\left(\sum_{j=1}^{k}\mathbf{z}_j, \epsilon\right).$$

From this it follows that

$$\sum_{j=1}^{k}\mathbf{f}_j(x) \in Z$$

and hence

$$x \in \left(\sum_{j=1}^{k}\mathbf{f}_j\right)^{*}(Z).$$

$x \in U \cap W \setminus \{w\}$ was arbitrary, so

$$U \cap W \setminus \{w\} \subseteq \left(\sum_{j=1}^{k}\mathbf{f}_j\right)^{*}(Z).$$

For every $Z \in \mathcal{T}_X$ such that $\sum_{j=1}^{k}\mathbf{z}_j \in Z$ we thus have a $W \in \mathcal{T}_X$ such that $w \in W$ and

$$U \cap W \setminus \{w\} \subseteq \left(\sum_{j=1}^{k}\mathbf{f}_j\right)^{*}(Z).$$

In other words,

$$\lim_{x\to w}\sum_{j=1}^{k}\mathbf{f}_j(x) = \sum_{j=1}^{k}\mathbf{z}_j.$$

□

1.13 Neighbourhoods

We can state some of the results above slightly more cleanly in terms of neighbourhoods.

Definition 1.13.1. If $(X, \mathcal{T})$ is a topological space, $x \in X$ and $V \in \wp(X)$ then V is said to be a *neighbourhood* of x if there is a $U \in \mathcal{T}$ such that $x \in U$ and $U \subseteq V$. It is called an *open neighbourhood* if, in addition, $V \in \mathcal{T}$. The set of neighbourhoods of x is denoted by $\mathcal{N}(x)$ and the set of open neighbourhoods of x is denoted by $\mathcal{O}(x)$.

Just as the notation for balls doesn't indicate the underlying metric space, which must therefore be understood from context, the notation for sets of neighbourhoods doesn't indicate the underlying topological space.

As you would expect, an open neighbourhood is simply a neighbourhood which is open. Unfortunately terminology isn't standard however. There are authors who use the word "neighbourhood" by itself to mean what is called an "open neighbourhood" above.

Neighbourhoods may or may not be open. In $\mathbf{R}$ with the usual metric $d(x, y) = |x - y|$ the interval $(-1, 1)$ is an open neighbourhood of 0, because not only is it a neighbourhood of 0 but it is also a neighbourhood of every other point in $(-1, 1)$. Indeed, if $x \in (-1, 1)$ then $B(x, 1 - |x|) \subseteq (-1, 1)$. The interval $[-1, 1]$, by contrast, is not an open neighbourhood of 0. It is a neighbourhood of 0, but it is not a neighbourhood of either -1 or $+1$ and so is not open.

The word "neighbourhood" is slightly misleading. A set is a neighbourhood of w if w and all points sufficiently close to w belong to the set, but points arbitrarily far away might also be in the set. For example, it's clear from the definition that X itself is a neighbourhood of w.

Lemma 1.13.2. *If $(X, \mathcal{T})$ is a topological space, $x \in X$ and $V \in \wp(X)$ then V is an open neighbourhood of x if and only if $x \in V$ and $V \in \mathcal{T}$.*

Proof. If V is an open neighbourhood of x then $V \in \mathcal{T}$ and there is a $U \in \mathcal{T}$ such that $x \in U$ and $U \subseteq V$. From $x \in U$ and $U \subseteq V$ it follows that $x \in V$.

Conversely, suppose that $x \in V$ and $V \in \mathcal{T}$. Let $U = V$. Then $x \in U$, $U \in \mathcal{T}$ and $U \subseteq V$. So V is a neighbourhood of x. It's an open neighbourhood since $V \in \mathcal{T}$. □

Proposition 1.13.3. *Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces. If $f : U \to Y$ is a function defined on a subset $U \subseteq X$, $w \in X$ and $z \in Y$ then the following three conditions are equivalent.*

(a)
$$\lim_{x \to w} f(x) = z,$$

(b) For all $T \in \mathcal{N}(z)$ there is a $W \in \mathcal{O}(w)$ such that if $x \in U \cap W \setminus \{w\}$ then $f(x) \in T$.

(c) For all $T \in \mathcal{N}(z)$ there is a $W \in \mathcal{O}(w)$ such that
$$U \cap W \setminus \{w\} \subseteq f^*(T).$$

Proof. Suppose that for all $T \in \mathcal{N}(z)$ there is a $W \in \mathcal{O}(w)$ such that
$$U \cap W \setminus \{w\} \subseteq f^*(T).$$
If $x \in U$ and $Z \in \mathcal{T}_Y$ is such that $z \in Z$ then $Z \in \mathcal{O}(z) \subseteq \mathcal{N}(z)$ and so there is there is a $W \in \mathcal{N}(w)$ such that
$$U \cap W \setminus \{w\} \subseteq f^*(Z).$$
By Definition 1.12.1 it follows that
$$\lim_{x \to w} f(x) = z.$$

Suppose, conversely, that
$$\lim_{x \to w} f(x) = z.$$
If $T \in \mathcal{N}(z)$ then there is a $Z \in \mathcal{T}_Y$ such that $z \in Z$ and $Z \subseteq T$. By Definition 1.12.1 there is a $W \in \mathcal{T}_X$ such that $w \in W$ and
$$U \cap W \setminus \{w\} \subseteq f^*(Z).$$
Since $f^*(Z) \subseteq f^*(T)$ it follows that
$$U \cap W \setminus \{w\} \subseteq f^*(T).$$
□

The set neighbourhoods of a point in a topological space has a number of useful properties.

Proposition 1.13.4. *Suppose $(X, \mathcal{T})$ is a topological space and $x \in X$.*

(a) $\mathcal{N}(x) \neq \varnothing$

(b) $\varnothing \notin \mathcal{N}(x)$.

(c) If $A \in \mathcal{N}(x)$ and $B \in \mathcal{N}(x)$ then there is a $C \subseteq A \cap B$ such that $C \in \mathcal{N}(x)$.

(d) If $A \in \mathcal{N}(x)$ and $A \subseteq B \subseteq X$ then $B \in \mathcal{N}(x)$.

Proof. We check the four properties in turn.

(a) $X \in \mathcal{N}(x)$.

(b) $x \notin \varnothing$.

(c) If $A, B \in \mathcal{N}(x)$ then there are $V, W \in \mathcal{T}$ such that $x \in V$ and $V \subseteq A$ and also $x \in W$ and $W \subseteq B$. Then $x \in A \cap B$ and $V \cap W \subseteq A \cap B$. Since $\mathcal{T}$ is a topology we also have $V \cap W \in \mathcal{T}$. Let $U = V \cap W$ and $C = A \cap B$. We've just seen that $U \in \mathcal{T}$, $x \in U$ and $U \subseteq C$. So $C \in \mathcal{N}(x)$.

(d) If $A \in \mathcal{N}(x)$ then there is a $V \in \mathcal{T}$ such that $x \in V$ and $V \subseteq A$. Since $A \subseteq B$ we then have $V \subseteq B$. So $B \in \mathcal{N}(x)$.

□

1.14 Directed sets

The notion of a directed set generalises some of the properties of the set of neighbourhoods of a point in a topological space. It doesn't generalise enough of them for all our purposes, which is why we will need to introduce filters and prefilters later, but directed sets are simpler, so we will start there.

Definition 1.14.1. A *directed set* is a pair $(D, \preccurlyeq)$ consisting of a set D and a relation $\preccurlyeq$ on D satisfying the following conditions.

(a) For all $a \in D$, $a \preccurlyeq a$.

(b) For all $a, b, c \in D$. if $a \preccurlyeq b$ and $b \preccurlyeq c$ then $a \preccurlyeq c$.

(c) For all $a, b \in D$ there is a $c \in D$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

Note that Condition 1.14.1c refers only to a pair of elements a and b in D but this can then be extended to any non-empty finite collection of elements by induction.

You've seen many examples of directed sets already, although they weren't described that way at the time.

Proposition 1.14.2. *The following are directed sets:*

(a) the set of positive integers with the relation $\leq$

(b) the set of positive integers with the relation $|$, where $a|c$ means that a divides c, i.e. that there is a positive integer b such that $ab = c$.

(c) the integers or the rationals or the real numbers with the relation $\leq$

(d) the integers or the rationals or the real numbers with the relation $\geq$

(e) the set $\wp(X)$, where X is any set, with the relation $\subseteq$

(f) the set $\wp(X)$, where X is any set, with the relation $\supseteq$

(g) the set F of finite subsets of a set X, with the relation $\subseteq$

(h) the set F of finite subsets of a set X, with the relation $\supseteq$

(i) the set N of non-empty subsets of a set X, with the relation $\subseteq$

(j) the set P of proper subsets of a set X, with the relation $\subseteq$

(k) the set of open (or closed) balls centred at a point x in a metric space (X, d) with the relation $\subseteq$

(l) the set of open (or closed) balls centred at a point x in a metric space (X, d) with the relation $\supseteq$

(m) The set $\mathcal{N}(x)$ of neighbourhoods, or the set $\mathcal{O}(x)$ or open neighbourhoods, of a point x in a topological space with the relation $\subseteq$

(n) or the set $\mathcal{O}(x)$ or open neighbourhoods, of a point x in a topological space with the relation $\supseteq$

Proof. In each of these examples the condition 1.14.1a follows from the fact that $a \leq a$ for any real number a, the fact that $a|a$ integer a, or the fact that $A \subseteq A$ for any set A. Similarly, 1.14.1b follows from the fact that $a \leq b$ and $b \leq c$ imply $a \leq c$ for any real numbers a, b and c, the fact that $a|b$ and $b|c$ imply $a|c$ for any integers, or the fact that $A \subseteq B$ and $B \subseteq C$ imply $A \subseteq C$ for any sets A, B and C. It therefore suffices to check the condition 1.14.1c.

(a) If a and b are positive integers then $\max(a, b)$ is a positive integer, $a \leq \max(a, b)$ and $b \leq \max(a, b)$.

(b) If a and b are positive integers then $\operatorname{lcm}(a, b)$ is a positive integer, where lcm denotes the least common multiple, $a| \operatorname{lcm}(a, b)$ and $b| \operatorname{lcm}(a, b)$.

(c) If a and b are integers then $\max(a, b)$ is an integer, $a \leq \max(a, b)$ and $b \leq \max(a, b)$. The same holds with integers replaced by rationals or reals.

(d) If a and b are integers then $\min(a, b)$ is an integer, $a \geq \min(a, b)$ and $b \geq \min(a, b)$. The same holds with integers replaced by rationals or reals.

(e) If A and B are subsets of X then $A \cup B$ is a subset of X, $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

(f) If A and B are subsets of X then $A \cap B$ is a subset of X, $A \supseteq A \cap B$ and $B \supseteq A \cap B$.

(g) If A and B are finite subsets of X then $A \cup B$ is a finite subset of X, $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

(h) If A and B are finite subsets of X then $A \cap B$ is a finite subset of X, $A \supseteq A \cap B$ and $B \supseteq A \cap B$.

(i) If A and B are non-empty subsets of X then $A \cup B$ is a non-empty subset of X, $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

(j) If A and B are proper subsets of X then $A \cap B$ is a proper subset of X, $A \supseteq A \cap B$ and $B \supseteq A \cap B$.

(k) If $B(x,r)$ and $B(x,s)$ are balls centred at x then $B(x,\max(r,s))$ is a ball centred at x, $B(x,r) \subseteq B(x,\max(r,s))$, and $B(x,s) \subseteq B(x,\max(r,s))$. The same construction works with closed balls in place of open balls.

(l) If $B(x,r)$ and $B(x,s)$ are balls centred at x then $B(x,\min(r,s))$ is a ball centred at x, $B(x,r) \supseteq B(x,\min(r,s))$, and $B(x,s) \supseteq B(x,\min(r,s))$. The same construction works with closed balls in place of open balls.

(m) If A and B are neighbourhoods of x then $A \cup B$ is a neighbourhood of x, $A \subseteq A \cup B$ and $B \subseteq A \cup B$. The same holds with open neighbourhoods in place of neighbourhoods.

(n) If A and B are neighbourhoods of x then $A \cap B$ is a neighbourhood of x, $A \supseteq A \cap B$ and $B \supseteq A \cap B$. The same holds with open neighbourhoods in place of neighbourhoods.

The c from 1.14.1c is generally not unique. For example, in 1.14.2b we could have used ab rather than $\operatorname{lcm}(a,b)$. Where possible I've chosen a c in the proofs above which is in some sense minimal, but I didn't have to, and it's not always possible to.

If $(D,\preccurlyeq)$ is a directed set and $\succcurlyeq$ is defined by $a \succcurlyeq b$ if and only if $b \preccurlyeq a$ then $a \succcurlyeq a$ for all $a \in D$ and if $a \succcurlyeq b$ and $b \succcurlyeq c$ then $a \succcurlyeq c$ for all $a,b,c \in D$. So $(D,\succcurlyeq)$ is a directed set if and only if for all $a,b \in D$ there is a c such that $a \succcurlyeq c$ and $b \succcurlyeq c$, i.e. such that $c \preccurlyeq a$ and $c \preccurlyeq b$. This holds for most of the examples above, and indeed most of them occur in pairs like this, but it doesn't hold for all of them. $(N,\supseteq)$ and $(P,\subseteq)$ are not directed sets, for example. □

We're interested in functions between directed sets which are compatible with the order relations on them.

Definition 1.14.3. If $(D,\preccurlyeq)$ and $(E,\preceq)$ are directed sets then a function $\tau\colon D \to E$ is called *monotone* if $\tau(a) \preceq \tau(b)$ whenever $a \preccurlyeq b$.

There are various ways to construct one directed set from another.

Proposition 1.14.4. *(a) If $(E,\preceq)$ and $S \subseteq E$ then $(S,\preceq)$ is a directed set if and only if for all $p,q \in S$ there is an $r \in S$ such that $p \preceq r$ and $q \preceq r$.*

(b) Suppose $(D,\preccurlyeq)$ and $(E,\preceq)$ are directed sets and $\tau\colon D \to E$ is a monotone function. Then $(\tau_*(D),\preceq)$ is a directed set.

(c) Suppose $(D,\preccurlyeq)$ is a directed set. For each $a \in D$ let

$$T_a = \{b \in D \colon a \preccurlyeq b\}$$

and let $\mathcal{E}$ be set of sets of the form T_a for some $a \in D$. Then $a \preccurlyeq b$ if and only if $T_a \supseteq T_b$ and $(\mathcal{E},\supseteq)$ is also a directed set.

Proof. (a) If $p \in S$ then $p \preceq p$ because $p \in E$ and $(E,\preceq)$ is a directed set. If $p,q,r \in S$, $p \preceq q$ and $q \preceq r$ then $p \preceq r$ because $p,q,r \in E$ and $(E,\preceq)$ is a directed set. Those are 1.14.1a and 1.14.1b. So $(S,\preceq)$ is a directed set if and only if it satisfies 1.14.1c

(b) By the previous part we only need to check that if $p,r \in \tau_*(D)$ then there is an $r \in \tau_*(D)$ such that $p \preceq r$ and $q \preceq r$. If $p,q \in \tau_*(D)$ then there are $a,b \in D$ such that $p = \tau(a)$ and $q = \tau(b)$. $(D,\preccurlyeq)$ is a directed set so there must be a $c \in D$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$. Let $r = \tau(c)$. Our assumption that τ is monotone implies that $\tau(a) \preceq \tau(c)$ and $\tau(b) \preceq \tau(c)$. In other words, $p \preceq r$ and $q \preceq r$.

(c) Let $E = \wp(D)$. By Proposition 1.14.2f $(E,\supseteq)$ is a directed set. Let $\tau\colon D \to E$ be defined by

$$\tau(a) = T_a.$$

Then $\mathcal{E} = \tau_*(D)$. Suppose $a \preccurlyeq b$. If $c \in T_b$ then $b \preccurlyeq c$ so $a \preccurlyeq c$, since $(D,\preccurlyeq)$ is a directed set, and hence $c \in T_a$. So every $c \in T_b$ belongs to T_a. In other words, $T_a \supseteq T_b$, i.e. $\tau(a) \supseteq \tau(b)$. Although we won't need it here, the converse is also true. If $\tau(a) \supseteq \tau(b)$, i.e. if $T_a \supseteq T_b$, then $b \in T_a$ and so $a \preccurlyeq b$. In other words, $a \preccurlyeq b$ if and only if $\tau(a) \supseteq \tau(b)$. The previous part therefore shows that $(\mathcal{E},\supseteq)$ is a directed set.

□

It follows from Propositions 1.14.2c and 1.14.4c, for example that the set of all semi-infinite intervals of the form $[a, +\infty)$ with the relation $\supseteq$ is a directed set.

1.15 Filters and prefilters

It's useful to have names for sets of subsets satisfying the conditions in Proposition 1.13.4, and also for sets of subsets satisfying all but the last condition.

Definition 1.15.1. Suppose X is a set $\mathcal{F} \in \wp(\wp(X))$. Then $\mathcal{F}$ is called a *filter* if it satisfies all of the following conditions.

(a) $\mathcal{F} \neq \varnothing$

(b) $\varnothing \notin \mathcal{F}$.

(c) If $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then there is a $C \subseteq A \cap B$ such that $C \in \mathcal{F}$.

(d) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$ then $B \in \mathcal{F}$.

$\mathcal{F}$ is called a *prefilter* if it satisfies the first three conditions.

Note that we don't assume a prefilter fails to satisfy the last condition; we simply don't assume that it does. So any filter is automatically a prefilter. Prefilters needn't be filters though, as we'll see in a moment.

Proposition 1.15.2. *If $\mathcal{F}$ is a prefilter or filter then $(\mathcal{F}, \supseteq)$ is a directed set. If $\mathcal{D} \in \wp(\wp(X))$ for some set X, $\mathcal{D} \neq \varnothing$, $\varnothing \notin \mathcal{D}$, and $(\mathcal{D}, \supseteq)$ is a directed set then $\mathcal{D}$ is a prefilter.*

Proof. Suppose $\mathcal{F}$ is a filter or prefilter. $A \supseteq A$ for all $A \in \mathcal{F}$ and if $A \supseteq B$ and $B \supseteq C$ then $A \supseteq C$ for all $A, B, C \in \mathcal{F}$. So the conditions 1.14.1a and 1.14.1b hold. If $A, B \in \mathcal{F}$ then there is, by 1.15.1c, a $C \in \mathcal{F}$ such that $C \subseteq A \cap B$. Then $A \supseteq C$ and $B \supseteq C$, so the condition 1.14.1c is satisfied. Therefore $(\mathcal{F}, \supseteq)$ is a directed set.

Suppose $\mathcal{D} \in \wp(\wp(X))$ for some set X, $\mathcal{D} \neq \varnothing$, $\varnothing \notin \mathcal{D}$, and $(\mathcal{D}, \supseteq)$ is a directed set. Then $\mathcal{D}$ satisfies the conditions 1.15.1a and 1.15.1b. If $A, B \in \mathcal{D}$ then by the condition 1.14.1c there is a $C \in \mathcal{D}$ such that $A \supseteq C$ and $B \supseteq C$, from which it follows that $C \subseteq A \cap B$, so the condition 1.15.1c is also satisfied and $\mathcal{D}$ is a filter. □

Proposition 1.15.3. *Suppose $(X, \mathcal{T})$ is a topological space and $x \in X$. Then $\mathcal{N}(x)$ is a filter and $\mathcal{O}(x)$ is a prefilter.*

Proof. The fact that $\mathcal{N}(x)$ is a filter is just Proposition 1.13.4. To show that $\mathcal{O}(x)$ is a prefilter we can use Proposition 1.15.2 above together with Proposition 1.14.2n. □

Note that $\mathcal{O}(x)$ is rarely a filter. In the case $X = \mathbf{R}$ we have $(-1, 1) \in \mathcal{O}(0)$ and $(-1, 1) \subseteq [-1, 1]$ but $[-1, 1] \notin \mathcal{O}(0)$ because $[-1, 1]$ is not open. On the other hand, if $\mathcal{T}$ is the discrete topology on a set X then $\mathcal{O}(x)$ is the set of all subsets containing x, which is the same as $\mathcal{N}(x)$, which we've just seen is a filter.

The proposition shows that $\mathcal{O}(x)$ is a prefilter if $x \in X$ and $(X, \mathcal{T})$ is a topological space, but not every filter arises in this way. Consider for example the set

$$\mathcal{E} = \{A \in \wp(\mathbf{R}) \colon \exists a \in \mathbf{R} \colon A = [a, +\infty)\},$$

which we already saw is a directed set. It's non-empty and does not contain the empty set so by Proposition 1.15.2 it is a prefilter.

Although the definition of a filter only requires that if $A, B \in \mathcal{F}$ then there is some $C \subseteq A \cap B$ such that $C \in \mathcal{F}$ this is in fact true of $C = A \cap B$.

Lemma 1.15.4. *If $\mathcal{F}$ is a filter on a set X and $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.*

Proof. The third condition of the definition just gives a $C \subseteq A \cap B$ such that $C \in \mathcal{F}$ but using the last condition we see that $C \subseteq A \cap B$ and $C \in \mathcal{F}$ imply $A \cap B \in \mathcal{F}$. □

Of course one can then show by induction that any finite intersection of elements of $\mathcal{F}$ is also an element of $\mathcal{F}$.

Another example of a filter, closely related to the definition of limits, is given by the following proposition.

Proposition 1.15.5. *If $(X, \mathcal{T})$ is a topological space, $w \in X$ and $U \in \wp(X)$ then the set $\mathcal{E}(w) \in \wp(\wp(X))$ consisting of sets of the form $U \cap W \setminus \{w\}$ where $W \in \mathcal{O}(w)$ is a prefilter if and only if w is a limit point of U.*

Proof. $(\mathcal{O}(w), \supseteq)$ is a directed by Proposition 1.14.2n. Define $\tau \colon \mathcal{O}(w) \to \wp(\wp(X))$ by

$$\tau(W) = U \cap W \setminus \{w\}.$$

If $V \supseteq W$ then $\tau(V) \supseteq W$. In other words τ is monotone. Now $\mathcal{E}(w) = \tau_*(\mathcal{O})$ by definition so $\mathcal{E}(w)$ is a directed set by Proposition 1.14.4b. It's non-empty because it's the image of the non-empty set $\mathcal{O}(w)$. By Proposition 1.15.2 it's a prefilter if and only if $\varnothing \notin \mathcal{E}(w)$ which happens if and only if w is a limit point of U, by Proposition 1.11.4. □

The prefilter $\mathcal{E}(w)$ from the proposition is never a filter, but there is a simple way to get a filter from a prefilter.

Definition 1.15.6. If X is a set and $\mathcal{E} \in \wp(\wp(X))$ then the *upward closure* of $\mathcal{E}$ is the set of $B \in \wp(X)$ such that there is an $A \in \mathcal{E}$ with $A \subseteq B$. $\mathcal{E}$ is called *upward closed* if it is equal to its upward closure.

Note that Condition 1.15.1d from the definition of a filter is simply the statement that $\mathcal{F}$ is upward closed.

Proposition 1.15.7. *(a) If X and Y are sets, $f \colon X \to Y$ is a function and $\mathcal{F} \in \wp(\wp(X))$ is upward closed then so is $f^{**}(\mathcal{F})$.*

(b) If X is set and $\mathcal{E} \in \wp(\wp(X))$ and $\mathcal{F}$ is the upward closure of $\mathcal{E}$ then $\mathcal{E} \subseteq \mathcal{F}$.

(c) If $\mathcal{E}$ is a prefilter on a set X and $\mathcal{F}$ is the upward closure of $\mathcal{E}$ then $\mathcal{E}$ is a filter if and only if $\mathcal{E} = \mathcal{F}$.

(d) If $\mathcal{E}$ is a prefilter on a set X and $\mathcal{F}$ is the upward closure of $\mathcal{E}$ then $\mathcal{F}$ is a filter on X.

(e) If $\mathcal{E}$ is a prefilter, $\mathcal{F}$ is the upward closure of $\mathcal{E}$, $\mathcal{G}$ is a filter and $\mathcal{E} \subseteq \mathcal{G}$ then $\mathcal{F} \subseteq \mathcal{G}$.

Proof. We prove each of the statements in turn.

(a) Suppose $A \subseteq B$ and $A \in f^{**}(\mathcal{F})$. Then $f^*(A) \in \mathcal{F}$. $f^*(A) \subseteq f^*(B)$ and $\mathcal{F}$ is upward closed so $f^*(B) \in \mathcal{F}$. In other words $B \in f^{**}(\mathcal{F})$. So if $A \subseteq B$ and $A \in f^{**}(\mathcal{F})$ then $B \in f^{**}(\mathcal{F})$. In other words, $f^{**}(\mathcal{F})$ is upward closed.

(b) If $B \in \mathcal{E}$ then there is an $A \in \mathcal{E}$ such that $A \subseteq B$, namely $A = B$.

(c) $\mathcal{E}$ is a prefilter so it satisfies conditions 1.15.1a, 1.15.1b and 1.15.1c. It is a filter if and only if it satisfies 1.15.1d. But 1.15.1d is just the statement that the upward closure of $\mathcal{E}$ is contained in $\mathcal{E}$. We just saw that the reverse inclusion always holds.

(d) We need to check conditions 1.15.1a, 1.15.1b, 1.15.1c and 1.15.1d.

$\mathcal{E} \neq \varnothing$ and $\mathcal{E} \subseteq \mathcal{F}$ so $\mathcal{F} \neq \varnothing$. This establishes 1.15.1a

If $\varnothing \in \mathcal{F}$ then there is an $S \in \mathcal{E}$ such that $S \subseteq T$. But $S \neq \varnothing$ because $\varnothing \notin \mathcal{E}$. So we have a non-empty subset of an empty set, which is impossible. So the assumption $\varnothing \in \mathcal{F}$ is false. This establishes 1.15.1b

If $S, T \in \mathcal{F}$ then there are $P, Q \in \mathcal{E}$ such that $P \subseteq S$ and $Q \subseteq T$. $\mathcal{E}$ is a prefilter so there is an $R \in \mathcal{E}$ such that $R \subseteq P \cap Q$. But $P \cap Q \subseteq S \cap T$ so $R \subseteq S \cap T$. But then $S \cap T \in \mathcal{F}$. This establishes 1.15.1c

If $S \in \mathcal{F}$ and $S \subseteq T$ then there is an $R \in \mathcal{E}$ such that $R \subseteq S$. But then $R \subseteq T$, so $T \in \mathcal{F}$. This establishes 1.15.1d

(e) If $B \in \mathcal{F}$ then there is an $A \in \mathcal{E}$ such that $A \subseteq B$. $\mathcal{E} \subseteq \mathcal{G}$ so $A \in \mathcal{G}$. But $\mathcal{G}$ is a filter and $A \subseteq B$ so $B \in \mathcal{G}$. We've just seen that if $B \in \mathcal{F}$ then $B \in \mathcal{G}$, so $\mathcal{F} \subseteq \mathcal{G}$.

□

We've already seen an example of this construction. $\mathcal{N}(x)$ is just the upward closure of $\mathcal{O}(x)$.

We can describe the Hausdorff property in terms of filters.

Theorem 1.15.8. *Suppose $(Y, \mathcal{T})$ is a topological space. The topology $\mathcal{T}$ on Y is Hausdorff if and only if for every filter $\mathcal{G}$ on Y there is at most one $z \in Y$ such that $\mathcal{N}(z) \subseteq \mathcal{G}$.*

Proof. Suppose $\mathcal{T}$ is Hausdorff. For any distinct $z_1, z_2 \in Y$ there are open sets V_1, V_2 such that $z_1 \in V_1$, $z_2 \in V_2$ and $V_1 \cap V_2 = \varnothing$. Now $V_1 \in \mathcal{O}(z_1) \subseteq \mathcal{N}(z_1)$ and $V_2 \in \mathcal{O}(z_2) \subseteq \mathcal{N}(z_2)$. If there were a filter $\mathcal{G}$ such that $\mathcal{N}(z_1) \subseteq \mathcal{G}$ and $\mathcal{N}(z_2) \subseteq \mathcal{G}$ then we would have $V_1 \in \mathcal{G}$ and $V_2 \in \mathcal{G}$. By Lemma 1.15.4 we would then have $\varnothing \in \mathcal{G}$, but $\mathcal{G}$ is a filter, so this is impossible. So there is at most one $z \in Y$ such that $\mathcal{N}(z) \subseteq \mathcal{G}$.

Suppose, on the other hand, that $\mathcal{T}$ is not Hausdorff, i.e. that there are distinct $z_1, z_2 \in Y$ such that $V_1 \cap V_2 \neq \varnothing$ for all $V_1 \in \mathcal{O}(z_1)$ and $V_2 \in \mathcal{O}(z_2)$. Let $\mathcal{F}$ be the set of sets of the form $V_1 \cap V_2$ where $V_1 \in \mathcal{O}(z_1)$ and $V_2 \in \mathcal{O}(z_2)$. I claim that $\mathcal{F}$ is a prefilter. $X \in \mathcal{F}$ so $\mathcal{F} \neq \varnothing$. On the other hand, we've just seen that the assumption that $\mathcal{T}$ is not Hausdorff implies that $\varnothing \notin \mathcal{F}$. If $A, B \in \mathcal{F}$ then there are $V_1, W_1 \in \mathcal{O}(z_1)$ and $V_2, W_2 \in \mathcal{O}(z_2)$ such that $A = V_1 \cap V_2$ and $B = W_1 \cap W_2$. Then

$$A \cap B = (V_1 \cap W_2) \cap (V_2 \cap W_2).$$

Now $V_1 \cap W_1 \in \mathcal{O}(z_1)$ and $V_1 \cap W_1 \in \mathcal{O}(z_1)$ so $A \cap B \in \mathcal{F}$. We've now checked that $\mathcal{F}$ has the three properties required for it to be a prefilter. Let $\mathcal{G}$ be the upward closure of $\mathcal{F}$, which is, by Proposition 1.15.7, a filter containing $\mathcal{F}$. If $T \in \mathcal{N}(z_1)$ then there is an $S \in \mathcal{O}(z_1)$ such that $S \subseteq T$. Now $S \in \mathcal{O}(z_1)$ and $Y \in \mathcal{O}(z_2)$ so $S \cap Y \in \mathcal{F}$. But $S \subseteq Y$ so $S \cap Y = S$. $\mathcal{G}$ is the upward closure of $\mathcal{F}$ and $S \subseteq T$ so $T \in \mathcal{G}$. We've just seen that if $T \in \mathcal{N}(z_1)$ then $T \in \mathcal{G}$, so $\mathcal{N}(z_1) \subseteq \mathcal{G}$. Similarly, $\mathcal{N}(z_2) \subseteq \mathcal{G}$. So there is more than one $z \in Y$ such that $\mathcal{N}(z) \subseteq \mathcal{G}$. □

Note that while for each filter $\mathcal{G}$ there is at most one $z \in Y$ such that $\mathcal{N}(z) \subseteq \mathcal{G}$ it's entirely possible that there is no such z. Consider, for example, the upward closure of the prefilter on $\mathbf{R}$ of semi-infinite intervals $[a, \infty)$ considered earlier. Calling this filter $\mathcal{G}$, there is no $z \in \mathbf{R}$ such that $\mathcal{N}(z) \subseteq \mathcal{G}$. If there were then we would have

$$(z-1, z+1) \in \mathcal{O}(z) \subseteq \mathcal{N}(z) \subseteq \mathcal{G}$$

and $[z+1, \infty) \in \mathcal{G}$ so

$$\varnothing = (z-1, z+1) \cap [z+1, \infty) \in \mathcal{G},$$

which is impossible because $\mathcal{G}$ is a filter.

1.16 Directed sets and limits

Definition 1.16.1. Suppose that $(D, \preccurlyeq)$ is a non-empty directed set, U is a set, $(Y, \mathcal{T})$ is a topological space, $z \in Y$, ω is a monotone function from $(D, \preccurlyeq)$ to $(\wp(U), \supseteq)$, and f is a function from U to Y. We say that z is the *limit* of f with respect to ω, written,

$$\lim_{\omega} f = z,$$

if $\mathcal{N}(z)$ is a subset of $f^{**}(\mathcal{F})$, where $\mathcal{F}$ is the upward closure of $\omega_*(D)$.

Theorem 1.16.2. *Suppose that $(D, \preccurlyeq)$ is a non-empty directed set, U is a set, $(Y, \mathcal{T})$ is a topological space, $z \in Y$, ω is a monotone function from $(D, \preccurlyeq)$ to $(\wp(U), \supseteq)$, and f is a function from U to Y. Let $\mathcal{F}$ be the upward closure of $\omega_*(D)$. The following statements are equivalent.*

(a) $\lim_{\omega} f = z$.

(b) For every $P \in \mathcal{O}(z)$ there is an $a \in D$ such that if $x \in \omega(a)$ then $f(x) \in P$.

(c) For every $P \in \mathcal{O}(z)$ there is an $a \in D$ such that $f_*(\omega(a)) \subseteq P$.

(d) For every $P \in \mathcal{O}(z)$ there is an $a \in D$ such that $f^*(P) \supseteq \omega(a)$.

(e) For every $P \in \mathcal{O}(z)$ there is a $B \in \mathcal{F}$ such that $f^*(P) = B$.

(f) For every $P \in \mathcal{O}(z)$, $f^*(P) \in \mathcal{F}$.

(g) For every $P \in \mathcal{O}(z)$, $P \in f^{**}(\mathcal{F})$.

(h) $\mathcal{O}(z) \subseteq f^{**}(\mathcal{F})$.

(i) $\mathcal{N}(z) \subseteq f^{**}(\mathcal{F})$.

(j) For every $P \in \mathcal{N}(z)$, $P \in f^{**}(\mathcal{F})$.

(k) For every $P \in \mathcal{N}(z)$, $f^*(P) \in \mathcal{F}$.

(l) *For every $P \in \mathcal{N}(z)$ there is a $B \in \mathcal{F}$ such that $f^*(P) = B$.*

(m) *For every $P \in \mathcal{N}(z)$ there is an $a \in D$ such that $f^*(P) \supseteq \omega(a)$.*

(n) *For every $P \in \mathcal{N}(z)$ there is an $a \in D$ such that $f_*(\omega(a)) \subseteq P$.*

(o) *For every $P \in \mathcal{N}(z)$ there is an $a \in D$ such that if $x \in \omega(a)$ then $f(x) \in P$.*

Proof. 1.16.2a and 1.16.2i are equivalent by the definition of the limit. 1.16.2c is equivalent to 1.16.2b by the definition of the image. 1.16.2d is equivalent to 1.16.2b by the definition of the preimage. 1.16.2e follows from 1.16.2d because if we set $B = f^*(P)$ then $\omega(a) \in \omega_*(D)$ and $\omega(a) \subseteq B$ so $B \in \mathcal{F}$, by the definition of the upward closure. 1.16.2f follows from 1.16.2e because if $B = f^*(P)$ and $B \in \mathcal{F}$ then $f^*(P) \in \mathcal{F}$. 1.16.2g follows from 1.16.2f by the definition of the preimage. 1.16.2h follows from 1.16.2g by the definition of inclusion of sets. 1.16.2i follows from 1.16.2h because if $Q \in \mathcal{N}(z)$ then there is an $P \in \mathcal{O}(z)$ such that $P \subseteq Q$. Then $P \in f^{**}(\mathcal{F})$ by 1.16.2h. $\mathcal{F}$ is the upward closure of $\omega_*(D)$ and hence is upward closed. It follows from Proposition 1.15.7b that $f^{**}(\mathcal{F})$ is upward closed and hence $Q \in f^{**}(\mathcal{F})$. 1.16.2j follows from 1.16.2i by the definition of inclusion of sets. 1.16.2k follows from 1.16.2j by the definition of the preimage. 1.16.2l follows from 1.16.2k because if we set $B = f^*(P)$ and $f^*(P) \in \mathcal{F}$ then $B \in \mathcal{F}$. 1.16.2m follows from 1.16.2l because if $B \in \mathcal{F}$ then there is, by the definition of the upward closure, an $A \in \omega_*(D)$ such that $A \subseteq B$ and hence, by the definition of the image, an $a \in D$ such that $\omega(a) \subseteq B$. 1.16.2o is equivalent to 1.16.2m by the definition of the preimage. 1.16.2o is equivalent to 1.16.2n by the definition of the image. 1.16.2b follows from 1.16.2o because $\mathcal{O}(z) \subseteq \mathcal{N}(z)$ by Proposition 1.15.7b. □

Theorem 1.16.3. *Suppose that $(D, \preccurlyeq)$ is a non-empty directed set, U is a set, $(Y, \mathcal{T})$ is a topological space, ω is a monotone function from $(D, \preccurlyeq)$ to $(\wp(U), \supseteq)$, and $\varnothing \notin \omega_*(D)$*

(a) *If f is a function from U to Y and $\mathcal{T}$ is Hausdorff then there is at most one $z \in Y$ such that $\lim_\omega f = z$.*

(b) *Suppose $Y = \mathbf{R}$ and $\mathcal{T}$ is the usual topology on $\mathbf{R}$. If f and g are functions from U to Y such that $f(x) \leq g(x)$ for all $x \in U$ and $\lim_\omega f$ and $\lim_\omega g$ exist then $\lim_\omega f \leq \lim_\omega g$.*

(c) *Suppose Y is a vector space and $\mathcal{T}$ is the topology associated to a norm q on Y. Suppose*

$$\mathbf{g} = \sum_{j=1}^{k} \alpha_j \mathbf{f}_j$$

where $\mathbf{f}_1, \ldots \mathbf{f}_k$ are functions from U to Y, $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$, and

$$\lim_\omega \mathbf{f}_j = \mathbf{z}_j,$$

where $\mathbf{z}_1, \ldots \mathbf{z}_k \in Y$. Then

$$\lim_\omega \mathbf{g} = \sum_{j=1}^{k} \alpha_j \mathbf{z}_j.$$

Also,

$$\lim_\omega q \circ \mathbf{g} \leq \sum_{j=1}^{k} |\alpha_j| \lim_\omega q \circ \mathbf{f}_j,$$

provided all the limits appearing in the inequality exist.

Proof. First note that $\omega_*(D)$ is the image of a non-empty set and so is non-empty and $\varnothing \notin \omega_*(D)$ by hypothesis. $(\omega_*(D), \supseteq)$ is a directed set by Proposition 1.14.4b and so $\omega_*(D)$ is a prefilter by Proposition 1.15.2. Its upward closure $\mathcal{F}$ is then a filter by Proposition 1.15.7d. We now prove the assertions of the theorem in order.

(a) By Theorem 1.15.8 there is at most one z such that $\mathcal{N}(z) \subseteq f^{**}(\mathcal{F})$. In view of the definition of the limit that's the same as saying there's at most one z such that $\lim_\omega f = z$.

(b) Suppose $\lambda > 0$. Let $\epsilon = \lambda/2$. Then $\epsilon > 0$. Let

$$z_1 = \lim_\omega f$$

and

$$z_2 = \lim_\omega g.$$

$B(z_1, \epsilon) \in \mathcal{N}(z_1)$ and $B(z_2, \epsilon) \in \mathcal{N}(z_2)$ so by Theorem 1.16.2 there are $a, b \in D$ such that if $x \in \omega(a)$ then $f(x) \in B(z_1, \epsilon)$ and if $x \in \omega(b)$ then $g(x) \in B(z_2, \epsilon)$. $(D, \preccurlyeq)$ is a directed set so there is a $c \in D$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$. $\varnothing \notin \omega_*(D)$ so $\omega(c)$ is non-empty and there is therefore an $x \in \omega(c)$. ω is monotone so $\omega(a) \supseteq \omega(c)$ and $\omega(b) \supseteq \omega(c)$. It follows that $x \in \omega(a)$ and $x \in \omega(b)$. Therefore $f(x) \in B(z_1, \epsilon)$ and $g(x) \in B(z_2, \epsilon)$. In other words, $|f(x) - z_1| < \epsilon$ and $|g(x) - z_2| < \epsilon$. Therefore

$$z_1 < f(x) + \epsilon \leq g(x) + \epsilon < z_2 + 2\epsilon = z_2 + \lambda.$$

So $\lim_\omega f < \lambda + \lim_\omega g$ for all positive λ, which is possible only if $\lim_\omega f \leq \lim_\omega g$.

(c) Suppose

$$P \in \mathcal{O}\left(\sum_{j=1}^{k} \alpha_j \mathbf{z}_j\right)$$

Then there is an $\epsilon > 0$ such that

$$B\left(\sum_{j=1}^{k} \mathbf{z}_j, \epsilon\right) \subseteq P.$$

Choose some

$$\kappa > \sum_{j=1}^{k} |\alpha_j|.$$

Then $\epsilon/\kappa > 0$. Since $\lim_\omega \mathbf{f}_j = \mathbf{z}_j$ and $B(\mathbf{z}_j, \epsilon/\kappa) \in \mathcal{O}(\mathbf{z}_j)$ there is an $a_j \in D$ such that if $x \in \omega(a_j)$ then $\mathbf{f}_j(x) \in B(\mathbf{z}_j, \epsilon/\kappa)$. $(D, \preccurlyeq)$ is a directed set so there's a $c \in D$ such that $a_j \preccurlyeq c$ for all j. ω is monotone so $\omega(a_j) \supseteq \omega(c)$. Suppose $x \in \omega(c)$. Then $x \in \omega(a_j)$ and hence $\mathbf{f}_j(x) \in B(\mathbf{z}_j, \epsilon/\kappa)$ for each j. In other words

$$q(\mathbf{f}_j(x) - \mathbf{z}_j) < \epsilon/\kappa$$

for each j. By the properties of norms then

$$q\left(\alpha_j\left(\mathbf{f}_j(x) - \mathbf{z}_j\right)\right) < \frac{|\alpha_j|\epsilon}{\kappa}.$$

Because of our choice of κ this implies

$$q\left(\sum_{j=1}^{k} \alpha_j\left(\mathbf{f}_j(x) - \mathbf{z}_j\right)\right) < \epsilon.$$

By the associative, commutative and distributive properties of vector spaces this is the same as

$$q\left(\sum_{j=1}^{k} \alpha_j \mathbf{f}_j(x) - \sum_{j=1}^{k} \alpha_j \mathbf{z}_j\right) < \epsilon$$

or

$$q\left(\mathbf{g}(x) - \sum_{j=1}^{k} \alpha_j \mathbf{z}_j\right) < \epsilon.$$

In other words,

$$\mathbf{g}(x) \in B\left(\sum_{j=1}^{k} \mathbf{z}_j, \epsilon\right)$$

and hence

$$\mathbf{g}(x) \in P.$$

In other words, for any $P \in \mathcal{O}\left(\sum_{j=1}^{k} \alpha_j \mathbf{z}_j\right)$ we've found a $c \in D$ such that if $x \in \omega(c)$ then $\mathbf{g}(x) \in P$. By Theorem 1.16.2 then

$$\lim_\omega \mathbf{g} = \sum_{j=1}^{k} \alpha_j \mathbf{z}_j.$$

(d) By the properties of norms,

$$q\left(\mathbf{g}(x)\right) \leq \sum_{j=1}^{k} |\alpha_j| q(\mathbf{f}_j(x))$$

for all $x \in U$. Applying the third part of the theorem,

$$\lim_\omega \sum_{j=1}^{k} |\alpha_j| q \circ \mathbf{f}_j = \sum_{j=1}^{k} |\alpha_j| \lim_\omega q \circ \mathbf{f}_j.$$

Then applying the second part,

$$\lim_\omega q \circ \mathbf{g} \leq \lim_\omega \sum_{j=1}^{k} |\alpha_j| q \circ \mathbf{f}_j.$$

Combining these

$$\lim_\omega q \circ \mathbf{g} \leq \sum_{j=1}^{k} |\alpha_j| \lim_\omega q \circ \mathbf{f}_j.$$

□

1.17 Theorems 1.12.3 and 1.12.4 revisited

As a corollary to the preceding theorem we have the following theorem which incorporates Theorems 1.12.3 and 1.12.4 together with various other useful results.

Theorem 1.17.1. *Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, that $U \in \wp(X)$ and that $w \in X$ is a limit point of U.*

(a) If f is a function from U to Y and $\mathcal{T}_Y$ is Hausdorff then there is at most one $z \in Y$ such that $\lim_{x \to w} f(x) = z$.

(b) Suppose $Y = \mathbf{R}$ and $\mathcal{T}$ is the usual topology on $\mathbf{R}$. If f and g are functions from U to Y such that $f(x) \leq g(x)$ for all $x \in U$ and $\lim_{x\to w} f(x)$ and $\lim_{x \to w} g(x)$ exist then $\lim_{x\to w} f(x) \leq \lim_{x\to w} g(x)$.

(c) Suppose Y is a vector space and $\mathcal{T}$ is the topology associated to a norm q on Y. Suppose

$$\mathbf{g} = \sum_{j=1}^{k} \alpha_j \mathbf{f}_j$$

where $\mathbf{f}_1, \ldots \mathbf{f}_k$ are functions from U to Y, $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$, and

$$\lim_{x\to w} \mathbf{f}_j(x) = \mathbf{z}_j,$$

where $\mathbf{z}_1, \ldots \mathbf{z}_k \in Y$. Then

$$\lim_{x\to w} \mathbf{g}(x) = \sum_{j=1}^{k} \alpha_j \mathbf{z}_j.$$

Also,

$$\lim_{x\to w} (q \circ \mathbf{g})(x) \leq \sum_{j=1}^{k} |\alpha_j| \lim_{x\to w} (q \circ \mathbf{f}_j)(x),$$

provided all the limits appearing in the inequality exist.

Proof. Let D be $\mathcal{O}(w)$ and let $\preccurlyeq$ be $\supseteq$. Then $(D, \preccurlyeq)$ is a directed set by Proposition 1.14.2n. It is clearly non-empty since $X \in \mathcal{O}(w)$. Let $\omega \colon D \to \wp(U)$ be defined by $\omega(W) = U \cap W \setminus \{w\}$. Then ω is a monotone function from the directed set $(D, \preccurlyeq) = (\mathcal{O}(w), \supseteq)$ to to the directed set $(\wp(U), \supseteq)$, as was observed in the proof of Proposition 1.15.5. $\varnothing \notin \omega_*(D)$ because w is a limit point of U. By Theorem 1.16.2

$$\lim_{\omega} f = z$$

for a function f from U to Y if and only if for all $Z \in \mathcal{O}(z)$ there is a $W \in D$ such that $\omega(W) \subseteq f^*(Z)$, i.e. if and only if for all $Z \in \mathcal{T}_Y$ such that $z \in Z$ there is a $W \in \mathcal{T}_X$ such that $w \in W$ and

$$U \cap W \setminus \{w\} \subseteq f^*(Z).$$

By Definition 1.12.1 this is the same as

$$\lim_{x\to w} f(x) = z.$$

We can therefore apply Theorem 1.16.3, replacing $\lim_\omega$ everywhere by $\lim_{x\to w}$. □

The alternate proof of Theorems 1.12.3 and 1.12.4 provided by the theorem above is much shorter than the proofs given when they were first stated, but only because it depends on many results proved in between. If those results served only to give new proofs of Theorems 1.12.3 and 1.12.4 it would be hard to argue that the reward was worth the effort. But in fact we can now give equally short proofs of the analogous theorems for a wide variety of types of limits.

1.18 Other types of limit

Limits at infinity are defined as follows.

Definition 1.18.1. Suppose $U \in \wp(\mathbf{R})$ is such that for every $a \in \mathbf{R}$ there is an $x \in U$ with $x \geq a$ and that $(Y, \mathcal{T})$ is a topological space. If f is a function from U to Y then we say that $z \in Y$ is the *limit* of f at $+\infty$, written

$$\lim_{x\to+\infty} f(x) = z$$

if for all $Z \in \mathcal{O}(z)$ there is an $a \in \mathbf{R}$ such that if $x \in U$ and $x \geq a$ then $f(x) \in Z$.

As in the theory of limits at finite points we allow our functions to be defined only on a subset. The usefulness of this will become clear shortly.

Theorem 1.18.2. *Suppose $U \in \wp(\mathbf{R})$ is such that for every $a \in \mathbf{R}$ there is an $x \in U$ with $x \geq a$ and that $(Y, \mathcal{T})$ is a topological space. If f is a function from U to Y then we say that $z \in Y$ is the limit of f at $+\infty$, written*

$$\lim_{x \to +\infty} f(x) = z$$

if for all $Z \in \mathcal{O}(z)$ there is an $a \in \mathbf{R}$ such that if $x \in U$ and $x \geq a$ then $f(x) \in Z$.

(a) If f is a function from U to Y and $\mathcal{T}$ is Hausdorff then there is at most one $z \in Y$ such that $\lim_{x \to +\infty} f(x) = z$.

(b) Suppose $Y = \mathbf{R}$ and $\mathcal{T}$ is the usual topology on $\mathbf{R}$. If f and g are functions from U to Y such that $f(x) \leq g(x)$ for all $x \in U$ and $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to +\infty} g(x)$ exist then $\lim_{x \to +\infty} f(x) \leq \lim_{x \to +\infty} g(x)$.

(c) Suppose Y is a vector space and $\mathcal{T}$ is the topology associated to a norm q on Y. Suppose

$$\mathbf{g} = \sum_{j=1}^{k} \alpha_j \mathbf{f}_j$$

where $\mathbf{f}_1, \ldots \mathbf{f}_k$ are functions from U to Y, $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$, and

$$\lim_{x \to +\infty} \mathbf{f}_j(x) = \mathbf{z}_j,$$

where $\mathbf{z}_1, \ldots \mathbf{z}_k \in Y$. Then

$$\lim_{x \to +\infty} \mathbf{g}(x) = \sum_{j=1}^{k} \alpha_j \mathbf{z}_j.$$

Also,

$$\lim_{x \to +\infty} (q \circ \mathbf{g})(x) \leq \sum_{j=1}^{k} |\alpha_j| \lim_{x \to +\infty} (q \circ \mathbf{f}_j)(x),$$

provided all the limits appearing in the inequality exist.

Proof. Let D be the set of all intervals of the form $[a, +\infty)$ and let $\preccurlyeq$ be $\supseteq$. Then $(D, \preccurlyeq)$ is a directed set by Proposition 1.14.4c. It is clearly non-empty. Let $\omega\colon D \to \wp(U)$ be defined by $\omega([a, +\infty) = U \cap [a, +\infty)$. Then ω is a monotone function from the directed set $(D, \preccurlyeq)$ to to the directed set $(\wp(U), \supseteq)$. $\varnothing \notin \omega_*(D)$ because of our assumption that for all $a \in \mathbf{R}$ there is an $x \in U$ with $x \geq a$. By Theorem 1.16.2

$$\lim_{\omega} f = z$$

for a function f from U to Y if and only if for all $Z \in \mathcal{O}(z)$ there is an $[a, +\infty \in D$ such that if $x \in \omega([a, +\infty))$ then $f(x) \in Z$. i.e. if and only if for all $Z \in \mathcal{O}(z)$ there is an $a \in \mathbf{R}$ such that if $x \in U$ and $x \geq a$ then $f(x) \in Z$. By the definition above this is the same as

$$\lim_{x \to +\infty} f(x) = z.$$

We can therefore apply Theorem 1.16.3, replacing $\lim_{\omega}$ everywhere by $\lim_{x \to +\infty}$. $\square$

This theorem applies in particular to $U = \mathbf{N}$, since clearly for any $a \in \mathbf{R}$ there is an $n \in \mathbf{N}$ such that $n \geq a$. A function with domain $\mathbf{N}$ is generally called a *sequence*. For historical reasons they are typically written with a subscript notation instead of the usual functional notation, but they are functions and the theorem applies to them.

$(\mathbf{N}, \leq)$ is a directed set. We can generalise the idea of a sequence from functions with domain $\mathbf{N}$ to functions with domain an arbitrary directed set.

Definition 1.18.3. A *net* is a function whose domain is a directed set. If $(D, \preccurlyeq)$ is directed set and f is a function, i.e. a net, from D to a topological space $(Y, \mathcal{T})$ then we say that $z \in Y$ is the limit of the net f, written

$$\lim f = z$$

if for all $Z \in \mathcal{O}(z)$ there is an $a \in D$ such that if $b \in D$ and $a \preccurlyeq b$ then $f(b) \in Z$.

The properties of limits of nets should by now be easy to guess.

Theorem 1.18.4. *Suppose that $(D, \preccurlyeq)$ is a non-empty directed set and that $(Y, \mathcal{T})$ is a topological space.*

(a) If f is a function from D to Y and $\mathcal{T}$ is Hausdorff then there is at most one $z \in Y$ such that $\lim f = z$.

(b) Suppose $Y = \mathbf{R}$ and $\mathcal{T}$ is the usual topology on $\mathbf{R}$. If f and g are functions from D to Y such that $f(x) \leq g(x)$ for all $x \in D$ and $\lim f$ and $\lim g$ exist then $\lim f \leq \lim g$.

(c) Suppose Y is a vector space and $\mathcal{T}$ is the topology associated to a norm q on Y. Suppose

$$\mathbf{g} = \sum_{j=1}^{k} \alpha_j \mathbf{f}_j$$

where $\mathbf{f}_1, \ldots \mathbf{f}_k$ are functions from D to Y, $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$, and

$$\lim \mathbf{f}_j = \mathbf{z}_j,$$

where $\mathbf{z}_1, \ldots \mathbf{z}_k \in Y$. Then

$$\lim \mathbf{g} = \sum_{j=1}^{k} \alpha_j \mathbf{z}_j.$$

Also,

$$\lim(q \circ \mathbf{g}) \leq \sum_{j=1}^{k} |\alpha_j| \lim(q \circ \mathbf{f}_j),$$

provided all the limits appearing in the inequality exist.

Proof. We take $U = D$ and take $\omega \colon D \to D$ to be the function

$$\omega(a) = \{b \in D \colon a \preccurlyeq b\}$$

as in Proposition 1.14.4. Then ω is a monotone function from the directed set $(D, \preccurlyeq)$ to the directed set $(\wp(D), \supseteq)$. $\varnothing \notin \omega_*(D)$ because $\omega(a)$ always contains at least the element a. By Theorem 1.16.2

$$\lim_{\omega} f = z$$

for a function f from D to Y if and only if for all $Z \in \mathcal{O}(z)$ there is an $a \in D$ such that if $b \in \omega(a)$ then $f \in Z$. i.e. if and only if for all $Z \in \mathcal{O}(z)$ there is an $a \in \mathbf{R}$ such that if $b \in D$ and $a \preccurlyeq b$ then $f(b) \in Z$. By the definition above this is the same as

$$\lim f = z.$$

We can therefore apply Theorem 1.16.3, replacing $\lim_{\omega}$ everywhere by lim. $\square$

Another familiar notion of limit from Real Analysis is the one-sided limit.

Definition 1.18.5. If $f \colon \mathbf{R} \to \mathbf{R}$ is a function, $w \in \mathbf{R}$ and $z \in \mathbf{R}$ then we say that z is the *limit* of f from the right at w, written

$$\lim_{x \searrow w} f(x) = z,$$

if for all $\epsilon > 0$ there is a $\delta > 0$ such that whenever $0 < x - w < \delta$ we have $|f(x) - z| < \epsilon$.

This can be generalised to cover functions defined on subsets $U \subseteq \mathbf{R}$ with values in a topological space $(Y, \mathcal{T})$. One can then prove properties similar to those already proved for other types of limit using Theorem 1.16.3. The set D in this case is $(w, +\infty)$. The relation $\preccurlyeq$ is $\geq$. The function ω is $\omega(b) = (w, b)$. The details, including the condition necessary to ensure that $\varnothing \notin \omega_*(D)$, are left as an exercise. Instead we move on to two classes limits which are not normally described as limit, but to which Theorem 1.16.3 applies.

1.19 Sums as limits

There are a number of constructions in Real Analysis which are not limits in the ordinary sense but which can be fit into the more general framework of the last few sections. One of these is the theory of sums, which we'll consider in this section and another is the theory of Riemann integrals, to be considered in the next section.

The classical definition of the sum of a function defined on $\mathbf{N}$ is as the limit of a sequence of partial sums. This definition uses the order structure of the natural numbers in an essential way, but we'll often want to consider sums of functions defined on general sets. The following definition covers that case.

Definition 1.19.1. Suppose S is a set, (Y, q) is a normed vector space and $\mathbf{u}\colon S \to Y$ is a function. We say that $\mathbf{z} \in Y$ is the sum of $\mathbf{u}$ over S, written

$$\sum_{S} \mathbf{u} = \mathbf{z}$$

or

$$\sum_{x \in S} \mathbf{u}(x) = \mathbf{z},$$

if for all $\epsilon > 0$ there is a finite subset $F \in \wp(S)$ such that whenever $G \in \wp(S)$ is finite and $F \subseteq G$ we have

$$q\left(\mathbf{z} - \sum_{x \in G} \mathbf{u}(x)\right) < \epsilon.$$

There are a few things to note about this definition. First of all, it is not circular. It defines sums in terms of sums, but the sums that a sum is defined in terms of are all finite, and finite sums in a vector space are already defined as part of the definition of a vector space. Or at least the sum of two elements is defined, but we can define finite sums inductively and show that they have the expected properties.

Second, the definition is consistent, in the sense that when applied to finite sums, which, as we've just noted, are already defined, the new definition agrees with the old one, since we can just take $F = S$.

Third, the definition is *not* fully consistent with the existing definition of sums over the natural numbers. If the sum

$$\sum_{n \in \mathbf{N}} \mathbf{u}(n) = z$$

in the sense of the definition above then the sum

$$\sum_{n=0}^{\infty} \mathbf{u}(n),$$

defined as usual as

$$\lim_{n \to \infty} \sum_{j=0}^{n} \mathbf{u}(j),$$

also exists and

$$\sum_{n=0}^{\infty} \mathbf{u}(n) = \mathbf{z}.$$

The converse, however, is false. There's a very good reason for this. It's well known that changing the order of summation for sums defined in the traditional manner can change the value of the sum, even in the case $Y = \mathbf{R}$. In other words, if $\beta\colon \mathbf{N} \to \mathbf{N}$ is a bijection then $\sum_{n=0}^{\infty} \mathbf{u}(n)$ and $\sum_{n=0}^{\infty} \mathbf{u}(\beta(n))$ can have different values. Definition 1.19.1 makes no reference to any order structure on S and so

$$\sum_{n \in \mathbf{N}} \mathbf{u}(n) = \sum_{n \in \mathbf{N}} \mathbf{u}(\beta(n))$$

and, more generally,

$$\sum_{x \in S} \mathbf{u}(x) = \sum_{x \in S} \mathbf{u}(x)$$

for any set S. There is no way to define sums which has the property above and agrees with the traditional limit of partial sums definition for sums over the natural numbers. There is one important case though where the equation

$$\sum_{n=0}^{\infty} \mathbf{u}(n) = \sum_{n=0}^{\infty} \mathbf{u}(\beta(n))$$

is known to hold: that of absolutely convergent series in the case $(Y, q) = (\mathbf{R}^m, \|\ \ \|)$. For such series the sum over the natural numbers in the sense of Definition 1.19.1 exists and is equal to the sum in the traditional sense.

Sums have many of the same properties as limits.

Theorem 1.19.2. *Suppose S is a set and (Y, q) is a normed vector space.*

(a) If $\mathbf{u}$ is a function from S to Y then there is at most one $\mathbf{z} \in Y$ such that $\sum_{x \in S} \mathbf{u}(x) = \mathbf{z}$.

(b) Suppose $Y = \mathbf{R}$ and $q = |\ \ |$. If u and v are functions from S to Y such that $u(x) \leq v(x)$ for all $x \in S$ and $\sum_{x \in S} u(x)$ and $\sum_{x \in S} v(x)$ exist then $\sum_{x \in S} u(x) \leq \sum_{x \in S} v(x)$.

(c) Suppose

$$\mathbf{v} = \sum_{j=1}^{k} \alpha_j \mathbf{u}_j$$

where $\mathbf{u}_1, \ldots \mathbf{u}_k$ *are functions from* S *to* Y, $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$, *and*

$$\sum_{x\in S} \mathbf{u}_j(x) = \mathbf{z}_j,$$

where $\mathbf{z}_1, \ldots \mathbf{z}_k \in Y$. *Then*

$$\sum_{x\in S} \mathbf{v}(x) = \sum_{j=1}^{k} \alpha_j \mathbf{z}_j.$$

Also,

$$\sum_{x\in S} (q \circ \mathbf{v})(x) \leq \sum_{j=1}^{k} |\alpha_j| \sum_{x\in S} (q \circ \mathbf{u}_j)(x),$$

provided all the sums appearing in the inequality exist.

Proof. We'd like to derive this from Theorem 1.16.3, but the statement of the theorem doesn't make the choice of $(D, \preccurlyeq)$, U, ω or f obvious. In fact we take D to be the set of finite subsets of S and $\preccurlyeq$ to be $\subseteq$. $(D, \preccurlyeq)$ is a directed set by Proposition 1.14.2g. D is non-empty because $\varnothing \in D$. U is also the set of finite subsets of S and $\omega\colon D \to \wp(U)$ is defined by

$$\omega(F) = \{G \in U : F \subseteq G\}.$$

This ω is monotone. $\varnothing \notin \omega_*(D)$ because $F \in \omega(F)$ for all $F \in D$. To a function $\mathbf{u}\colon S \to Y$ we associate a function $\mathbf{f}\colon U \to Y$ by

$$\mathbf{f}(G) = \sum_{x\in G} \mathbf{u}(x).$$

By Definition 1.19.1

$$\sum_{x\in S} \mathbf{u}(x) = \mathbf{z}$$

if and only for all $\epsilon > 0$ there is a finite subset $F \in \wp(S)$ such that whenever $G \in \wp(S)$ is finite and $F \subseteq G$ we have

$$q\left(\mathbf{z} - \sum_{x\in G} \mathbf{u}(x)\right) < \epsilon.$$

With the choices introduced above, we can restate this as

$$\sum_{x\in S} \mathbf{u}(x)$$

if and only for all $\epsilon > 0$ there is an $F \in D$ such that whenever $G \in \omega(F)$ we have

$$\mathbf{f}(G) \in B(\mathbf{z}, \epsilon).$$

If for all $\epsilon > 0$ there is an $F \in D$ such that whenever $G \in \omega(F)$ we have

$$\mathbf{f}(G) \in B(\mathbf{z}, \epsilon)$$

then for all $P \in \mathcal{O}(\mathbf{z})$ there is there is an $F \in D$ such that whenever $G \in \omega(F)$ we have

$$\mathbf{f}(G) \in P$$

because $B(\mathbf{z}, \epsilon) \in \mathcal{O}(\mathbf{z})$. Conversely, if for all $P \in \mathcal{O}(\mathbf{z})$ there is there is an $F \in D$ such that whenever $G \in \omega(F)$ we have

$$\mathbf{f}(G) \in P$$

then for all $\epsilon > 0$ there is an $F \in D$ such that whenever $G \in \omega(F)$ we have

$$\mathbf{f}(G) \in B(\mathbf{z}, \epsilon)$$

because and $P \in \mathcal{O}(\mathbf{z})$ contains $B(\mathbf{z}, \epsilon)$ for some $\epsilon > 0$. So we can restate the definition of the sum as

$$\sum_{x\in S} \mathbf{u}(x) = \mathbf{z}$$

if and only if for all $P \in \mathcal{O}(\mathbf{z})$ there is there is an $F \in D$ such that whenever $G \in \omega(F)$ we have

$$\mathbf{f}(G) \in P.$$

By Theorem 1.16.2 this is equivalent to

$$\lim_{\omega} \mathbf{f} = \mathbf{z}.$$

So $\sum_{x\in S} \mathbf{u}(x)$ if and only if $\lim_{\omega} \mathbf{f} = \mathbf{z}$. The statements of this theorem then follow from those of Theorem 1.16.3. $\square$

It would have been a bit easier to derive this theorem from Theorem 1.18.4 rather than from Theorem 1.16.3. Sums in the sense of Definition 1.19.1 aren't limits of *sequences* of finite sums but they are limits of *nets* of finite sums. I've chosen the somewhat longer route above though because it more closely parallels what we'll do with integrals.

1.20 Riemann integrals as limits

Doing this example in detail would require more notation and terminology than I want to introduce here so I will give only a sketch of the procedure.

Definition 1.20.1. $\mathcal{P} \in \wp(\wp(\mathbf{R}))$ is called a *partition* of the interval I if

(a) $\mathcal{P}$ is finite,

(b) every element of $\mathcal{P}$ is an interval,

(c) $\bigcup_{J \in \mathcal{P}} = I$, and

(d) for every distinct $J, K \in \mathcal{P}$, $J \cap K = \varnothing$ unless $J = K$.

A more usual way to express this is to give the intervals in terms of their endpoints and to order the elements of $\mathcal{P}$. For example, if $I = [a, b)$ then $\mathcal{P} = \{J_1, \ldots, J_m\}$ is a partition of I where $J_k = [c_k, d_k)$ and $c_1 = a$, $d_k = b$ and $d_k = c_{k+1}$ for $1 \leq k < m$. That's easier to visualise, but less convenient for proving theorems.

One minor technical note is that it's more convenient to allow empty intervals as elements of a partition than to exclude them. This will require treating empty intervals as a special case in a few places but will remove the need to check that intervals are non-empty in more places.

Every interval I has at least one partition, namely the *trivial partition* $\mathcal{P} = \{I\}$.

A fact which is geometrically obvious but trickier to prove than one might expect is that the sum of the lengths of the intervals in a partition of I is equal to the length of I. For this section I'll just assume that fact without proof.

Definition 1.20.2. The partition $\mathcal{Q}$ of I is called a *refinement* of the partition $\mathcal{P}$ of I if for each $K \in \mathcal{Q}$ there is a $J \in \mathcal{P}$ such that $J \supseteq K$.

Note that the definition only requires that there exists such a J, but this J is in fact unique if K is non-empty. The subset of $\mathcal{Q}$ consisting of those K such that $J \supseteq K$ is a partition of J.

Definition 1.20.3. If $\mathcal{P}$ and $\mathcal{Q}$ are partitions of I then their *common refinement* is the set of sets of the form $J \cap K$ where $J \in \mathcal{P}$ and $K \in \mathcal{Q}$.

The name is justified by the following proposition.

Proposition 1.20.4. *The common refinement of two partitions is a partition and is a refinement of each of them.*

Proof. Suppose $\mathcal{R}$ is the common refinement of the partitions $\mathcal{P}$ and $\mathcal{Q}$ of the interval I. Define $i \colon \mathcal{P} \times \mathcal{Q} \to \mathcal{R}$ by $i(J, K) = J \cap K$. $\mathcal{R}$ was defined as the image of i, so i is a surjection. The product of finite sets is finite and the image of a finite set under a surjection is finite, so $\mathcal{R}$ is finite. The intersection of intervals is an interval, so every element of $\mathcal{R}$ is an interval.

$$\begin{aligned}
\bigcup_{L \in \mathcal{R}} L &= \bigcup_{J \in \mathcal{P}, K \in \mathcal{Q}} J \cap K \\
&= \left(\bigcup_{J \in \mathcal{P}} \right) \cap \left(\bigcup_{K \in \mathcal{Q}} \right) \\
&= I \cap I = I.
\end{aligned}$$

If $L_1, L_2 \in \mathcal{R}$ then $L_1 = J_1 \cap K_1$ and $L_2 = J_2 \cap K_2$ where $J_1, J_2 \in \mathcal{P}$ and $K_1, K_2 \in \mathcal{Q}$. Then

$$L_1 \cap L_2 = (J_1 \cap K_1) \cap (J_2 \cap K_2) = (J_1 \cap J_2) \cap (K_1 \cap K_2)$$

and the sets $J_1 \cap J_2$ and $K_1 \cap K_2$ are each empty unless $J_1 = J_2$ and $K_1 = K_2$, so their intersection $L_1 \cap L_2$ is empty unless $L_1 = L_2$. This shows that $\mathcal{R}$ is a partition of I.

$\mathcal{R}$ is a refinement of both $\mathcal{P}$ and $\mathcal{Q}$ because if $L \in \mathcal{R}$ then there are $J \in \mathcal{P}$ and $K \in \mathcal{R}$ such that $L = J \cap K$. Then $J \supseteq L$ and $K \supseteq L$. $\square$

Corollary 1.20.5. *The set of partitions of an interval together with the relation $\preccurlyeq$, where $\mathcal{P} \preccurlyeq \mathcal{Q}$ if $\mathcal{Q}$ is a refinement of $\mathcal{P}$ is a directed set.*

Definition 1.20.6. A *system of weights* is a finite subset $F \subseteq \mathbf{R}$ together with a function $w\colon F \to \mathbf{R}$ such that $w(x) \geq 0$ for each $x \in F$. The weights are said to be *compatible* with a partition $\mathcal{P}$ of an interval I if for each $J \in \mathcal{P}$ the length of J is equal to $\sum_{x \in F \cap J} w(x)$.

Proposition 1.20.7. *If $\mathcal{Q}$ is a refinement of $\mathcal{P}$ and (F, w) is a system of weights compatible with $\mathcal{Q}$ then it's also compatible with $\mathcal{P}$.*

Proof. For each $J \in \mathcal{P}$ we consider the subset $\mathcal{Q}_J$ of $\mathcal{Q}$ consisting of those K such that $J \supseteq K$. As noted previously, $\mathcal{Q}_J$ is a partition of J. For each $x \in J$ there is exactly one $K \in \mathcal{Q}_J$ such that $x \in K$. Therefore for each $x \in F \cap J$ there is one $K \in \mathcal{Q}_J$ such that $x \in F \cap K$. It follows that

$$\sum_{x \in F \cap J} w(x) = \sum_{K \in \mathcal{Q}_J} \sum_{x \in F \cap K} w(x).$$

The inner sum is the length of K because w is compatible with $\mathcal{Q}$ so the outer sum is the sum of the lengths of the intervals in $\mathcal{Q}_J$, which is just the length of J because $\mathcal{Q}_J$ is a partition of J. So $\sum_{x \in F \cap J} w(x)$ is the length of J for all $J \in \mathcal{P}$, which is the definition of a compatible system of weights for $\mathcal{P}$. □

An immediate consequence of the preceding proposition is the following:

Proposition 1.20.8. *Let D be the set of partitions of an interval I. and U the set of systems of weights. Let $\omega\colon D \to \wp(U)$ be the function which takes each partition to the set of systems of weights compatible with it. Then ω is monotone when considered as a function from the directed set $(D, \preccurlyeq)$ to $(\wp(U), \supseteq)$, where $\preccurlyeq$ is the refinement relation from Corollary 1.20.5.*

Proof. If $\mathcal{P} \preccurlyeq \mathcal{Q}$ then $\mathcal{Q}$ is a refinement of $\mathcal{Q}$. By the preceding proposition every system of weights compatible with $\mathcal{Q}$ is also compatible with $\mathcal{P}$. In other words, the set of systems of weights compatible with $\mathcal{P}$ is a superset of the set of weights compatible with $\mathcal{Q}$. □

Definition 1.20.9. If $\mathbf{u}$ is a function from an interval I to a normed vector space (Y, q) and (F, w) is a system of weights then the *Riemann sum* of $\mathbf{u}$ with respect to (F, w) over the interval I is

$$\sum_{x \in F \cap I} w(x)\mathbf{u}(x).$$

We now have everything we need to define integrals.

Definition 1.20.10. Suppose I is an interval, (Y, q) is a normed vector space and $\mathbf{u}\colon I \to Y$ is a function. We say that $\mathbf{z} \in Y$ is the *Riemann integral* of $\mathbf{u}$ over I, written

$$\int_{x \in I} \mathbf{u}(x)\, dx = \mathbf{z}$$

or just

$$\int_I \mathbf{u} = \mathbf{z},$$

if for every $\epsilon > 0$ there is a partition $\mathcal{P}$ of I such that for any partition $\mathcal{Q}$ and any system of weights (F, w) compatible with $\mathcal{Q}$ the Riemann sum of $\mathbf{u}$ with respect to (F, w) over I is an element of $B(\mathbf{z}, \epsilon)$.

An alternate characterisation uses limits in the sense of Definition 1.16.1.

Proposition 1.20.11. *If $\mathbf{u}$ is a function from an interval I to a normed vector space (Y, q) then*

$$\int_{x \in I} \mathbf{u}(x)\, dx = \mathbf{z}$$

if and only if

$$\lim_{\omega} \mathbf{f} = \mathbf{z}$$

where ω is the function from Proposition 1.20.8 and $\mathbf{f}\colon U \to Y$ is the function which takes the system of weights (F, w) to the Riemann sum of $\mathbf{u}$ with respect to (F, w).

Proof. By Definition 1.20.10

$$\int_{x \in I} \mathbf{u}(x)\, dx$$

if and only for all $\epsilon > 0$ there is a partition $\mathcal{P}$ such that whenever $\mathcal{Q}$ is a refinement of $\mathcal{P}$ and (F, w) is compatible with $\mathcal{Q}$ we have

$$\sum_{x \in F \cap I} w(x)\mathbf{u}(x) \in B(\mathbf{z}, \epsilon).$$

With notation as in Proposition 1.20.8 we can restate this as
$$\int_{x\in I} \mathbf{u}(x)\,dx = \mathbf{z}$$
if and only for all $\epsilon > 0$ there is an $\mathcal{P} \in D$ such that whenever $\mathcal{P} \preccurlyeq \mathcal{Q}$ $(F, w) \in \omega(\mathcal{Q})$ we have
$$\sum_{x\in F\cap I} w(x)\mathbf{u}(x) \in B(\mathbf{z}, \epsilon).$$
If $\int_{x\in I} \mathbf{u}(x)\,dx = \mathbf{z}$ and $P \in \mathcal{O}(\mathbf{z})$ then there is an $\epsilon > 0$ such that $B(\mathbf{z}, \epsilon) \subseteq P$ and hence
$$\sum_{x\in F\cap I} w(x)\mathbf{u}(x) \in P.$$
By Theorem 1.16.2 this implies
$$\lim_{\omega} = \mathbf{z}.$$
So if $\int_{x\in I} \mathbf{u}(x)\,dx = \mathbf{z}$ then $\lim_{\omega} = \mathbf{z}$.

Conversely, if $\lim_{\omega} \mathbf{f} = \mathbf{z}$ then for every $P \in \mathcal{O}(z)$ there is a $\mathcal{P} \in D$ such that whenever $\mathcal{P} \preccurlyeq \mathcal{Q}$ $(F, w) \in \omega(\mathcal{Q})$ we have
$$\sum_{x\in F\cap I} w(x)\mathbf{u}(x) \in P,$$
again by Theorem 1.16.2.
$$\lim_{\omega} = \mathbf{z}.$$
$B(\mathbf{z}, \epsilon) \in \mathcal{O}(\mathbf{z})$ for every $\epsilon > 0$. So for every $\epsilon > 0$ there is a $\mathcal{P} \in D$ such that whenever $\mathcal{P} \preccurlyeq \mathcal{Q}$ $(F, w) \in \omega(\mathcal{Q})$ we have
$$\sum_{x\in F\cap I} w(x)\mathbf{u}(x) \in B(\mathbf{z}, \epsilon).$$
In other words, for all $\epsilon > 0$ there is a partition $\mathcal{P}$ such that whenever $\mathcal{Q}$ is a refinement of $\mathcal{P}$ and (F, w) is compatible with $\mathcal{Q}$ we have
$$\sum_{x\in F\cap I} w(x)\mathbf{u}(x) \in B(\mathbf{z}, \epsilon).$$
So if $\lim_{\omega} = \mathbf{z}$ then $\int_{x\in I} \mathbf{u}(x)\,dx = \mathbf{z}$. □

Because this proposition expresses integrals directly in terms of limits in the sense of Definition 1.16.1 the following theorem becomes very straightforward to prove.

Theorem 1.20.12. *Suppose I is an interval and (Y, q) is a normed vector space.*

(a) If $\mathbf{u}$ is a function from I to Y then there is at most one $\mathbf{z} \in Y$ such that $\int_{x\in I} \mathbf{u}(x)\,dx = \mathbf{z}$.

(b) Suppose $Y = \mathbf{R}$ and $q = |\ \ |$. If u and v are functions from I to Y such that $u(x) \leq v(x)$ for all $x \in I$ and $\int_{x\in I} u(x)\,dx$ and $\int_{x\in I} v(x)\,dx$ exist then $\int_{x\in I} u(x)\,dx \leq \int_{x\in I} v(x)\,dx$.

(c) Suppose
$$\mathbf{v} = \sum_{j=1}^{k} \alpha_j \mathbf{u}_j$$
where $\mathbf{u}_1, \ldots \mathbf{u}_k$ are functions from I to Y, $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$, and
$$\int_{x\in I} \mathbf{u}_j(x)\,dx = \mathbf{z}_j,$$
where $\mathbf{z}_1, \ldots \mathbf{z}_k \in Y$. Then
$$\int_{x\in I} \mathbf{v}(x)\,dx = \sum_{j=1}^{k} \alpha_j \mathbf{z}_j.$$
Also,
$$\int_{x\in I} (q \circ \mathbf{v})(x)\,dx \leq \sum_{j=1}^{k} |\alpha_j| \int_{x\in I} (q \circ \mathbf{u}_j)(x)\,dx,$$
provided all the integrals appearing in the inequality exist.

Proof. At this point there is nothing left to prove. We just use Proposition 1.20.11 to translate all statements about integrals in the hypotheses of the theorem to statements about limits, which gives us the hypotheses of the corresponding parts of Theorem 1.16.3. That theorem then gives us conclusions about limits, which then use Proposition 1.20.11 again to translate into statements about integrals, which are exactly the conclusions of this theorem. □

2 Sets and Cardinality

The usual axioms of set theory are chosen to be minimal rather than comprehensible. From those axioms one then proves a number of elementary properties, which are often more intuitive than the axioms they're derived from. These are the elementary properties of unions, intersections, relative complements, power sets, ordered pairs, (finite) Cartesian products, etc. I'm going to assume those are familiar enough and will not review them here.

2.1 Injections, surjections, bijections

A function $f: X \to Y$ is called an *injection* if $f(x) = f(y)$ implies $x = y$. It's called a *surjection* for all $y \in Y$ there is an $x \in X$ such that $f(x) = y$. In terms of the image f is a surjection if and only if $f_*(X) = Y$. It's called a *bijection* if it is both an injection and a surjection. If f is a bijection then it has an inverse, i.e. a function $g: Y \to X$ such that $f \circ g$ is the identity on Y and $g \circ f$ is the identity on X. The function g is defined by saying that $g(y)$ is the unique $x \in X$ such that $f(x) = y$. There's at least one such x because f is a surjection and at most one such x because f is an injection. A function g which satisfies the first of these conditions but not necessarily the second is called a right inverse of f while one which satisfies the second but not necessarily the first is called a left inverse.

The composition of two injections is an injection, the composition of two surjections is a surjection and the composition of two bijections is a bijection. These facts follow immediately from the definitions.

The following three propositions relate injections, surjections and bijections to the various types of inverse.

Proposition 2.1.1. *If X is non-empty then $f: X \to Y$ is an injection if and only if there is a function $g: X \to Y$ such that $g \circ f$ is the identity on X.*

In other words g is a left inverse to f, although it won't be a right inverse unless f is also a surjection.

Proof. The "if" part is proved as follows. Suppose there is such a g. If $f(x) = f(y)$ then $g(f(x)) = g(f(y))$. But $g \circ f$ is the identity so $x = y$. So if $f(x) = f(y)$ then $x = y$. In other words, f is an injection.

The "only if" is proved as follows. Suppose that f is an injection. If $z \in Y$ then there is at most one $x \in X$ such that $f(x) = z$ since if $f(y) = z$ then $f(x) = f(y)$ and so $x = y$. For any $z \in Y$ for which there is such an x we define $g(z)$ to be this x. By assumption X is non-empty. In other words there is a $w \in X$. For any $z \in Y$ for which there is no $x \in X$ with $f(x) = z$ we define $g(z) = w$. Then $g(f(x)) = x$ for all $x \in X$ so $g \circ f$ is the identity. □

Proposition 2.1.2. *$f: X \to Y$ is a surjection if there is a $g: Y \to X$ such that $f \circ g$ is the identity on Y and $g \circ f$ is the identity*

Proof. If $y \in Y$ then $y = f(x)$ where $x = g(y)$. □

Proposition 2.1.3. *$f: X \to Y$ is a bijection if and only if there is a $g: Y \to X$ such that $f \circ g$ is the identity on Y and $g \circ f$ is the identity on X.*

Proof. The "only if" part was proved immediately after the definition of bijection. For the "if" part we combine the preceding two propositions. Suppose that $f \circ g$ is the identity on Y and $g \circ f$ is the identity on X. By Proposition 2.1.1 f is an injection. By Proposition 2.1.2 it is a surjection. It is therefore a bijection.

There's one small gap in this proof. If X is empty we cannot apply Proposition 2.1.1. But if X is empty then there can't be a function $g: Y \to X$ unless Y is also empty. In this case f is vacuously an injection and a surjection, and so is a bijection. □

You might have noticed something missing in one of the theorems above. All of these statements were "if and only if" except for Proposition 2.1.2 which has merely "if". Can this be improved to 'if and only if"? This is a question we'll return to in a later section.

2.2 Finite sets

If you already have the natural numbers available then it's tempting to define the finite sets to be those for which there's a bijection to a set of the form $\{1, \ldots, n\}$. This reverses our usual intuition though, which is that natural numbers arise from the need to count things, i.e. that they arise as the cardinalities of finite sets, and that the arithmetic operations on them reflect natural operations on sets, with addition representing disjoint unions and multiplication representing Cartesian products. If you want to view the natural numbers this way then you need a pre-existing notion of finiteness of sets. The standard one is that a set X is *finite* if and only if every injection from X to X is a surjection. A set is called *infinite* if and only if it is not finite. The set $\mathbf{N}$ is infinite, for example, because the function $f(n) = n + 1$ is an injection but is not a surjection.

Proposition 2.2.1. *If $A \subseteq B$ and B is finite then A is finite.*

Proof. This is equivalent to the statement that if A is infinite then B is infinite, and it's that statement which we'll prove. Suppose then that $A \subseteq B$ and A is infinite, i.e. that there there is an injection $f\colon A \to A$ which is not a surjection. We can then defined $g\colon B \to B$ by $g(x) = f(x)$ if $x \in A$ and $g(x) = x$ if $x \in B \setminus A$. This g is also then an injection but not a surjection. So B is infinite. $\square$

A corollary of this is that the intersection of any collection of sets is finite if any one of them is, since the intersection must be a subset of that one.

Proposition 2.2.2. *The union of any finite set of finite sets is finite.*

This is harder prove than one might expect. It's painful enough if you've defined finiteness in terms of a bijection with a set of the form $\{1, \ldots, n\}$ and have the basic properties of the natural numbers available but it's particularly unpleasant to prove from the definition given above, i.e. that a set is finite if every injection is a surjection. Giving a proof here would take us too far into axiomatic set theory, so we'll just take this as given. In fact I already did at several points in the previous chapter.

Proposition 2.2.3. *The union any finite set of finite sets is finite is that the Cartesian product of two finite sets is finite.*

Proof. Indeed we can write the Cartesian product as

$$X \times Y = \bigcup_{x \in X} \{x\} \times Y.$$

There's an obvious bijection from Y to $\{x\} \times Y$, namely the one which takes y to (x, y), and Y is finite so each of the sets $\{x\} \times Y$ is finite. So $X \times Y$ is a union of finitely many finite sets and hence is finite. $\square$

Even though I followed the standard practice from the foundations of mathematics of defining finiteness independently of the natural numbers we are not in fact going to prove the existence of the set of natural numbers but simply assume it, along with all its familiar properties. Since we have the natural numbers available, so we can state and prove the following proposition, which gives an alternate criterion for finiteness.

Proposition 2.2.4. *A set X is infinite if and only if there is an injection $f\colon \mathbf{N} \to X$.*

Proof. We first prove the "if" part and then the "only if" part.

Suppose X is infinite. By the definition of infinite, there is an injection $h\colon X \to X$ which is not a surjection. Choose $y \in X$ such that there is no $x \in X$ with $h(x) = y$. Define $f\colon \mathbf{N} \to X$ inductively by

$$f(0) = y, \qquad f(k + 1) = h(f(k)).$$

We can prove by induction on m the statement that if $i, j \leq m$ and $f(i) = f(j)$ then $i = j$. This is clearly true for $m = 0$. For the inductive step we assume it holds for a given m and prove it with m replaced by $m+1$. Assume then that $i, j \leq m+1$ and $f(i) = f(j)$. By the inductive hypothesis if $i, j \leq m$ and $f(i) = f(j)$ then $i = j$. If we have $i, j \leq m + 1$ but not $i, j \leq m$ then at least one of i, j is equal to $m + 1$. Suppose that $i \leq m + 1$, $j = m + 1$ and $f(i) = f(j)$. Then

$$f(j) = h(f(m)).$$

We know that $i \neq 0$ because $f(0) = y$ and there is no x such that $f(x) = y$. Therefore $i = l + 1$ for some $l \leq m$. From $f(i) = f(j)$ we see that

$$f(l + 1) = f(m + 1)$$

and hence

$$h(f(l)) = h(f(m)).$$

h in an injection so $f(l) = f(m)$. But $l, m \le m$ so the inductive hypothesis means that $l = m$, and therefore $i = j$. The argument for the case $i = m+1$, $j \le m+1$ is similar. So if $i, j \le m+1$ and $f(i) = f(j)$ then $i = j$, completing the induction. For every $i, j \in \mathbf{N}$ there is an m such that $i, j \le m$, namely $m = \max(i, j)$. So we find that for all $i, j \in \mathbf{N}$ if $f(i) = f(j)$ then $i = j$. In other words, f is an injection. So we have an injection $f\colon \mathbf{N} \to X$.

Suppose that there is an injection $f\colon \mathbf{N} \to X$. We define $h\colon X \to X$ as follows. f is an injection so there is for each x at most one m such that $f(x) = m$. If there is such an m we define

$$h(x) = f(m+1).$$

If there isn't then we define $h(x) = x$. Then h is an injection. In other words, if $h(x) = h(y)$ then $x = y$. This is proved by considering the possible cases for x and y. If there is no m such that $x = f(m)$ and no n such that $y = f(n)$ then $h(x) = x$ and $h(y) = y$ so $x = y$. If there is no m such that $x = f(m)$ but there is an n such that $y = f(n)$ then $h(x) = x$ and $h(y) = f(n+1)$ so $x = f(n+1)$. But this is impossible because x was not in the image of f. Similarly, the case where there is an m such that $x = f(m)$ but there is no n such that $y = f(n)$ does not occur. In the last case, where $x = f(m)$ and $y = f(n)$, we have

$$f(m+1) = h(x) = h(y) = f(n+1).$$

f is an injection so

$$m + 1 = n + 1.$$

But then $m = n$ so

$$x = f(m) = f(n) = y.$$

In all cases $h(x) = h(y)$ implies $x = y$ so h is an injection.

Suppose $x \in X$. If there is no m such that $x = f(m)$ then $h(x) = x$ so there is no m such that $h(x) = f(m)$. In particular, $h(x) \ne f(0)$. If there is an m such that $x = f(m)$ then

$$h(x) = f(m+1).$$

f is an injection and $m + 1 \ne 0$ so $h(x) \ne f(0)$. So for all $x \in X$ we have $h(x) \ne f(0)$. h is therefore not a surjection.

We've now constructed an injection $h\colon X \to X$ which is not a surjection. Therefore X is infinite. $\square$

2.3 Equivalence classes

An *equivalence relation* on a set X is a relation $\sim$ with the following three properties.

(a) For all $x \in X$, $x \sim x$.

(b) For all $x, y \in X$, if $x \sim y$ then $y \sim x$.

(c) For all $x, y, z \in X$, if $x \sim y$ and $y \sim z$ then $x \sim z$.

There are two trivial ways to construct an equivalence relation on a set. One is to say that $x \sim y$ for all $x, y \in X$ and the other is to say that $x \sim y$ if and only if $x = y$.

Of course we're mostly interested in non-trivial equivalence relations. For example, there's an equivalence relation on $\mathbf{N} \times \mathbf{N}$ defined by saying that

$$(a, b) \sim (c, d)$$

if and only if

$$a + d = b + c.$$

We can verify immediately that this satisfies the three conditions above.

$$(a, b) \sim (a, b)$$

because

$$a + b = b + a.$$

If

$$(a, b) \sim (c, d)$$

, i.e. if

$$a + d = b + c$$

then

$$(c, d) \sim (a, b),$$

because

$$c + b = d + a.$$

Finally, if

$$(a, b) \sim (c, d)$$

and

$$(c, d) \sim (e, f)$$

then

$$a + d = b + c$$

and

$$c + f = d + e$$

so

$$a + f + c + d = b + e + c + d$$

and hence

$$a + f = b + e.$$

In other words,

$$(a, b) \sim (e, f).$$

Here we've used the fact that addition in $\mathbf{N}$ is right cancelable, i.e. that if $x + z = y + z$ then $x = y$.

For each $x \in X$ the set

$$\{y \in X : x \sim y\}$$

is called the *equivalence class* of x under the relation $\sim$. If the relation is obvious then it's just called the equivalence class of x. We can define a function $X \to \wp(X)$ by saying that it takes each $x \in X$ to its equivalence class. The image of this function is the set of equivalence classes. In the case of the equivalence relation on the set $\mathbf{N} \times \mathbf{N}$ above the equivalence class of (a, b) is the set of (c, d) such that $a+d = b+c$. We could also write this as $d - c = b - a$ but that requires $\mathbf{Z}$ rather than $\mathbf{N}$ so that subtraction is well defined and in fact the main point of this equivalence relation is that it can be used to construct $\mathbf{Z}$ from $\mathbf{N}$. The usual construction is to *define* the integers as the set of equivalence classes in $\mathbf{N} \times \mathbf{N}$ with respect to this equivalence relation.

We can describe equivalence relations in terms of sets. Given an equivalence relation $\sim$ on a set X the set

$$E = \{(x, y) \in X \times X : x \sim y\}$$

has the three following properties.

(a) For all $x \in X$, $(x, x) \in E$.

(b) For all $x, y \in X$, if $(x, y) \in E$ then $(y, x) \in E$.

(c) For all $x, y, z \in X$, if $(x, y) \in E$ and $(y, z) \in E$ then $(x, z) \in E$.

Conversely, if $E \in \wp(X \times X)$ has the properties above and a relation $\sim$ on X is defined by saying that $x \sim y$ if and only if $(x, y) \in E$ then $\sim$ is an equivalence relation. In fact logicians *define* equivalence relations as subsets of satisfying these three conditions, much as they define functions as graphs. Mathematicians generally prefer to keep these as separate, but closely related, concepts. The sets corresponding to our two trivial equivalence relations on X are the product $X \times X$ and the diagonal

$$\Delta_X = \{(x, y) \in X \times X : x = y\}.$$

More generally, we can define a set $R \in \wp(X \times X)$ for any relation $\vdash$ via

$$R = \{(x, y) \in X \times X : x \vdash y\}$$

and we can define a relation $\vdash$ on X for any $R \in \wp(X \times X)$ by $x \vdash y$ if and only if $(x, y) \in R$.

It's often useful to compare relations.

Definition 2.3.1. We say that a relation $\propto$ is *stronger* than a relation $\vdash$ if $x \propto y$ implies $x \vdash y$ and that $\propto$ is *weaker* than $\vdash$ if $x \vdash y$ implies $x \propto y$.

These terms are slightly awkward in that any relation is both stronger than and weaker than itself. It would be more accurate to call them "at least as strong as" and "at least as weak as" rather than "stronger" and "weaker" but for reasons of brevity one uses the shorter but somewhat misleading terms.

If Q is the subset of $X \times X$ consisting of those (x, y) for which $x \propto y$ and R is the subset of $X \times X$ consisting of those (x, y) for which $x \vdash y$ then $\propto$ is stronger than $\vdash$ if and only if $Q \subseteq R$ and $\propto$ is weaker than $\vdash$ if and only if $Q \supseteq R$.

If we have a set $\mathcal{E} \in \wp(\wp(X \times X))$ with the property that each $E \in \mathcal{E}$ satisfies the three conditions

(a) For all $x \in X$, $(x, x) \in E$.

(b) For all $x, y \in X$, if $(x, y) \in E$ then $(y, x) \in E$.

(c) For all $x, y, z \in X$, if $(x, y) \in E$ and $(y, z) \in E$ then $(x, z) \in E$.

then so does their intersection.

(a) For all $x \in X$, $(x, x) \in \bigcup_{E \in \mathcal{E}} E$.

(b) For all $x, y \in X$, if $(x, y) \in \bigcup_{E \in \mathcal{E}} E$ then $(y, x) \in \bigcup_{E \in \mathcal{E}} E$.

(c) For all $x, y, z \in X$, if $(x, y) \in \bigcup_{E \in \mathcal{E}} E$ and $(y, z) \in \bigcup_{E \in \mathcal{E}} E$ then $(x, z) \in \bigcup_{E \in \mathcal{E}} E$.

We'll check this only for the last condition, since the others are similar but easier. Suppose

$$(x, y) \in \bigcap_{E \in \mathcal{E}} E$$

and

$$(y, z) \in \bigcap_{E \in \mathcal{E}} E.$$

Then for each $E \in \mathcal{E}$ we have $(x, y) \in E$ and $(y, z) \in E$. Our assumption was that the conditions above are satisfied for each $E \in \mathcal{E}$, so $(x, z) \in E$. This holds for each $E \in \mathcal{E}$ so

$$(x, z) \in \bigcap_{E \in \mathcal{E}} E.$$

In other words, For all $x, y, z \in X$, if

$$(x, y) \in \bigcap_{E \in \mathcal{E}} E$$

and

$$(y, z) \in \bigcap_{E \in \mathcal{E}} E$$

then

$$(x, z) \in \bigcap_{E \in \mathcal{E}} E.$$

In terms of relations, if we have a set of equivalence relations then there is an equivalence relation which is stronger than all of them. We define this equivalence relation by saying that two elements are equivalent with respect to this relation if and only if they are equivalent with respect to all the equivalence relations in our set.

For any set $R \in \wp(X \times X)$ we can form the set $\mathcal{E}$ of all $E \in \wp(X \times X)$ such that $R \subseteq E$ and E satisfies the three conditions above. There is at least one such E, namely $X \times X$. As we've just seen, this intersection also satisfies all three conditions. In addition, it's a superset of R. In terms of relations what we've just seen is the following.

Proposition 2.3.2. *Given any relation $\vdash$ on X there is an equivalence relation $\sim$ which is weaker than $\vdash$ but is stronger than any other equivalence relation weaker than $\vdash$.*

As an example of this construction, consider a set Z together with an injection $h \colon Z \to Z$. Define a relation $\vdash$ on X by $w \vdash h(w)$ for all $w \in Z$. Let $\sim$ be strongest equivalence relation weaker than $\vdash$, as above. The equivalence classes with respect to $\sim$ have two interesting properties. For any $w \in C$ we have $h(w) \in C$. Also, in each class C there is, for all $z \in C$, with at most one exception, a $w \in C$ such that $z = h(w)$. In other words the restriction of h to C is an injection from C to C and either is a surjection or very nearly a surjection. The first of these properties is an immediate consequence of the fact that $w \vdash h(w)$ and h is an injection. The second is more complicated. Suppose y is an element of C for which there is no $x \in C$ with $h(x) = y$. There's then also no $x \in X$ such that $h(x) = y$ because any such element would satisfy $x \vdash y$ and hence $x \sim y$. In other words, it would be an element of C. Consider the set N consisting of y and all the elements obtained from it by repeated applications of h. All the elements of N are then elements of C because they are equivalent to y with respect to $\sim$. Furthermore, all but y itself are $h(w)$ for some $w \in C$. We can define a different equivalence relation $\bowtie$ on Z by saying that $p \bowtie q$ if either both p and q are elements of N or neither is. Suppose that $p \vdash q$, i.e. that $q = h(p)$. If $p \in N$ then $q \in N$ so $p \bowtie q$. Suppose $p \notin N$ and $q \in N$. Since $q \in N$ either $q = y$ or there is an $w \in N$ such that $h(w) = q$. The former is impossible because y is not $h(x)$ for any $x \in X$ while the latter is impossible since $h(p) = q = h(w)$ implies $p = w$ but $p \notin N$ while $w \in N$. This contradiction shows that there are no p and q such that $p \vdash q$ with $p \notin N$ and $q \in N$. In other words, if $p \notin N$ then $q \notin N$, or, equivalently $p \bowtie q$. So $p \vdash q$ implies $p \bowtie q$ in all cases. In other words, $\bowtie$ is weaker than $\vdash$. $\sim$ was defined to be the strongest equivalence relation weaker then $\vdash$ so it's stronger than $\bowtie$. Therefore if $z \sim y$ then $z \bowtie y$. In other words, if $z \in C$ then either both y and z are elements of N or neither is. But we know $y \in N$ so $z \in N$. So $C \subseteq N$. Now y was the only element of N which was not of the form $h(w)$ for a $w \in N$ so it's the only element of C which is not of the form $h(w)$ for some $w \in N$. For any other $z \in C$ there is such a w, and this w is an element of C because $w \sim h(w) = z \in C$.

2.4 Cantor's Theorem

For any set X there is a natural injection from X to $\wp(X)$, namely $f(x) = \{x\}$. This f is not a surjection though. More interestingly, neither is any other function from X to $\wp(X)$. That is the content of Cantor's Theorem:

Theorem 2.4.1. *Suppose X is a set and $f \colon X \to \wp(X)$ is a function. Then f is not a surjection.*

Proof. Define

$$A = \{y \in X : y \notin f(y)\}.$$

Suppose $f(x) = A$. Is $x \in A$? If it is then $x \notin f(x)$ by the definition of A. But $f(x) = A$ and $x \in A$, so this is impossible. On the other hand, if $x \notin A$ then $x \in f(x)$ by the definition of A. But $f(x) = A$ and $x \notin A$, so this is impossible as well. So the assumption that $f(x) = A$ leads to a contradiction in either case and therefore there is no such x. $\square$

When we discuss cardinality later in this chapter we'll see that Cantor's Theorem implies the existence of different sizes of infinite sets. There are in fact more infinite numbers than finite numbers, in a sense which can be made precise.

2.5 The Schröder-Bernstein Theorem

The Schröder-Bernstein Theorem, below, is required in order to be able to compare cardinalities in a reasonable way, as we'll see when we get to that section.

Theorem 2.5.1. *Suppose $f \colon X \to Y$ and $g \colon Y \to X$ are injections. Then there is a bijection $h \colon X \to Y$.*

Proof. There are some technical difficulties related to the fact that X and Y might have elements in common. For this reason it's helpful to introduce the disjoint union of the two sets, which is roughly a set which contains a copy of X and a copy of Y, not overlapping with it, and nothing else. We do this by introducing an index set I to label which copy an element belongs to. Since we have a disjoint union of two sets we need an index set with two elements. Any set with two elements will do equally well. We could, for example, take $I = \{1, 2\}$. We could also take $I = \wp(\wp(\varnothing))$. We could take $I = \{X, Y\}$ if $X \neq Y$. More often than not we want to apply the theorem with $X \neq Y$ and these labels are particularly convenient in that case, but we have no reason to exclude the case $X = Y$ and so shouldn't make that choice here. In any case, having chosen a set I with two elements, we choose distinct a, b in I and define

$$Z = (\{a\} \times X) \cup (\{b\} \times Y).$$

This is a subset of $I \times (X \cup Y)$. Note that

$$(\{a\} \times X) \cap (\{b\} \times Y) = \varnothing$$

since $a \neq b$. So Z is the disjoint union of subsets $\{a\} \times X$ and $\{b\} \times Y$. There is an obvious bijection from X to $\{a\} \times X$, the one which takes x to (a, x), and an obvious bijection from Y to $\{b\} \times Y$, the one which takes y to (b, y). This makes precise the notion introduced earlier of a set composed of a copy of X and a copy of Y and nothing else.

Having defined Z we can now define a function $h \colon Z \to Z$ by

$$h(a, x) = (b, f(x))$$

and

$$h(b, y) = (a, g(y)).$$

This h is an injection. We can prove this as follows. If

$$h(c, s) = h(d, t)$$

for $(c, s), (d, t) \in Z$ then there are four possibilities for c and d, i.e. $c = a$ and $d = a$, $c = a$ and $d = b$, $c = b$ and $d = a$, or $c = b$ and $d = b$. The middle two possibilities are incompatible with

$$h(c, s) = h(d, t),$$

because of how h was defined. If $c = a$ and $d = a$ then

$$h(c, s) = h(d, t)$$

means

$$(a, f(s)) = (a, f(t)).$$

This requires $f(s) = f(t)$ and hence $s = t$ because f is an injection. But then

$$(c, s) = (d, t).$$

If $c = b$ and $d = b$ then

$$h(c, s) = h(d, t)$$

means

$$(b, g(s)) = (b, g(t)).$$

This requires $g(s) = g(t)$ and hence $s = t$ because g is an injection. But then $(c, s) = (d, t)$. So for any $(c, s), (d, t) \in Z$ we have that

$$h(c, s) = h(d, t)$$

implies $(c, s) = (d, t)$. In other words, h is an injection.

The next step is to introduce a relation $\vdash$ on Z such that $w \vdash z$ if and only if $z = h(w)$, as in the example at the end of the section on equivalence relations. In other words,

$$(a, x) \vdash (b, f(x))$$

and

$$(b, y) \vdash (a, g(y)).$$

As in that section we can define $\sim$ to be the strongest relation weaker than $\vdash$. As we saw in that section, for each equivalence class C there is at most one z which is not of the form $h(w)$ for any $w \in C$. In other words, there are three possibilities for an equivalence class C:

(I) For every $z \in C$ there is a $w \in Z$ such that $z = h(w)$.

(II) There is one $z \in C$ for which there is no such w and it is of the form (a, x) for $x \in X$.

(III) There is one $z \in C$ for which there is no such w and it is of the form (b, y) for $y \in Y$.

The next step is to define a function $i\colon \{a\} \times X \to \{b\} \times Y$. We do this within each equivalence class, and we do it based on the classification of equivalence classes into the three categories above. For a class C of type I or of type II we define

$$i(a, x) = h(a, x)$$

for all $(a, x) \in C$. For a class C of type III we define $i(a, x)$ for $(a, x) \in C$ to be (b, y) where $y \in Y$ is the unique element such that $g(y) = x$. There is such an element because $z = (a, x)$ is not of the form (b, y) and so $z = h(w)$ for some $w \in Z$, it must be of the form (b, y) because h interchanges $\{a\} \times X$ and $\{b\} \times Y$, and it's unique because g is an injection.

Whichever type of equivalence class C is, we have $i(a, x) \in C$ for every $(a, x) \in C$. For type I and type II classes this is true because

$$(a, x) \vdash (b, f(x))$$

and hence

$$(a, x) \sim (b, f(x)).$$

For type III it is true because

$$(b, y) \vdash (a, g(y))$$

and hence

$$(b, y) \sim (a, g(y)).$$

Similarly, we define a function $j\colon \{b\} \times Y \to \{a\} \times X$. For a class C of type I or of type II we define $j(b, y)$ to be (a, x) where $x \in X$ is the unique element such that $f(x) = y$. There is such an element because $z = (b, y)$ is not of the form (a, x) and so $z = h(w)$ for some $w \in Z$ and it's unique because f is an injection. For a class C of type III we define

$$j(b, y) = h(b, y).$$

Just as we had $i(a, x) \in C$ for every $(a, x) \in C$, we have $j(b, y) \in C$ for every $(b, y) \in C$.

Now $i \circ j$ is the identity on $\{b\} \times Y$ and $j \circ i$ is the identity on $\{a\} \times X$, as can be checked from the definitions of i and j for each of the three types. It follows that i is a bijection from $\{a\} \times X$ to $\{b\} \times Y$. If we compose this on one side with the bijection from X to $\{a\} \times X$ which takes x to (a, x) and on the other side with the bijection from $\{b\} \times Y$ to Y which takes (b, y) to y then we get a bijection from X to Y. □

2.6 Zorn's Lemma

We know return the question of whether every surjection has a right inverse. Informally one might argue that if f is a surjection then there is for each $y \in Y$ an $x \in X$ such that $f(x) = y$. If f is not an injection then there might be more than one such x, but if there are then we just choose one. Formally we need an axiom or theorem which says we can do that. Historically the original such axiom was called the Axiom of Choice. There are a number of other axioms one could adopt which accomplish the same thing though and the most commonly used is called, somewhat inaccurately, Zorn's Lemma. Both words in its name are, from a modern point of view, inaccurate. The statement is originally due to Kuratowski and although it was originally a lemma proved using the Axiom of Choice it is now taken as an axiom and the former "Axiom of Choice" is no longer an axiom but rather a consequence of "Zorn's Lemma".

To state Zorn's Lemma cleanly we need some definitions.

Definition 2.6.1. Suppose that S is a set. A relation $\preccurlyeq$ on S is called a *partial order* if the following three conditions are satisfied.

(a) If $a \in S$ then $a \preccurlyeq a$.

(b) If $a, b, c \in S$, $a \preccurlyeq b$ and $b \preccurlyeq c$ then $a \preccurlyeq c$.

(c) If $a, b \in S$, $a \preccurlyeq b$ and $b \preccurlyeq a$ then $a = b$. A pair $(S, \preccurlyeq)$ is called a *partially ordered set*.

Note that two of the three conditions match the definition of a directed set, but the third does not. Note also that the restriction of a partial order to any subset is still a partial order, unlike the case of directed sets, where there was an additional condition which needed to be satisfied.

Definition 2.6.2. A partial order $\preccurlyeq$ on a set S is called a *total order* if for every $a, b \in S$ either $a \preccurlyeq b$ or $b \preccurlyeq a$.

The "or" here is not an exclusive or. It's possible that $a \preccurlyeq b$ and $b \preccurlyeq a$, although 2.6.1c shows that this happens only if $a = b$.

Definition 2.6.3. If $(S, \preccurlyeq)$ is a partially ordered set then $a \in S$ is called a *maximal* element if $b \in S$ and $a \preccurlyeq b$ imply $a = b$. $b \in S$ is called a *minimal* element if $a \in S$ and $a \preccurlyeq b$ imply $a = b$. $b \in S$ is called a *greatest* element if $a \preccurlyeq b$ for all $a \in S$. $a \in S$ is called a *least* element if $a \preccurlyeq b$ for all $b \in S$.

Definition 2.6.4. If $(S, \preccurlyeq)$ is a partially ordered set and $R \subseteq S$ then $b \in S$ is called an *upper bound* for R if $a \preccurlyeq b$ for all $a \in R$. $a \in S$ is called a *lower bound* for R if $a \preccurlyeq b$ for all $a \in R$.

Lemma 2.6.5. *Suppose $(S, \preccurlyeq)$ is a partially ordered set and that whenever $R \in \wp(S)$ is such that the restriction of $\preccurlyeq$ to R is a total order on R the set R has an upper bound. Then S has a maximal element.*

Although I've followed traditional practice in calling it a lemma it is in fact an axiom of set theory and therefore not something we can or should prove. Zorn's Lemma is not obviously true. Neither is it obviously false. In some sense that's a good thing. There are now many statements which are known to be equivalent to Zorn's Lemma, in the sense that they can be proved from Zorn's Lemma if it's taken as an axiom but it can be proved from any of them if they are taken as axioms instead. There is in fact a book full of them. Some of these equivalent statements appear to be obviously true, like the statement that any product of non-empty sets is non-empty, which is one way of formulating the Axiom of Choice. Some appear to be obviously false. Since one can't have equivalent statements one of which is true and one of which is false this says more about the perils of obviousness than it does about set theory.

If $\preccurlyeq$ is a partial order then so is its opposite order $\succcurlyeq$, define by $a \succcurlyeq b$ if and only if $b \preccurlyeq a$. The effect of replacing a partial order with its opposite is to interchange the notions of minimal and maximal, least and greatest, and upper and lower bounds. So Zorn's Lemma is equivalent to the statement that if $(S, \preccurlyeq)$ is a partially ordered set and whenever $R \in \wp(S)$ is such that the restriction of $\preccurlyeq$ to R is a total order on R the set R has a lower bound then S has a minimal element. This is obtained by applying Zorn's Lemma to the partially ordered set $(S, \succcurlyeq)$.

2.7 Applications of Zorn's Lemma

Although the statement of Zorn's Lemma is somewhat opaque it's well adapted to proving existence theorems. A good example is the following.

Proposition 2.7.1. *Suppose X and Y are sets. There is an injection $f : X \to Y$ or there is an injection $g : Y \to X$.*

There might of course be both, in which case the Schröder-Bernstein Theorem tells us there's a bijection.

Proof. Consider the set S consisting of all $Z \in \wp(X \times Y)$ such that for all $x \in X$ there is at most one $y \in Y$ such that $(x, y) \in Z$ and for all $y \in Y$ there is at most one $x \in X$ such that $(x, y) \in Z$. We order S by inclusion. In other words, we take $\preccurlyeq$ to be $\subseteq$. Suppose $R \in \wp(S)$ and $\subseteq$ is a total order on R. Let $B = \bigcup_{Z \in R} Z$. If $x \in X$ and (x, y_1) and (x, y_2) are in B then, by the definition of the union, there are Z_1 and Z_2 in R such that $(x, y_1) \in Z_1$ and $(x, y_2) \in Z_2$. $\subseteq$ is, by assumption, a total order on R, so either $Z_1 \subseteq Z_2$ or $Z_2 \subseteq Z_1$. In the former case $(x, y_1) \in Z_2$. Since $(x, y_2) \in Z_2$ and $Z_2 \in S$ there is at most one $y \in Y$ such that $(x, y) \in Z_2$. In other words, $y_1 = y_2$. In the latter case $(x, y_1) \in Z_1$. Because $Z_1 \in S$ we must have $y_1 = y_2$. So if (x, y_1) and (x, y_2) are in B then $y_1 = y_2$. In other words, for each $x \in X$ there is at most one $y \in Y$ such that $(x, y) \in B$. A similar argument shows that there is, for each $y \in Y$, at most one $x \in X$ such $(x, y) \in B$. Combining these facts we see that $B \in S$. B is therefore an upper bound for R with respect to the total order $\subseteq$. So any $R \in \wp(S)$ on which $\subseteq$ is a total order has an upper bound. The hypotheses of Zorn's Lemma are therefore satisfied. The conclusion must be as well. There is therefore a maximal $M \in S$, i.e. a $M \in S$ such that if $Z \in S$ and $M \subseteq Z$ then $M = Z$.

Suppose that both of the following statements are false.

(a) There is for every $x \in X$ a $q \in Y$ such that $(x, q) \in M$.

(b) There is for every $y \in Y$ a $p \in X$ such that $(p, y) \in M$.

In other words suppose there is an $x \in X$ is such that there is no $q \in Y$ for which $(x, q) \in M$ and there is a $y \in Y$ is such that there is no $p \in X$ for which $(p, y) \in M$. Define

$$Z = M \cup \{(x, y)\}.$$

Then $Z \in S$ and $M \subseteq Z$. It follows that $M = Z$. But $(x, y) \notin M$. So our assumption that (a) and (b) are both false leads to a contraction. Therefore at least one of them is true.

In Case (a) we define $f(x)$ to be the unique $q \in Y$ such that $(x, q) \in M$. Its uniqueness follows from the fact that $M \in S$. If

$$f(x_1) = f(x_2)$$

for some $x_1, x_2 \in X$ then let $y = f(x_1)$. Then $(x_1, y) \in M$ and $(x_2, y) \in M$. But $M \in S$ so $x_1 = x_2$. So if

$$f(x_1) = f(x_2)$$

then $x_1 = x_2$. In other words, f is an injection. In Case (b) we define $g(y)$ to be the unique $p \in X$ such that $(p, y) \in M$. Its uniqueness follows from the fact that $M \in S$. If

$$g(y_1) = g(y_2)$$

for some $y_1, y_2 \in Y$ then let $x = g(y_1)$. Then $(x, y_1) \in M$ and $(x, y_2) \in M$. But $M \in S$ so $y_1 = y_2$. So if

$$g(y_1) = g(y_2)$$

then $y_1 = y_2$. In other words, g is an injection. So we have an injection $f : X \to Y$ in one case and an injection $g : Y \to X$ in the other case. □

Another application of Zorn's Lemma is to resolve the question, which was left open in an earlier section, of whether surjections have right inverses.

Proposition 2.7.2. *Suppose $f : X \to Y$ is a surjection. Then there is a function $g : Y \to X$ such that $f \circ g$ is the identity on Y.*

Proof. The idea of the proof is the same as for the preceding proposition. We construct functions by constructing their graphs and construct the graph by appealing to Zorn's Lemma to get a maximal element in a set of subsets of the Cartesian product.

Let S be the set of $Z \in \wp(X \times Y)$ such that

(a) for each $y \in Y$ there is at most one $x \in X$ such that $(x, y) \in Z$, and

(b) if there is such an x then $f(x) = y$.

Suppose $R \in \wp(S)$ is such that for any $Z_1, Z_2 \in R$ either $Z_1 \subseteq Z_2$ or $Z_2 \subseteq Z_1$. Let $B = \bigcup_{Z \in R} Z$. Suppose $(x_1, y) \in B$ and $(x_2, y) \in B$. Then there are $Z_1, Z_2 \in R$ such that $(x_1, y) \in Z_1$ and $(x_2, y) \in Z_2$. If $Z_1 \subseteq Z_2$ then $(x_1, y) \in Z_2$ and $(x_2, y) \in Z_2$. $Z_2 \in S$ so $x_1 = x_2$. There is therefore at most one $x \in X$ such that $(x, y) \in B$. Also, if there is such an x then $f(x) = y$ because $(x, y) \in Z$ for some $Z \in R$. A similar argument with the 1's and 2's reversed shows that if $Z_2 \subseteq Z_1$ then again there is at most one $x \in X$ such that $(x, y) \in B$ and that if there is such an x then $f(x) = y$. So $B \in S$. For any $Z \in R$ we have $Z \subseteq B$ so B is an upper bound. The hypotheses of Zorn's Lemma are therefore satisfied for the partially ordered set $(S, \subseteq)$ and therefore S has a maximal element, which we can call M.

We'd like to know that for all $y \in Y$ there is an x such that $(x, y) \in M$. Suppose otherwise and choose a y for which there is no such x. f is a surjection so there is an x such that $f(x) = y$. Let

$$Z = M \cup \{(x, y)\}.$$

Then $M \subseteq Z$ and $Z \in S$. M is maximal so $M = Z$. But $(x, y) \notin M$, so our assumption was false. In other words, for all $y \in Y$ there is an x such that $(x, y) \in M$. Since $M \in S$ there is only one such x and $f(x) = y$.

Define $g(y)$ for each $y \in Y$ to be the unique $x \in X$ such that (x, y) in M. The existence and uniqueness were just established above. Then

$$f(g(y)) = f(x) = y,$$

so $f \circ g$ is the identity on Y. □

Corollary 2.7.3. *Suppose X is a set and $\sim$ is an equivalence relation on X. Let $\mathcal{E}$ be the set of equivalence classes with respect to $\sim$. There there is a function $g\colon \mathcal{E} \to X$ such that for any equivalence class $C \in \mathcal{E}$*

$$\{y \in X : y \sim g(C)\} = C.$$

This function thus chooses a single representative from each equivalence class.

Proof. Let $f\colon X \to \mathcal{E}$ be the function which takes each element to its equivalence class, i.e.

$$f(x) = \{y \in X : y \sim x\}.$$

f is a surjection because each equivalence class is the equivalence class of some element. By the proposition f therefore has a right inverse g. The equation

$$(f \circ g)(C) = C$$

is just

$$\{y \in X : y \sim g(C)\} = C.$$

□

The following proposition is often used in defining functions on sets of equivalence classes.

Proposition 2.7.4. *Suppose X and Y are sets and $f\colon X \to Y$ is a function. Suppose $\sim$ is an equivalence relation on X, $\mathcal{E}$ is the set of equivalence classes with respect to $\sim$ and $p\colon X \to \mathcal{E}$ is the function which takes each element of x to its equivalence class with respect to $\sim$. Suppose $\bowtie$ is an equivalence relation on Y, $\mathcal{F}$ is the set of equivalence classes with respect to $\bowtie$ and $q\colon Y \to \mathcal{F}$ is the function which takes each element of y to its equivalence class with respect to $\bowtie$. Suppose that $f(s) \bowtie f(t)$ whenever $s \sim t$. Then there is a unique function $g\colon \mathcal{E} \to \mathcal{F}$ such that*

$$q \circ f = g \circ p.$$

Proof. By the corollary above there is an $i\colon \mathcal{E} \to X$ such that $p \circ i$ is the identity on $\mathcal{E}$. For all $x \in X$ we have

$$p((i \circ p)(x)) = p(i(p(x))) = (p \circ i)(p(x)) = p(x).$$

In view of the definition of p this means that $(i \circ p)(x)$ and x belong to the same equivalence class with respect to $\sim$, i.e that

$$(i \circ p)(x) \sim x.$$

Our assumption on f then implies that

$$f((i \circ p)(x)) \bowtie f(x)$$

or

$$(f \circ (i \circ p))(x) \bowtie f(x).$$

In view of the definition of q this means that

$$q((f \circ (i \circ p)))(x) = q(f(x))$$

or

$$(q \circ (f \circ (i \circ p)))(x) = (q \circ f)(x).$$

Since composition is associative we can rewrite this as

$$((q \circ f \circ i) \circ p)(x) = (q \circ f)(x).$$

Let

$$g = q \circ f \circ i.$$

Then

$$(g \circ p)(x) = (q \circ f)(x).$$

This holds for all x so

$$g \circ p = q \circ f.$$

This establishes the existence of the function g from the statement of the proposition.

To establish the uniqueness of g, suppose that g_1 and g_2 are such that

$$g_1 \circ p = q \circ f$$

and

$$g_2 \circ p = q \circ f.$$

Composing with i,

$$g_1 \circ p \circ i = q \circ f \circ i$$

and

$$g_2 \circ p \circ i = q \circ f \circ i.$$

But $p \circ i$ is the identity, so

$$g_1 = g_1 \circ p \circ i = q \circ f \circ i = g_2 \circ p \circ i = g_2.$$

□

As an example, consider the set $\mathbf{N} \times \mathbf{N}$ with the relation $\sim$ considered earlier. This was the relation where

$$(a, b) \sim (c, d)$$

if and only if

$$a + d = b + c.$$

The set $\mathcal{E}$ of equivalence classes "is" the set of integers in the standard construction of the integers. One needs to define all the usual arithmetic functions on it. The additive inverse function from $\mathcal{E}$ to itself, for example, is defined to be the function g which the proposition above associates to the function $f \colon \mathbf{N} \times \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ defined by

$$f(a, b) = (b, a).$$

In this case $Y = X = \mathbf{N} \times \mathbf{N}$ and $\bowtie$ is the same as $\sim$. The proposition has a hypothesis, $f(s) \bowtie f(t)$ whenever $s \sim t$, which needs to be checked. In our context this means

$$(b, a) \sim (d, c)$$

wherever

$$(a, b) \sim (c, d),$$

i.e.

$$b + c = a + d$$

whenever

$$a + d = b + c,$$

which is certainly true.

2.8 Cardinality

We now have enough results at our disposal to talk about cardinality of sets.

Definition 2.8.1. If there is a bijection $f \colon X \to Y$ then we write

$$\#X = \#Y.$$

If there is an injection $f \colon X \to Y$ then we write

$$\#X \leq \#Y$$

or

$$\#Y \geq \#X.$$

The term equinumerous is sometimes used in the first case. In other words X and Y are said to be equinumerous if there is a bijection from one to the other. This term is not widely used however. There appears to be no word in common usage to describe the second situation.

We'll leave unresolved the question of what sort of object $\#X$ is. We'll call it a *cardinal number*, but the only meaning attached to it will be via equations or inequalities like the ones above, where we compare to cardinals. We'll never have occasion to consider a single cardinal number in isolation.

The symbols $=$ and $\leq$ behave as you would expect.

Proposition 2.8.2. *For any sets X, Y, W and Z*

(a) $\#X = \#X$, $\#X \leq \#X$ *and* $\#X \geq \#X$.

(b) $\#X = \#Y$ *if and only if* $\#Y = \#X$.

(c) $\#X \leq \#Y$ *if and only if* $\#Y \geq \#X$.

(d) *If* $\#X = \#Y$ *then* $\#X \leq \#Y$ *and* $\#X \geq \#Y$.

(e) $\#X \leq \#Y$ *or* $\#Y \leq \#X$.

(f) *If* $\#X \leq \#Y$ *and* $\#X \geq \#Y$ *then* $\#X = \#Y$.

(g) *If* $\#X = \#\,Y$ *and* $\#Y = \#Z$ *then* $\#X = \#Z$.

(h) *If* $\#X \leq \#\,Y$ *and* $\#Y \leq \#Z$ *then* $\#X \leq \#Z$.

(i) $\#(X \times Y) = \#(Y \times X)$

(j) *If* $\#W = \#X$ *and* $\#Y = \#Z$ *then* $\#(W \times Y) = \#(X \times Z)$.

(k) *If* $\#W \leq \#X$ *and* $\#Y \leq \#Z$ *then* $\#(W \times Y) \leq \#(X \times Z)$.

(l) *If* $X \subseteq Y$ *then* $\#X \leq \#Y$.

(m) $\#(X \cap Y) \leq \#X \leq \#(X \cup Y)$ *and* $\#(X \cap Y) \leq \#Y \leq \#(X \cup Y)$.

(n) X *is infinite if and only if* $\#\mathbf{N} \leq \#X$.

(o) *If* X *is finite then* $\#X \leq \#\mathbf{N}$ *but* $\#X \neq \#\mathbf{N}$.

(p) *If* X *is finite and* Y *is infinite then* $\#X \leq \#Y$ *but* $\#X \neq \#Y$.

(q) *If* $f: X \to Y$ *is a surjection then* $\#X \geq \#Y$.

(r) $\#X \leq \#\wp(X)$ *but* $\#X \neq \#\wp(X)$.

Proof. All of these are proved by converting them to statements about bijections and injections. I'll just list those statements and, where they are not obvious, prove them.

(a) The identity function is a bijection and an injection.

(b) Any bijection has an inverse, which is also a bijection.

(c) Both statements were defined to mean that there is an injection from X to Y.

(d) Every bijection is an injection.

(e) This is Proposition 2.7.1

(f) This is the Schröder-Bernstein Theorem.

(g) The composition of bijections is a bijection.

(h) The composition of injections is an injection.

(i) The function $f(x, y) = (y, x)$ is a bijection.

(j) If $f: W \to X$ and $g: Y \to Z$ are bijections then so is
$$h: W \times Y \to X \times Z,$$
defined by
$$h(w, y) = (f(w), g(y)).$$

(k) If $f: W \to X$ and $g: Y \to Z$ are injections then so is
$$h: W \times Y \to X \times Z,$$
defined by
$$h(w, y) = (f(w), g(y)).$$

(l) The function $i: X \to Y$ defined by $i(x) = x$ is an injection.

(m)
$$X \cap Y \subseteq X \subseteq X \cup Y$$
and
$$X \cap Y \subseteq Y \subseteq X \cup Y$$
so we can apply the preceding part.

(n) This is Proposition 2.2.4.

(o) By 2.8.2e we have $\#\mathbf{N} \leq \#X$ or $\#X \leq \#\mathbf{N}$. We just saw that the former happens if and only if X is infinite and our X is finite so $\#X \leq \#\mathbf{N}$. If there were a bijection $g \colon X \to \mathbf{N}$ then the function $h \colon X \to X$ defined by

$$h(x) = f(g(x) + 1),$$

where f is the inverse of g would be an injection which is not a surjection, so X would be infinite. So there is not such bijection. It follows that $\#X \neq \#\mathbf{N}$.

(p) $\#X \leq \#Y$ follows from 2.8.2h, 2.8.2n and 2.8.2o. If $\#X = \#Y$ then by 2.8.2b, 2.8.2d we would have $\#Y \leq \#X$. Y is infinite so by 2.8.2h and 2.8.2n we would have $\#N \leq \#X$. But then X would also be infinite by 2.8.2n. It's not, so $\#X \neq \#Y$

(q) If $f \colon X \to Y$ is a surjection then there is, by Proposition 2.7.2, a g such that $f \circ g$ is the identity on Y. This g is, by Proposition 2.1.1, an injection.

(r) $f(x) = \{x\}$ is an injection from X to $\wp(X)$ so $\#X \leq \#\wp(X)$. The fact that $\#X \neq \#\wp(X)$ is Cantor's Theorem.

□

2.9 Countable sets

After the finite sets the next nicest class are the countable sets. There is an unfortunate divergence of terminology regarding countable sets. The most common convention appears to be that X is said to be countable if and only if $\#X = \#\mathbf{N}$ and is said to be at most countable if $\#X \leq \#\mathbf{N}$. A less commonly used convention is that X is said to be countable if and only if $\#X \leq \#\mathbf{N}$ and is said to be countably infinite if $\#X = \#\mathbf{N}$. Everyone agrees that X is uncountable when $\#X \geq \#\mathbf{N}$ but $\#X \neq \#\mathbf{N}$. One advantage of the less commonly used convention is that all sets are either countable or uncountable, as one would expect. With the other convention finite sets are neither countable nor uncountable. Another advantage of the less commonly used convention is that $\#X \leq \#\mathbf{N}$ appears more commonly in the hypotheses and conclusions of theorems than $\#X = \#\mathbf{N}$ and it's convenient for the more commonly used condition to have the shorter name. For those reasons I'm going to use the less commonly used convention in these notes, but when communicating with other people you should be aware that most of them use the other convention. The safest, although most awkward, option when communicating with someone whose conventions you don't know is to say "countably infinite" for $\#X = \#\mathbf{N}$, to say "at most countable" for $\#X \leq \#\mathbf{N}$, and not to say "countable" at all.

Definition 2.9.1. A set X is said to be *countable* if $\#X \leq \#\mathbf{N}$ and *countably infinite* if $\#X = \#\mathbf{N}$. It is said to be *uncountable* if it is not countable.

The phrase countably infinite is justified by the following proposition.

Proposition 2.9.2. *X is countably infinite if and only if it is countable and infinite.*

Proof. Suppose X is countably infinite, i.e. that $\#X = \#\mathbf{N}$. Then $\#X \leq \#\mathbf{N}$ by 2.8.2d and so is countable. $\#X \geq \#\mathbf{N}$, also by 2.8.2d, which means $\#\mathbf{N} \leq \#X$, by 2.8.2c, and therefore X is infinite, by 2.8.2n.

Suppose X is countable, i.e. $\#X \leq \#\mathbf{N}$, and X is infinite. Then $\#\mathbf{N} \leq \#X$ by 2.8.2n and $\#X \geq \#\mathbf{N}$ by 2.8.2c. It then follows from 2.8.2f that $\#X = \#\mathbf{N}$, i.e. that X is countably infinite. □

The following facts are useful when you want to use the fact that one sets is known to be countable to prove that another set is countable.

Proposition 2.9.3. *(a) If $f \colon X \to Y$ is an injection and Y is countable then X is countable.*

(b) If $X \subseteq Y$ and Y is countable then X is countable.

(c) If $g \colon Y \to X$ is a surjection and Y is countable then X is countable.

(d) If X and Y are countable then so is $X \times Y$.

(e) Suppose $\mathcal{A}$ is a countable set and then each $X \in \mathcal{A}$ is a countable set. Then $\cup_{X\in\mathcal{A}} X$ is countable.

(f) If X and Y are countable then so is $X \cup Y$.

Proof. (a) Because there's an injection $f\colon X \to Y$ we have $\#X \leq \#Y$. Y is countable so $\#Y \leq \#\mathbf{N}$. Therefore $\#X \leq \#\mathbf{N}$ by proposition 2.8.2h. So X is countable.

(b) This is the previous part applied to the inclusion function $f\colon X \to Y$ defined by $f(x) = x$.

(c) By 2.8.2q $\#Y \geq \#X$. By 2.8.2c $\#X \leq \#Y$. Y is countable so $\#Y \leq \#\mathbf{N}$. By 2.8.2h $\#X \leq \#\mathbf{N}$. Therefore X is countable.

(d) Since X and Y are countable there are injections $f\colon X \to \mathbf{N}$ and $g\colon Y \to \mathbf{N}$. Define $i\colon \mathbf{N}\times\mathbf{N} \to \mathbf{N}$ by

$$i(m,n) = \frac{(m+n)(m+n+1)}{2} + n.$$

Note that $i(m,n)$ is an integer, despite the 2 in the denominator, because from any two successive integers one of them must be even. Define $j\colon \mathbf{N} \to \mathbf{N}\times\mathbf{N}$ as by

$$j(p) = \left(\frac{q(q+3)}{2} - p, p - \frac{q(q+1)}{2}\right)$$

where q is the smallest natural number such that

$$p \leq \frac{q(q+3)}{2}.$$

There is at least one natural number q satisfying this inequality, namely p itself, and so there is a least such natural number. Because it's the least such number

$$\frac{(q-1)(q+2)}{2} < p$$

and hence

$$\frac{q(q+1)}{2} = \frac{(q-1)(q+2)}{2} + 1 \leq p$$

So $j(p) \in \mathbf{N}\times\mathbf{N}$. It's straightforward to check that $j \circ i$ is the identity on $\mathbf{N}$, i.e. that

$$j(i(m,n)) = (m,n)$$

since the smallest q such that

$$\frac{(m+n)(m+n+1)}{2} + n \leq \frac{q(q+3)}{2}$$

or, equivalently,

$$\frac{(m+n)(m+n+1)}{2} + n < \frac{(q+1)(q+2)}{2}$$

is $q = m+n$. $i \circ j$ is also the identity on $\mathbf{N}$ since if

$$(m,n) = j(p)$$

then

$$m + n = \frac{q(q+3)}{2} - p + p - \frac{q(q+1)}{2} = q$$

and

$$n = p - \frac{q(q+1)}{2}$$

so

$$i(m,n) = \frac{q(q+1)}{2} + p - \frac{q(q+1)}{2} = p.$$

So i and j are bijections.

We can now define $h\colon X \times Y \to \mathbf{N}$ by

$$h(x,y) = i(f(x), g(y)).$$

If

$$h(x_1, y_1) = h(x_2, y_2)$$

then

$$i(f(x_1), g(y_2)) = i(f(x_1), g(y_2))$$

and, since i is an injection,

$$(f(x_1), g(y_2)) = (f(x_1), g(y_2)),$$

which is possible only if

$$f(x_1) = f(x_2)$$

and

$$g(y_1) = g(y_2).$$

Since f and g are injections it follows that $x_1 = x_2$ and $y_1 = y_2$, and therefore

$$(x_1, y_1) = (x_2, y_2).$$

h is therefore an injection and so $X \times Y$ is countable.

(e) The hypotheses mean that there is an injection $f\colon \mathcal{A} \to \mathbf{N}$ and that there is, for each $X \in \mathcal{A}$, an injection $g_X\colon X \to \mathbf{N}$. Define

$$i\colon \bigcup_{X\in\mathcal{A}} X \to \mathbf{N}$$

by saying that $i(x)$ is the least natural number n for which there's an $X \in \mathcal{A}$ with $x \in X$ and $f(X) = n$. There is at least one such n because $x \in X$ for some $X \in \mathcal{A}$ by the definition of the union and therefore there is a least such n. If $i(x) = n$ then there is an $x \in X$ such that $x \in X$ and $f(X) = n$, by definition. There is in fact only one such X. If there were two, X_1 and X_2 then we would have

$$f(X_1) = f(X_2),$$

but f is an injection so then

$$X_1 = X_2.$$

It therefore makes sense to define

$$j\colon \bigcup_{X\in\mathcal{A}} X \to X$$

by saying that $j(x) = X$ where $X \in \mathcal{A}$ is the unique element such that $x \in X$ and

$$f(X) = i(x).$$

So

$$f(j(x)) = i(x).$$

Now define

$$h\colon \bigcup_{X\in\mathcal{A}} X \to \mathbf{N}\times\mathbf{N}$$

by

$$h(x) = (i(x), g_{j(x)}(x)).$$

This is an injection. Indeed if

$$h(x_1) = h(x_2)$$

then

$$i(x_1) = i(x_2).$$

Also

$$f(j(x_1)) = i(x_1) = i(x_2) = f(j(x_2))$$

and f is an injection so

$$j(x_1) = j(x_2).$$

Let X be their common value. Then

$$h(x_1) = (i(x_1), g_X(x_1))$$

and

$$h(x_2) = (i(x_2), g_X(x_2)$$

so

$$g_X(x_1) = g_X(x_2).$$

But g_X is an injection,

$$x_1 = x_2.$$

(f) This is just the previous part applied to the special case $\mathcal{A} = \{X, Y\}$.

□

We can use the previous proposition to prove that various familiar sets are countable.

Proposition 2.9.4. *Every finite set is countable. The set* $\mathbf{N}$ *of natural numbers, the set* $\mathbf{Z}$ *of rational numbers and the set* $\mathbf{Q}$ *of rational numbers are countable. So is the set of algebraic numbers, i.e solutions of polynomial equations of positive degree with rational coefficients.*

Proof. $\mathbf{N}$ is infinite so by 2.8.2p $\# \leq \#\mathbf{N}$ if X is finite.

$\#\mathbf{N} \leq \#\mathbf{N}$ by 2.8.2a so $\mathbf{N}$ is countable.

$\mathbf{N}\times\mathbf{N}$ is countable by 2.9.3d. The function $g\colon \mathbf{N}\times\mathbf{N} \to \mathbf{Z}$ defined by $g(m, n) = m - n$ is a surjection so $\mathbf{Z}$ is countable by 2.9.3c

$$\mathbf{Z} \setminus \{0\} \subseteq \mathbf{Z}$$

so $\mathbf{Z} \setminus \{0\}$ is countable by 2.9.3b. $\mathbf{Z} \times (\mathbf{Z} \setminus \{0\})$ is countable by 2.9.3d. The function

$$g\colon \mathbf{Z} \times (\mathbf{Z} \setminus \{0\}) \to \mathbf{Q}$$

defined by

$$g(p, q) = p/q$$

is a surjection so $\mathbf{Q}$ is countable by 2.9.3c.

$\mathbf{Q}^n$ is countable by induction on n, using 2.9.3d and the fact that

$$\mathbf{Q}^{n+1} = \mathbf{Q}^n \times \mathbf{Q}.$$

$\mathbf{Q}\setminus\{0\}$ is countable by 2.9.3b $(\mathbf{Q} \setminus \{0\}) \times \mathbf{Q}^n$ is countable by 2.9.3b. There's a natural surjection from this set to the set of polynomials of degree n with rational coefficients, which just takes an $n+1$-tuple of rational numbers to the polynomial with those numbers as coefficients, so the set of polynomials of degree n is countable by 2.9.3c. The set of polynomials of positive degree with rational coefficients is therefore countable by 2.9.3e. There are at most n solutions to a polynomial of degree n so the set of solutions to any polynomial of positive degree is finite and hence countable. Applying 2.9.3e again shows that the set of all numbers which are a solution to some such polynomial is again countable. □

2.10 The Cantor Set

Definition 2.10.1. Let $f\colon \wp(\mathbf{N}) \to \mathbf{R}$ be defined by

$$f(A) = \frac{2}{3} \sum_{j\in A} 3^{-j}.$$

The *Cantor Set* is $f_*(\wp(\mathbf{N}))$.

This set will be a recurring example, and counterexample, in these notes. It has a number of interesting topological and measure-theoretic properties but in this section we are only concerned with its set-theoretic properties.

Proposition 2.10.2. *The function* $f\colon \wp(\mathbf{N}) \to \mathbf{R}$ *defined by*

$$f(A) = \frac{2}{3} \sum_{j\in A} 3^{-j}$$

is an injection.

Proof. Suppose $A \neq B$. Let m be the least element in $A \triangle B$. Either $m \in A$ and $m \notin B$ or vice versa. We'll assume the former. The argument that follows will apply in the other case if A and B are swapped everywhere.

$$\frac{3}{2} f(A) = \sum_{j\in A} 3^{-j} = \sum_{\substack{j\in A\\ j<m}} 3^{-j} + 3^{-m} + \sum_{\substack{j\in A\\ j>m}} 3^{-j}$$

and

$$\frac{3}{2} f(B) = \sum_{j\in B} 3^{-j} = \sum_{\substack{j\in B\\ j<m}} 3^{-j} + \sum_{\substack{j\in B\\ j>m}} 3^{-j}.$$

$$\sum_{\substack{j\in A\\ j>m}} 3^{-j} \geq 0$$

because it's a sum of non-negative terms. So

$$\frac{3}{2} f(A) \geq \sum_{\substack{j\in A\\ j<m}} 3^{-j} + 3^{-m}.$$

$$\sum_{\substack{j\in B\\ j>m}} 3^{-j} + \sum_{\substack{j\notin B\\ j>m}} 3^{-j} = \sum_{j>m} 3^{-j} = \frac{1}{2} 3^{-m}.$$

$$\sum_{\substack{j\notin B\\ j>m}} 3^{-j} \geq 0$$

because it's a sum of non-negative terms, so

$$\sum_{\substack{j\in B\\ j>m}} 3^{-j} \leq \frac{1}{2} 3^{-m}$$

and

$$\frac{3}{2} f(B) \leq \sum_{\substack{j\in B\\ j<m}} 3^{-j} + \frac{1}{2} 3^{-m}$$

If $j < m$ then $j \in A$ if and only if $j \in B$ because m was the least element of $A \triangle B$. So

$$\sum_{\substack{j\in A\\ j<m}} 3^{-j} = \sum_{j\in B} 3^{-j}.$$

It follows that

$$\frac{3}{2} f(A) \geq \frac{3}{2} f(B) + \frac{1}{2} 3^{-m}$$

and hence

$$f(A) \geq f(B) + 3^{-m-1} > f(B).$$

So $f(A) \neq f(B)$. Therefore f is an injection. □

Corollary 2.10.3. *The Cantor set is uncountable.*

Proof. Let C be the Cantor Set and let $g\colon \wp(\mathbf{N}) \to C$ be defined by $g(A) = f(A)$ for all $A \in \wp(\mathbf{N})$. This is well defined because if $A \in \wp(\mathbf{N})$ then $f(A) \in C$, by the definition of f. Also, every element of C is $f(A)$ for some A, again by the definition of C, so g is a surjection g is an injection because f is. So g is a bijection from $\wp(\mathbf{N})$ to C. Therefore

$$\#\wp(\mathbf{N}) = \#C.$$

By 2.8.2r

$$\#\mathbf{N} \leq \#\wp(\mathbf{N})$$

but

$$\#\mathbf{N} \neq \#\wp(\mathbf{N}).$$

It follows that

$$\#\mathbf{N} \leq \#C$$

but

$$\#\mathbf{N} \neq \#C.$$

If $\#C \leq \#\mathbf{N}$ then we have a contradiction to 2.8.2f, so $\#C \leq \#\mathbf{N}$ is not true. In other words, C is not countable. □

Are f and g the same function? They appear to be because they have the same domain and g was defined by $g(A) = f(A)$ for all A in this domain, but they aren't. g is a surjection but f is not.

The Cantor Set will appear again in later chapters but for the purposes of this chapter it is solely a step in the proof of the following theorem.

Theorem 2.10.4. *The set* $\mathbf{R}$ *of real numbers is uncountable. So is the set* $\mathbf{R} \setminus \mathbf{Q}$ *of irrational numbers.*

Proof. If C is the Cantor Set then $C \subseteq \mathbf{R}$. If $\mathbf{R}$ were countable then by 2.9.3b C would also be countable, and we've just seen that it isn't.

$\mathbf{Q}$ is countable by Proposition 2.9.4. If $\mathbf{R} \setminus \mathbf{Q}$ were also countable then

$$\mathbf{R} = \mathbf{Q} \cup (\mathbf{R} \setminus \mathbf{Q})$$

would also be countable, by Proposition 2.9.3f, but we've just seen that it isn't. □

2.11 Disjoint unions, products

The construction of the disjoint union of two sets which was needed for the proof of the Schröder-Bernstein Theorem can be generalised to an arbitrary collection. There's also a similar construction of the product of an arbitrary collection of sets.

As before, we use a set of labels to distinguish the various copies. The sets themselves are unsuitable as labels because we could require more than one copy of the same set. The natural numbers are unsuitable because we could have an uncountable number of sets. The solution is to let the set of labels be an arbitrary set. One then needs a function to associate the correct set to each label. That's the purpose of the following definition.

Definition 2.11.1. An *indexed collection of sets* is a function $j\colon L \to \mathcal{A}$, where $\mathcal{A}$ is a set of sets, i.e. such that if $S \in \mathcal{A}$ then S is a set.

From a logician's point of view the condition that each $S \in \mathcal{A}$ is a set is redundant. Everything is a set. From a mathematician's point of view though some of these sets are really sets, in the sense that expect to apply the usual operations on sets to them, and some are only accidentally sets, because we happened to construct them that way but don't intend to operate on them as sets once they've been constructed. We routinely write things like $X \subseteq \mathbf{R}$, for example, but never write $X \subseteq \sqrt{2}$ even though technically both $\mathbf{R}$ and $\sqrt{2}$ are defined as sets and therefore can have subsets. From a logician's point of view then the definition above is pointless, since indexed collection of sets is merely another word for function. From a mathematician's point of view though the use of the term indexed collection of sets serves as a statement of intent. If we say $j\colon L \to \mathcal{A}$ is an indexed collection of sets then we mean that $j(\lambda)$ for each $\lambda \in L$ is not just a set in some accidental sense but rather is something which we want to think of and act on as a set. In other words, that it's something like $\mathbf{R}$ rather than $\sqrt{2}$.

Definition 2.11.2. The disjoint union of an indexed collection of sets $j\colon L \to \mathcal{A}$ is the set of all ordered pairs of the form (λ, x), where $\lambda \in L$ and $x \in j(\lambda)$.

In other words, the disjoint union is

$$\left\{ (\lambda, x) \in L \times \bigcup_{S \in \mathcal{A}} S \colon x \in j(\lambda) \right\}.$$

If $L = \{a, b\}$ and j is defined by $j(a) = X$ and $j(b) = Y$ then we get exactly the disjoint union of X and Y constructed earlier.

If $j\colon L \to \mathcal{A}$ is an indexed collection of sets and $\lambda \in L$ then there is a natural injection from $j(\lambda)$ to the disjoint union of j, given by $i_\lambda(x) = (\lambda, x)$. We met this injection several times in the case of the disjoint union of X and Y, where it was the function $i_a(x) = (a, x)$ or $i_b(y) = (b, y)$. In general the injection above is referred to as the inclusion at λ. If j is injective then we can safely refer to it as the inclusion of the set $j(\lambda)$ in the disjoint union but if j is not injective then there are $\lambda, \mu \in L$ such that $j(\lambda) = j(\mu)$ and referring to the inclusion of this set in the disjoint union is ambiguous, since it could refer to the inclusion at λ or the inclusion at μ.

Two other notions related to indexed collections of sets are those of choice functions and products.

Definition 2.11.3. If $j\colon L \to \mathcal{A}$ is an indexed collection of sets then a *choice function* for j is a function $f\colon L \to \bigcup_{S \in \mathcal{A}} S$ such that $f(\lambda) \in j(\lambda)$ for each $\lambda \in L$.

Definition 2.11.4. The *product* of an indexed collection of sets $j\colon L \to \mathcal{A}$ is the set of all choice functions for it.

For example, consider the indexed collection considered earlier, where $L = \{a, b\}$ and j is defined by $j(a) = X$ and $j(b) = Y$. A choice function for j is a function f on $\{a, b\}$ such that $f(a) \in X$ and $f(b) \in Y$. To any such choice function we can associate the element $(f(a), f(b))$ in $X \times Y$. Conversely, given any $(x, y) \in X \times Y$ we can define a choice function f by $f(a) = x$ and $f(b) = y$. So while it's not quite correct to say that the product in the sense of the definition *is* the Cartesian product $X \times Y$, there is a natural bijection between the product of j and $X \times Y$. For simplicity we generally treat these two as the same object. This is similar to the way we generally treat the sets $(X \times Y) \times Z$ and $X \times (Y \times Z)$ as if they were the same set when in reality they are two different sets related by the bijection which takes $((x, y), z)$ to $(x, (y, z))$.

As another example, suppose $L = \{1, \ldots, n\}$ and $j(k) = \mathbf{R}$ for each $k \in L$. The choice functions for j are precisely the functions f such that $f(k) \in \mathbf{R}$ for all $k \in \{1, \ldots, n\}$. In other words, they are the functions from $\{1, \ldots, n\}$ to $\mathbf{R}$. We can associate to such a function an element $(f(j), \ldots, f(n)) \in \mathbf{R}^n$. Conversely, to every element $(x_1, \ldots, x_n)$ we can associate the choice function f such that $f(k) = x_k$ for each k. Again, we generally treat the product as if it were $\mathbf{R}^n$ rather than just being connected to it by the bijection above.

Other than familiarity there's nothing special about the sets $\{1, \ldots, n\}$ and $\mathbf{R}$. More generally if X and Y are any sets then we can define the indexed collection of sets $L = X$ and $j(x) = Y$. A choice function for j is then just a function from X to Y.

If $j\colon L \to \mathcal{A}$ is an indexed collection of sets and $\lambda \in L$ then there is a natural surjection π_λ from the product of j to $j(\lambda)$, given by $\pi_\lambda(f) = f(\lambda)$. In other words, it takes the choice function f to its value at λ. In the case $L = \{a, b\}$ considered above if we identify the product with the Cartesian product then this function is $\pi_a(x, y) = x$ and $\pi_b(x, y) = y$. In other words, π_a and π_b are the usual projection functions. More generally, for any indexed collection of sets we refer to π_λ as the projection at λ. If the function j is an injection then it's permissible to speak of the projection onto $j(\lambda)$ but if there are distinct $\lambda, \mu \in L$ such that $j(\lambda) = j(\mu)$ then this terminology is ambiguous since the projection onto this set could refer to the projection at λ or the projection at μ. An extreme example is the case $L = \{1, \ldots, n\}$, $j(k) = \mathbf{R}$ considered above, where we saw that the product could be identified with $\mathbf{R}^n$. Referring to the projection of $\mathbf{R}^n$ onto $\mathbf{R}$ is ambiguous while the projection at k refers unambiguously to the function $\pi_k((x_1, \ldots, x_n)) = x_k$, more commonly called the k'th projection or the projection onto the k'th factor.

If $j(\lambda) = \varnothing$ for some $\lambda \in L$ then the product is empty. Indeed there are no choice functions because there is no function f with $f(\lambda) \in \varnothing$. The converse of this statement is the Axiom of Choice. The name is historical but for us it's a theorem rather than an axiom.

Theorem 2.11.5. *If $j: L \to \mathcal{A}$ is an indexed collection of sets and $j(\lambda) \neq \varnothing$ for each $\lambda \in L$ then the product of j is non-empty.*

Proof. In view of the definition of the product, to say that the product is non-empty is the same as saying that there is a choice function. We show this using Zorn's Lemma.

A pair (D, f) will be called a partial choice function for j if $D \subseteq L$, and f is a function on D such that $f(\lambda) \in j(\lambda)$ for all $\lambda \in D$. Let S be the set of all partial choice functions for j. We order it by $(D_1, f_1) \preccurlyeq (D_2, f_2)$ if $D_1 \subseteq D_2$ and f_1 is the restriction of f_2 to D_1. If $R \in \wp(S)$ is totally ordered with respect to $\preccurlyeq$ then there is an upper bound for R. This upper bound is $(\hat{D}, \hat{f})$ where $\hat{D}$ is the is the set of $\lambda \in L$ such that $\lambda \in D$ for some $(D, f) \in R$ and $\hat{f}(\lambda) = f(\lambda)$. Some work is required to establish that $\hat{f}$ is well defined, since there could be more than one $(D, f) \in R$ with $\lambda \in D$. However if $(D_1, f_1) \in R$ and $(D_2, f_2) \in R$ are such that $\lambda \in D_1$ and $\lambda \in D_2$ then either $(D_1, f_1) \preccurlyeq (D_2, f_2)$ or $(D_2, f_2) \preccurlyeq (D_1, f_1)$. In the former case f_1 is the restriction to D_1 of f_2 so $f_1(\lambda) = f_2(\lambda)$. In the latter case f_2 is the restriction to D_2 of f_1 so $f_2(\lambda) = f_1(\lambda)$. So in either case $f_1(\lambda) = f_2(\lambda)$ and it doesn't matter which we use in defining $\hat{f}(\lambda)$. From the definitions it's clear that $(D, f) \preccurlyeq (\hat{D}, \hat{f})$ for all $(D, f) \in R$. So $(\hat{D}, \hat{f})$ is indeed an upper bound. Zorn's Lemma therefore applies and we have a maximal element in S. If (D, f) is this maximal element and $D \neq L$ then we would be able be to find a larger element $(\tilde{D}, \tilde{f})$ by picking a $\lambda \in L \setminus D$ and an $x \in j(\lambda)$ and setting $\tilde{D} = D \cup \{\lambda\}$ and defining $\tilde{f}$ by $\tilde{f}(\lambda) = x$ and $\tilde{f}(\mu) = f(\mu)$ for $\mu \in L$. □

The theorem could have been proved by constructing an appropriate set and then defining the choice function in such a way that that set is its graph, as was done for Propositions 2.7.1 and 2.7.2. In more detail, we could have defined S to be the set of subsets Z of $L \times \bigcup_{\lambda \in L} i(\lambda)$ such that for every $\lambda \in L$ there is at most one $x \in \bigcup_{\lambda \in L} j(\lambda)$ such that $(\lambda, x) \in Z$ and such that $x \in j(\lambda)$ for all $(\lambda, x) \in Z$. This S then has a maximal element M by Zorn's Lemma and the fact that it is maximal implies that for all $\lambda \in L$ there is a unique $x \in j(\lambda)$ such that $(\lambda, x) \in M$. If we then define $f(\lambda)$ to be this x then f is a choice function.

One could also prove Proposition 2.7.2 in a manner more similar to the proof of this theorem, with the elements of the set S being pairs consisting of a subset V of Y and a function k from V to X such that $f \circ k$ is the identity on V.

The situation for Proposition 2.7.1 is more complicated because we don't know when defining S whether we're constructing a function f from X to Y or a function g from Y to X. It's still possible to give an argument like the one for the theorem, but the definition of S is more complicated. Elements of S are of the form (U, V, j, k) where $U \subseteq X$, $V \subseteq Y$, j is a bijection from U to V and k is its inverse.

3 Topological spaces

3.1 Definitions

As already stated in Definition 1.11.1, a *topology* on a set X is a $\mathcal{T} \in \wp(\wp(X))$ such that

(a) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$,

(b) If $V \in \mathcal{T}$ and $W \in \mathcal{T}$ then $V \cap W \in \mathcal{T}$.

(c) If $\mathcal{E} \subseteq \mathcal{T}$ then $\bigcup_{V \in \mathcal{E}} V \in \mathcal{T}$.

A pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X is called a *topological space*.

The set of topologies on X will be denoted $\mathbf{T}(X)$.

A number of examples of topologies were given in Section 1.11 and we will see more examples later. A subset $U \in \wp(X)$ is called *open* if $U \in \mathcal{T}$ and *closed* if $X \setminus U \in \mathcal{T}$.

Two results we will need later are the following.

Lemma 3.1.1. *Suppose X and Y are sets and $f: X \to Y$ is a function. Then*

$$\mathbf{T}(X) \subseteq f^{***}(\mathbf{T}(Y)).$$

Proof. Suppose $\mathcal{T}_X \in \mathbf{T}(X)$. Define $\mathcal{T}_Y$ by

$$\mathcal{T}_Y = f^{**}(\mathcal{T}_X).$$

Then $\mathcal{T}_Y$ is a topology on Y, i.e. $\mathcal{T}_Y \in \mathbf{T}(Y)$. To see this, first observe that $A \in \mathcal{T}_Y$ if and only if $f^*(A) \in \mathcal{T}_X$, by the definition of $\mathcal{T}_Y$ and of the preimage under f^*.

$$f^*(\varnothing) = \varnothing \in \mathcal{T}_X$$

so $\varnothing \in \mathcal{T}_Y$.

$$f^*(Y) = X \in \mathcal{T}_X$$

so $X \in \mathcal{T}_Y$. If $V, W \in \mathcal{T}_Y$ then

$$f^*(V \cap W) = f^*(V) \cap f^*(W)$$

and $f^*(V), f^*(W) \in \mathcal{T}_X$, so $f^*(V \cap W) \in \mathcal{T}_X$ and hence $V \cap W \in \mathcal{T}_Y$. If $\mathcal{E} \subseteq \mathcal{T}_Y$ then

$$f^*\left(\bigcup_{V\in\mathcal{E}} V\right) = \bigcup_{V\in\mathcal{E}} f^*(V)$$

and $f^*(V) \in \mathcal{T}_X$ for all $V \in \mathcal{E}$ so

$$f^*\left(\bigcup_{V\in\mathcal{E}} V\right) \in \mathcal{T}_X$$

and hence

$$\bigcup_{V\in\mathcal{E}} V \in \mathcal{T}_Y.$$

So $\mathcal{T}_Y$ satisfies all the conditions to be a topology on Y.

So if $\mathcal{T}_X \in \mathbf{T}(X)$ then $f^{**}(\mathcal{T}_X) \in \mathbf{T}(Y)$. In other words, $\mathcal{T}_X \in f^{***}(\mathbf{T}(Y))$. Since this holds for all $\mathcal{T}_X \in \mathbf{T}(X)$ we have $\mathbf{T}(X) \subseteq f^{***}(\mathbf{T}(Y))$. □

Lemma 3.1.2. *Suppose X and Y are sets and $f\colon X \to Y$ is a function. Then*

$$\mathbf{T}(Y) \subseteq ((f^*)_*)^*(\mathbf{T}(X)).$$

Proof. Suppose $\mathcal{T}_Y \in \mathbf{T}(Y)$. Define $\mathcal{T}_X$ by

$$\mathcal{T}_X = (f^*)_*(\mathcal{T}_Y).$$

$\mathcal{T}_X$ is the set of subsets U of X for which there is a $V \in \mathcal{T}_Y$ with $U = f^*(V)$. $\varnothing \in \mathcal{T}_Y$ because $\mathcal{T}_Y$ is a topology on Y and $\varnothing = f^*(\varnothing)$ so $\varnothing \in \mathcal{T}_X$. $Y \in \mathcal{T}_Y$ because $\mathcal{T}_Y$ is a topology on Y and $X = f^*(Y)$ so $X \in \mathcal{T}_X$. If $U_1, U_2 \in \mathcal{T}_X$ then there are $V_1 \cap V_2 \in \mathcal{T}_X$ such that $U_1 = f^*(V_1)$ and $U_2 = f^*(V_2)$. But

$$U_1 \cap U_2 = f^*(V_1) \cap f^*(V_2) = f^*(V_1 \cap V_2).$$

$\mathcal{T}_Y$ is a topology on Y and $V_1, V_2 \in \mathcal{T}_Y$ so $V_1 \cap V_2 \in \mathcal{T}_Y$ and $U_1 \cap U_2 \in \mathcal{T}_X$. Suppose $\mathcal{E} \subseteq \mathcal{T}_X$

$$\bigcup_{U\in\mathcal{E}} U = \bigcup_{U\in\mathcal{E}, V\in\mathcal{T}_Y, U=f^*(V)} U = f^*(W)$$

where

$$W = \bigcup_{U\in\mathcal{E}, V\in\mathcal{T}_X, U=f^*(V)} V.$$

$\mathcal{T}_Y$ is a topology on Y and W is a union of elements in $\mathcal{T}_Y$ and so $W \in \mathcal{T}_Y$. Therefore

$$\bigcup_{U\in\mathcal{E}} U \in \mathcal{T}_X.$$

We've just seen that $\varnothing \in \mathcal{T}_X$, $X \in \mathcal{T}_X$, the intersection of any two elements of $\mathcal{T}_X$ is an element of $\mathcal{T}_X$ is an element of $\mathcal{T}_X$ and the union of any set of elements of $\mathcal{T}_X$ is an element of $\mathcal{T}_X$. So $\mathcal{T}_X$ is a topology on X.

We've now established that if $\mathcal{T}_Y \in \mathbf{T}(Y)$ then $(f^*)_*(\mathcal{T}_Y) \in \mathbf{T}(X)$ or, equivalently,

$$\mathcal{T}_Y \in (((f^*)_*)^*)(\mathbf{T}(X)).$$

This holds for every $\mathcal{T}_Y \in \mathbf{T}(Y)$ so

$$\mathbf{T}(Y) \subseteq (((f^*)_*)^*)(\mathbf{T}(X)).$$

□

Another definition which we've already seen is that of a Hausdorff topology from Definition 1.11.2. A topology $\mathcal{T}$ on X is said to be *Hausdorff* if for every $x, y \in X$ such that $x \neq y$ there are $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $V \cap W = \varnothing$.

A simple consequence of the definition is the following.

Proposition 3.1.3. *If $\mathcal{T}$ is a Hausdorff topology on X and $x \in X$ then $\{x\}$ is closed.*

Proof. Let

$$U = \bigcup_{W\in\mathcal{T}, x\notin W} W.$$

Suppose $y \in X \setminus \{x\}$. There are then $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $V \cap W = \varnothing$. From $x \in V$ and $V \cap W = \varnothing$ it follows that $x \notin W$. So W

belongs to the union which defines U. From $y \in W$ it then follows that $y \in U$. We've just seen that if $y \in X \setminus \{x\}$ then $y \in U$, so

$$X \setminus \{x\} \subseteq U.$$

But each W in the union which defines U is a subset of $X \setminus \{x\}$ because of the condition $x \notin W$. So

$$U \subseteq X \setminus \{x\}$$

and therefore

$$X \setminus \{x\} = U.$$

U is union of elements of $\mathcal{T}$ and hence is an element of $\mathcal{T}$, so $X \setminus \{x\} \in \mathcal{T}$. Therefore $\{x\}$ is closed. □

If $\mathcal{T}$ is not Hausdorff then sets of the form $\{x\}$ needn't be closed. In the trivial topology on a set with more than one element, for example, sets of the form $\{x\}$ are never closed. There are however examples of non-Hausdorff topologies in which all sets of the form $\{x\}$ are closed. The Zariski topology furnishes an example.

3.2 Interior, closure and boundary

Given a topology on a set we can define the interior, closure and boundary of any subset.

Definition 3.2.1. Suppose $(X, \mathcal{T})$ is a topological space and $A \in \wp(X)$. The *interior* of A, denoted A°, is the union of all open sets contained in A:

$$A^\circ = \bigcup_{U \in \mathcal{T}, U \subseteq A} U.$$

The *closure* of A, denoted $\overline{A}$, is the intersection of all closed sets containing A.

$$\overline{A} = \bigcap_{X \setminus V \in \mathcal{T}, A \subseteq V} V.$$

The *boundary* of A, denoted ∂A, is the relative complement of the interior of A in the closure of A:

$$\partial A = \overline{A} \setminus A^\circ.$$

As an example, the interior of $[0, 1)$ is $(0, 1)$, its closure is $[0, 1]$ and its boundary is $\{0, 1\}$.

The following proposition summarises some elementary properties of the interior, closure and boundary.

Proposition 3.2.2. *Suppose $(X, \mathcal{T})$ is a topological space, $x \in X$ and $A, B \in \wp(X)$.*

(a) A° is open and $\overline{A}$ is closed.

(b) $A^\circ \subseteq A \subseteq \overline{A} \subseteq X$

(c) If $A \subseteq B$ then $A^\circ \subseteq B^\circ$ and $\overline{A} \subseteq \overline{B}$.

(d) If $A \subseteq B$ and A is open then $A \subseteq B^\circ$.

(e) If $A \subseteq B$ and B is closed then $\overline{A} \subseteq B$.

(f) A is open if and only if $A = A^\circ$.

(g) B is closed if and only if $B = \overline{B}$.

(h) $(A^\circ)^\circ = A^\circ$ and $\overline{(\overline{A})} = \overline{A}$.

(i) $(X \setminus A)^\circ = X \setminus \overline{A}$ and $\overline{X \setminus A} = X \setminus A^\circ$.

(j) $\partial (X \setminus A) = \partial A$.

(k) The following three statements are equivalent:

(i) $x \in A^\circ$.

(ii) There is a $W \in \mathcal{O}(x)$ such that $W \subseteq A$.

(iii) There is a $W \in \mathcal{N}(x)$ such that $W \subseteq A$.

(l) The following three statements are equivalent:

(i) $x \in \overline{A}$.

(ii) For every $W \in \mathcal{O}(x)$, $W \cap A \neq \varnothing$.

(iii) For every $W \in \mathcal{N}(x)$, $W \cap A \neq \varnothing$.

(m) The following three statements are equivalent:

(i) $x \in \partial A$.

(ii) For every $W \in \mathcal{O}(x)$, $W \cap A \neq \varnothing$ and $W \cap (X \setminus A) \neq \varnothing$.

(iii) For every $W \in \mathcal{N}(x)$, $W \cap A \neq \varnothing$ and $W \cap (X \setminus A) \neq \varnothing$.

Proof. We prove these in turn.

(a) A° is a union of open sets and so is open. $\overline{A}$ is an intersection of closed sets and so is closed. $X \setminus A^*$ is closed because A° is open. $\partial A = \overline{A} \cap (X \setminus A^\circ)$ is the intersection of two closed sets and hence is closed.

(b) Any union of subsets of A is a subset of A and any intersection of supersets of A is a superset of A. Any intersection of subsets of X is a subset of X.

(c) Suppose $A \subseteq B$. Then $U \in \mathcal{T}$ and $U \subseteq A$ imply $U \in \mathcal{T}$ and $U \subseteq B$, so

$$A^\circ = \bigcup_{U \in \mathcal{T}, U \subseteq A} U \subseteq \bigcup_{U \in \mathcal{T}, U \subseteq B} U = B^\circ.$$

Also, if $X \setminus V \in \mathcal{T}$ and $B \subseteq V$ then $X \setminus V \in \mathcal{T}$ and $A \subseteq V$, so

$$\overline{A} = \bigcap_{X \setminus V \in \mathcal{T}, A \subseteq V} V \subseteq \bigcap_{X \setminus V \in \mathcal{T}, B \subseteq V} V = \overline{B}.$$

(d) If A is open and $A \subseteq B$ then A is one of the sets whose union defines B°, so $A \subseteq B^\circ$.

(e) If B is closed and $A \subseteq B$ then B is one of the sets whose intersection defines $\overline{A}$, so $\overline{A} \subseteq B$.

(f) Suppose A is open. Let $B = A$. Then $A \subseteq B$ so $A \subseteq B^\circ$ by (d). In other words, $A \subseteq A^\circ$. But $A^\circ \subseteq A$ by (b), so $A = A^\circ$.

Suppose, conversely, that $A = A^\circ$. A° is open by (a), so A is open.

(g) Suppose B is closed. Let $A = B$. Then $A \subseteq B$ so $\overline{A} \subseteq B$ by (e). In other words $\overline{B} \subseteq B$. But $B \subseteq \overline{B}$ by (b), so $B = \overline{B}$.

Suppose, conversely, that $B = \overline{B}$. $\overline{B}$ is closed by (a), so B is closed.

(h) By definition

$$A^\circ = \bigcup_{U \in \mathcal{P}} U$$

where

$$\mathcal{P} = \{U \in \mathcal{T} \colon U \subseteq A\}.$$

If $U \in \mathcal{P}$ then $U \subseteq A$ and hence $U^\circ \subseteq A^\circ$ by (c). But $U = U^\circ$ by (f), so $U \subseteq A^\circ$. On the other hand, $A^\circ \subseteq A$ by (b) so if $U \in \mathcal{T}$ and $U \subseteq A^\circ$ then $U \subseteq A$ and therefore $U \in \mathcal{P}$. So

$$\mathcal{P} = \{U \in \mathcal{T} \colon U \subseteq A^\circ\}$$

and hence

$$A^\circ = \bigcup_{U \in \mathcal{T}, U \subseteq A^\circ} U = (A^\circ)^\circ$$

By definition

$$\overline{A} = \bigcup_{V \in \mathcal{Q}} V$$

where

$$\mathcal{Q} = \{V \in \wp(X) \colon X \setminus V \in \mathcal{T}, A \subseteq V\}.$$

If $V \in \mathcal{Q}$ then $A \subseteq V$ and hence $\overline{A} \subseteq \overline{V}$ by (c). Also, $X \setminus V \in \mathcal{T}$ and V is closed, so $\overline{V} = V$ by (g). It follows that $\overline{A} \subseteq V$. Conversely, if $X \setminus V \in \mathcal{T}$ and $\overline{A} \subseteq V$ then $A \subseteq V$ because $A \subseteq \overline{A}$ by (b). Therefore

$$\mathcal{Q} = \left\{V \in \wp(X) \colon X \setminus V \in \mathcal{T}, \overline{A} \subseteq V\right\}$$

and hence

$$\overline{A} = \bigcap_{X \setminus V \in \mathcal{T}, \overline{A} \subseteq V} V = \overline{(\overline{A})}.$$

(i) By definition

$$(X \setminus A)^\circ = \bigcup_{U \in \mathcal{T}, U \subseteq X \setminus A} U$$

and

$$X \setminus \overline{A} = X \setminus \left(\bigcap_{X \setminus V \in \mathcal{T}, A \subseteq V} V \right).$$

The latter is equivalent to

$$X \setminus \overline{A} \bigcup_{X \setminus V \in \mathcal{T}, A \subseteq V} (X \setminus V).$$

If we reparameterise the union using $U = X \setminus V$ then

$$X \setminus \overline{A} \quad \bigcup_{U \in \mathcal{T}, A \subseteq X \setminus U} U.$$

But $A \subseteq X \setminus U$ if and only if $U \subseteq X \setminus A$ so this is the same as $(X \setminus A)^\circ$. So

$$(X \setminus A)^\circ = X \setminus \overline{A}.$$

This holds for any $A \in \wp(X)$ so it also holds with A replaced by $X \setminus A$.

$$(X \setminus (A \setminus X))^\circ = X \setminus \overline{(X \setminus A)},$$

or, more simply,

$$A^\circ = X \setminus \overline{(X \setminus A)}.$$

From this it follows that

$$X \setminus A^\circ = \overline{(X \setminus A)}.$$

(j)

$$\begin{aligned}\partial(X \setminus A) &= \overline{(X \setminus A)} \setminus (X \setminus A)^\circ \\ &= (X \setminus A^\circ) \setminus \left(X \setminus \overline{A}\right) \\ &= \overline{A} \setminus A^\circ = \partial A.\end{aligned}$$

(k) If $x \in A^\circ$ then there is, by the definition of A° as a union, a $U \in \mathcal{T}$ such $x \in U$ and $U \subseteq A$. Take W to be this U. Then $W \in \mathcal{O}(x)$ and $W \subseteq A$. $\mathcal{O}(x) \subseteq \mathcal{N}(x)$ to this W is also in $\mathcal{N}(x)$. Suppose, conversely, that there is a $W \in \mathcal{N}(x)$ such that $W \subseteq A$. By the definition of $\mathcal{N}(x)$ there is a $U \in \mathcal{T}$ such that $x \in U$ and $U \subseteq W$. It follows that $U \subseteq A$. Thus U is one of the sets appearing in the union which defines A°, so $U \subseteq A^\circ$ and hence $x \in A^\circ$.

(l) Suppose $W \in \mathcal{O}(x)$. If $W \cap A = \varnothing$ then $A \subseteq X \setminus W$. $X \setminus W$ is closed so it is one of the sets V whose intersection defines $\overline{A}$. So $\overline{A} \subseteq X \setminus W$. But $W \in \mathcal{O}(x)$ so $x \in W$ and hence $x \notin X \setminus W$. Therefore $x \notin \overline{A}$. So if $W \cap A = \varnothing$ then $x \notin \overline{A}$. Therefore if $x \in \overline{A}$ then $W \cap A \neq \varnothing$. So for every $W \in \mathcal{O}(x)$, $W \cap A \neq \varnothing$.

Next, suppose that for every $W \in \mathcal{O}(x)$, $W \cap A \neq \varnothing$. If $W \in \mathcal{N}(x)$ then there is a $U \in \mathcal{T}$ such that $x \in U$ and $U \subseteq W$. Then $U \in \mathcal{O}(x)$. So $U \cap A \neq \varnothing$. $U \cap A \subseteq W \cap A$ so $W \cap A \neq \varnothing$. So for every $W \in \mathcal{N}(x)$, $W \cap A \neq \varnothing$.

Suppose finally that for every $W \in \mathcal{N}(x)$, $W \cap A \neq \varnothing$. Set $W = X \setminus \overline{A}$. $A \subseteq \overline{A}$ by (b) so $W \cap A \subseteq W \cap \overline{A} = \varnothing$ so $W \notin \mathcal{N}(x)$. $\overline{A}$ is closed by (a) so then $W \in \mathcal{T}$. The only way we can have $W \in \mathcal{T}$ but $W \notin \mathcal{N}(x)$ is if $x \notin W$, i.e. if $x \in \overline{A}$.

(m)

$$\partial A = \overline{A} \setminus A^\circ = \overline{A} \cap (X \setminus A^\circ) = \overline{A} \cap \overline{X \setminus A}$$

by (i). So $x \in \partial A$ if and only if $x \in \overline{A}$ and $x \in \overline{X \setminus A}$. By (l) this happens if and only if for every $W \in \mathcal{O}(x)$ we have $W \cap A \neq \varnothing$ and $W \cap (X \setminus A) \neq \varnothing$. Or if and only for all $W \in \mathcal{N}(x)$ we have $W \cap A \neq \varnothing$ and $W \cap (X \setminus A) \neq \varnothing$.

□

The behaviour of balls in a metric space with respect to interior, closure and boundary is not exactly as one might expect. $B(x,r) \subseteq \bar{B}(x,r)^\circ$ and $\overline{B(x,r)} \subseteq \bar{B}(x,r)$ as consequences of Propositions 3.2.2d and3.2.2e respectively. In $\mathbf{R}^n$ with the usual metric one can replace these inclusions with equations, but this is not possible in general. If d is the discrete metric on a set X and $x \in X$ then $B(x,1) = \{x\}$ while $\bar{B}(x,r)^\circ = X$ and $\overline{B(x,1)} = \{x\}$ while $\bar{B}(x,1) = X$. In this case also the sphere of radius 1 about x is not equal to the boundary of either the open ball of radius 1 or the closed ball of radius 1 about x.

3.3 Closure and limits

Recall Definition 1.18.3: A net is a function whose domain is a directed set. If $(D, \preccurlyeq)$ is directed set and f is a function, i.e. a net, from D to a topological space $(Y, \mathcal{T})$ then we say that $z \in Y$ is the limit of the net f, written

$$\lim f = z$$

if for all $Z \in \mathcal{O}(z)$ there is an $a \in D$ such that if $b \in D$ and $a \preccurlyeq b$ then $f(b) \in Z$.

The following proposition relates nets to closures.

Proposition 3.3.1. *Suppose f is a net from a directed set $(D, \preccurlyeq)$ to a topological space $(Y, \mathcal{T})$, $A \subseteq Y$, $f(a) \in A$ for all $a \in D$, and $\lim f = z$. Then $z \in \overline{A}$.*

Proof. For every $Z \in \mathcal{O}(z)$ there is an $a \in D$ such that if $b \in D$ and $a \preccurlyeq b$ then $f(b) \in Z$. In particular $f(a) \in Z$. By hypothesis $f(a) \in A$. So $f(a) \in Z \cap A$ and $Z \cap A \neq \varnothing$. We've just seen that every $Z \in \mathcal{O}(z)$, $Z \cap A \neq \varnothing$. By Proposition 3.2.2l then $z \in \overline{A}$. □

This proposition has a sort of converse.

Proposition 3.3.2. *Suppose $(Y, \mathcal{T})$ is a topological space, $A \subseteq Y$ and $z \in \overline{A}$. Then there is a directed set $(D, \preccurlyeq)$ and a function $f : D \to Y$ such that $f(a) \in A$ for all $a \in D$ and $\lim f = z$.*

Proof. We can take $(D, \preccurlyeq)$ to be $(\mathcal{O}(z), \subseteq)$. This is a directed set by Proposition 1.14.2j. Define

$$X = \{(W, y) \in \mathcal{O}(z) \times Y : y \in W \cap A\}$$

and define $g : X \to \mathcal{O}(z)$ by $g(W, y) = W$. and $h : X \to Y$ by $h(W, y) = y$. $z \in \overline{A}$ so $W \cap A \neq \varnothing$ by Proposition 3.2.2l. In other words, there is a $y \in Y$ such that $g(W, y) = W$. g is therefore a surjection. By Proposition 2.7.2 it has a right inverse. In other words, there is a function $i : \mathcal{O}(z) \to X$ such that $g \circ i$ is the identity on $\mathcal{O}(z)$. Define $f : \mathcal{O}(z) \to Y$ by $f = h \circ i$. For any $W \in \mathcal{O}(z)$ then let $(V, y) = i(W)$. $V = h(i(W)) = W$ so $i(W) = (W, y)$ and $f(W) = h(i(W)) = y$. $(W, y) \in X$ so $y \in W \cap A$. Therefore $f(W) \in W \cap A$. In particular

$$f(W) \in A$$

for all $W \in \mathcal{O}(z)$. Also, if $V, W \in \mathcal{O}(z)$ and $V \subseteq W$ then $f(V) \in V \subseteq W$. So for all $W \in \mathcal{O}(z)$ if $W \subseteq V$ then $f(V) \in W$. Therefore

$$\lim f = z.$$

□

Every sequence is a net with domain $(D, \preccurlyeq) = (\mathbf{N}, \leq)$. so we obtain the following proposition as a corollary to Proposition 3.3.1.

Proposition 3.3.3. *Suppose $\alpha : \mathbf{N} \to Y$ is a sequence with values in a topological space $(Y, \mathcal{T})$, $A \subseteq Y$, $\alpha_n \in A$ for all $n \in \mathbf{N}$, and $\lim_{n\to\infty} \alpha_n = z$. Then $z \in \overline{A}$.*

Can we also replace nets with sequences in Proposition 3.3.2? Not without further restrictions on the topological space $(Y, \mathcal{T})$. There are many special cases where we can, though, including the case where the topology $\mathcal{T}$ is metrisable, as we will see in a later section.

3.4 Dense subsets

Definition 3.4.1. Suppose $(X, \mathcal{T})$ is a topological space. A subset $A \in \wp(X)$ is called *dense* if $\overline{A} = X$.

Proposition 3.2.2 gives the following properties of dense subsets.

Proposition 3.4.2. *Suppose $(X, \mathcal{T})$ is a topological space and $A, B \in \wp(X)$.*

(a) If $A \subseteq B$ and A is dense then so is B.

(b) The only dense closed set is X.

(c) A is dense if any only if the interior of $X \setminus A$ is empty.

(d) The following three statements are equivalent:

(i) A is dense.

(ii) For every $x \in X$ and $W \in \mathcal{O}(x)$, $W \cap A \neq \varnothing$.

(iii) For every $x \in X$ and $W \in \mathcal{N}(x)$, $W \cap A \neq \varnothing$.

Proof. Each of these follows from one of the parts of Proposition 3.2.2.

(a) By Proposition 3.2.2c, if $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$. If A is dense then $\overline{A} = X$, so $X \subseteq \overline{B}$. But $\overline{B} \subseteq X$ so $\overline{B} = X$. In other words, B is dense.

(b) If A is dense then $\overline{A} = X$. By Proposition 3.2.2g A is closed if and only if $A = \overline{A}$, i.e. if and only if $A = X$.

(c) A is dense if and only if $A = \overline{A}$, i.e. if and only if $X \setminus \overline{A} = \varnothing$. By Proposition 3.2.2i $(X \setminus A)^\circ = X \setminus \overline{A}$.

(d) A is dense if and only if $\overline{A} = X$, i.e. if and only if $x \in \overline{A}$ for every $x \in X$. We can rewrite the condition $x \in \overline{A}$ in two alternate forms using Proposition 3.2.2l.

□

As an example $\mathbf{Q}$ is a dense subset of $\mathbf{R}$. This is most easily seen from the last part of the Proposition. For every $x \in \mathbf{R}$ every neighbourhood of x contains an open ball about x. Balls in $\mathbf{R}$ are non-empty open intervals and every non-empty open interval contains a rational number.

Propositions 3.3.1 and 3.3.2 gives a characterisation of dense subsets in terms of nets.

Proposition 3.4.3. *Suppose $(Y, \mathcal{T})$ is a topological space and $A \subseteq Y$. A is dense if and only if for every $z \in Y$ there is a directed set $(D, \preccurlyeq)$ and a net $f \colon D \to Y$ such that $f(a) \in A$ for all $a \in D$ and $\lim f = z$.*

Proof. Suppose A is dense. Then $X = \overline{A}$. If $z \in Y$ then $z \in \overline{A}$ so by Proposition 3.3.2 there is a directed set $(D, \preccurlyeq)$ and a net $f \colon D \to Y$ such that $f(a) \in A$ for all $a \in D$ and $\lim f = z$.

Suppose, conversely that for every $z \in Y$ there is a directed set $(D, \preccurlyeq)$ and a net $f \colon D \to Y$ such that $f(a) \in A$ for all $a \in D$ and $\lim f = z$. By Proposition 3.3.1 then $z \in \overline{A}$. So for every $z \in Y$ we have $z \in \overline{A}$ and hence $Y \subseteq \overline{A}$. But $\overline{A} \subseteq Y$ so $\overline{A} = Y$ and hence A is dense. □

If we replace nets by sequences then we still have the "if" part of the proposition but don't have the "only if" part, because we have Proposition 3.3.3 but don't have a converse for it without further restrictions on $(Y, \mathcal{T})$.

3.5 Comparison of topologies

Definition 3.5.1. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X then $\mathcal{T}_1$ is a said to be *stronger* than $\mathcal{T}_2$ if $\mathcal{T}_1 \supseteq \mathcal{T}_2$ and *weaker* if $\mathcal{T}_1 \subseteq \mathcal{T}_2$

As with relations, the terms stronger and weaker are used in a non-strict sense. Each topology is both stronger and weaker than itself. Also, it's entirely possible to have two topologies neither of which is stronger than the other. For example neither of the topologies $\mathcal{T}_1 = \{\varnothing, \{1\}, \{1, 2\}\}$ or $\mathcal{T}_2 = \{\varnothing, \{2\}, \{1, 2\}\}$ on $\{1, 2\}$ is stronger than the other. There is a strongest topology on any set, the discrete topology, and a weakest topology, the trivial topology. Of course we are usually interested in topologies in between these two.

As a non-trivial example of the notions of stronger and weaker topologies we have the following.

Proposition 3.5.2. *The Zariski topology on $\mathbf{R}^n$ is weaker than the metric topology. It is strictly weaker in the sense that the Zariski topology is a proper subset of the metric topology.*

Proof. The metric topology is the set of open sets in the sense of Definition 1.10.1. In other words U is open if and only if for all $\mathbf{x} \in U$ there's an $r > 0$ such that $B(\mathbf{x}, r) \subseteq U$. The balls here are with respect to the metric coming from the Euclidean norm. In other words $B(\mathbf{x}, r)$ is the set of $\mathbf{y}$ such that $\|\mathbf{y} - \mathbf{x}\| < r$.

The Zariski topology on $\mathbf{R}^n$ was defined in Section 1.11 to be the set of subsets of $\mathbf{R}^n$ whose relative complements were the zero sets of finite sets of polynomials. In other words, U is an element of the Zariski topology if there are $p_1, \ldots, p_m$ such that

$$\mathbf{R}^n \setminus U = \{\mathbf{x} \in \mathbf{R}^n \colon p_1(\mathbf{x}) = \cdots = p_m(\mathbf{x}) = 0\}.$$

This is equivalent to

$$U = \{\mathbf{x} \in \mathbf{R}^n \colon \ p_1(\mathbf{x}) \neq 0 \text{ or } \cdots \text{ or } p_m(\mathbf{x}) \neq 0 \ \}.$$

If $\mathbf{x} \in U$ then there is some j such that $p_j(\mathbf{x}) \neq 0$. Let $\epsilon = |p_j(\mathbf{x})|$. Then $\epsilon > 0$. Polynomials are continuous so there is therefore a $\delta > 0$ such that $|p_j(\mathbf{y}) - p_j(\mathbf{x})| < \epsilon$ whenever $\|\mathbf{y} - \mathbf{x}\| < \delta$, and hence $p_j(\mathbf{y}) \neq 0$. So $B(\mathbf{x}, \delta) \subseteq U$. There is therefore an open ball about each point in U which is contained in U. In other words, U belongs to the metric topology on $\mathbf{R}^n$.

An example of set in the metric topology but not the Zariski topology is $B(\mathbf{0}, 1)$. There is no finite set of polynomials whose set of common zeroes is $\mathbf{R}^n \setminus B(\mathbf{0}, 1)$. □

Proposition 3.5.3. *If* $\mathbf{A} \in \wp(\mathbf{T}(X))$ *is a set of topologies on* X *and* $\mathcal{M} = \bigcap_{\mathcal{T}\in\mathbf{A}} \mathcal{T}$ *then* $\mathcal{M}$ *is a topology on* X*.*

Because $\mathcal{M} \subseteq \mathcal{T}$ for all $\mathcal{T} \in \mathbf{A}$ the topology $\mathcal{M}$ is weaker than every topology in $\mathbf{A}$.

Proof. To show that $\mathcal{M}$ is a topology it suffices to check the conditions 1.11.1a, 1.11.1b and 1.11.1c.

(a) $\varnothing \in \mathcal{T}$ for all $\mathcal{T} \in \mathbf{A}$ so $\varnothing \in \mathcal{M}$. Similarly, $X \in \mathcal{T}$ for all $\mathcal{T} \in \mathbf{A}$ so $X \in \mathcal{M}$.

(b) If $V, W \in \mathcal{M}$ then $V, W \in \mathcal{T}$ for all $\mathcal{T} \in \mathbf{A}$. Each $\mathcal{T}$ is a topology, so $V \cap W \in \mathcal{T}$. Since this holds for all $\mathcal{T} \in \mathbf{A}$ and $\mathcal{W} = \bigcap_{\mathcal{T}\in\mathbf{A}} \mathcal{T}$ we have $V \cap W \in \mathcal{M}$.

(c) Suppose $\mathcal{E} \subseteq \mathcal{M}$. Then $\mathcal{E} \subseteq \mathcal{T}$ for all $\mathcal{T} \in \mathbf{A}$. Each $\mathcal{T}$ is a topology, so $\bigcup_{V\in\mathcal{E}} V \in \mathcal{T}$. Since this holds for all $\mathcal{T} \in \mathbf{A}$ and $\mathcal{W} = \bigcap_{\mathcal{T}\in\mathbf{A}} \mathcal{T}$ we have $\bigcup_{V\in\mathcal{E}} V \in \mathcal{M}$.

□

The main purpose of the proposition above is to prove the following.

Proposition 3.5.4. *For any* $\mathcal{A} \in \wp(\wp(X))$ *there is a unique topology* $\mathcal{M}$ *such that*

(a) $\mathcal{A} \subseteq \mathcal{M}$*, and*

(b) if $\mathcal{T}$ *is a topology on* X *and* $\mathcal{A} \subseteq \mathcal{T}$ *then* $\mathcal{M} \subseteq \mathcal{T}$*.*

Proof. Let $\mathcal{A}$ be the set of all topologies $\mathcal{T}$ on X such that $\mathcal{A} \subseteq \mathcal{T}$. There is at least one $\mathcal{T} \in \mathbf{A}$, namely the discrete topology $\mathcal{T} = \wp(X)$. By the proposition above $\mathcal{M} = \bigcap_{\mathcal{T}\in\mathbf{A}} \mathcal{T}$ is a topology on X. If $\mathcal{T}$ is a topology on X and $\mathcal{A} \subseteq \mathcal{T}$ then $\mathcal{T} \in \mathbf{A}$ by the definition of $\mathbf{A}$ and $\mathcal{M} \subseteq \mathcal{T}$ by the definition of the intersection. This establishes the existence of topology $\mathcal{M}$ as promised by the statement of the proposition.

To establish the uniqueness, assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are topologies on X such that

(a) $\mathcal{A} \subseteq \mathcal{M}_j$, and

(b) if $\mathcal{T}$ is a topology on X and $\mathcal{A} \subseteq \mathcal{T}$ then $\mathcal{M}_j \subseteq \mathcal{T}$

for $j = 1$ and $j = 2$. By (a) with $j = 1$ we know that $\mathcal{A} \subseteq \mathcal{M}_1$. By (b) with $j = 2$ we know that if $\mathcal{T}$ is a topology on X and $\mathcal{A} \subseteq \mathcal{T}$ then $\mathcal{M}_2 \subseteq \mathcal{T}$. Applying that to $\mathcal{T} = \mathcal{M}_1$ gives $\mathcal{M}_2 \subseteq \mathcal{M}_1$. The same argument with the 1's and 2's swapped gives $\mathcal{M}_1 \subseteq \mathcal{M}_2$. Combining those inclusions gives $\mathcal{M}_1 = \mathcal{M}_2$. □

We call the topology $\mathcal{M}$ of the proposition the topology *generated* by the set $\mathcal{A}$ and we say that the set $\mathcal{A}$ is a *subbase* for the topology $\mathcal{M}$.

The meaning of the theorem is that for any set $\mathcal{A}$ of subsets of X there is a topology in which all the sets in A are open and that among all such topologies there is a weakest one. We will often use this to define topologies. There's also a strongest one, but that's not very interesting: it's just the discrete topology.

A simple but useful consequence of the definitions is the following.

Proposition 3.5.5. *Suppose* $\mathcal{A}_1, \mathcal{A}_2 \in \wp(\wp(X))$*,* $\mathcal{T}_1$ *is the topology generated by* $\mathcal{A}_1$ *and* $\mathcal{T}_2$ *is the topology generated by* $\mathcal{A}_2$*. If* $\mathcal{A}_1 \subseteq \mathcal{A}_2$ *then* $\mathcal{T}_1 \subseteq \mathcal{T}_2$*.*

Proof. $\mathcal{A}_1 \subseteq \mathcal{A}_2$ by hypothesis and $\mathcal{A}_2 \subseteq \mathcal{T}_2$ by the definition of the topology generated by a set of sets. So $\mathcal{A}_1 \subseteq \mathcal{T}_2$. $\mathcal{T}_2$ is a topology containing $\mathcal{A}_1$ and $\mathcal{T}_1$ is the weakest topology containing $\mathcal{A}_1$ so $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$. In other words, $\mathcal{T}_1 \subseteq \mathcal{T}_2$. □

Under very weak hypotheses we can describe the topology generated by a subbase more explicitly.

Proposition 3.5.6. *Suppose* X *is a set and* $\mathcal{A} \in \wp(\wp(X)))$*. If* X *is a union of elements of* $\mathcal{A}$ *then the elements of the topology generated by* $\mathcal{A}$ *are the unions of intersections of finitely many elements of* $\mathcal{A}$*.*

Proof. Let $\mathcal{M}$ be the set whose elements are the unions of intersections of finitely many elements of $\mathcal{A}$. If $U \in \mathcal{A}$ then U is an intersection of finitely many elements of $\mathcal{A}$, since we can just take an intersection of only the one set U. Similarly, $U \in \mathcal{M}$ because we can form a union with only the set U. So $\mathcal{A} \subseteq \mathcal{M}$.

$\varnothing \in \mathcal{M}$ as an empty union. $X \in \mathcal{M}$ because X was assumed to be a union of elements of $\mathcal{A}$. If $V, W \in \mathcal{M}$

then

$$V = \bigcup_{P \in \mathcal{E}} P$$

and

$$W = \bigcup_{Q \in \mathcal{M}} Q$$

where $\mathcal{E}$ and $\mathcal{M}$ are sets intersections of finitely many elements of $\mathcal{A}$. Then

$$V \cap W = \bigcup_{P \in \mathcal{E}, Q \in \mathcal{M}} P \cap Q.$$

Each $P \cap Q$ is an intersection of finitely many elements of $\mathcal{A}$, so $V \cap W \in \mathcal{M}$. Any union of unions of intersections of finitely many elements of $\mathcal{A}$ is a union of intersections of finitely many elements of $\mathcal{A}$, so any union of elements of $\mathcal{M}$ is an element of $\mathcal{M}$. So $\mathcal{M}$ is a topology on X.

Proposition 3.5.4 shows that there is only one topology $\mathcal{M}$ on X which contains $\mathcal{A}$ and is weaker than any other topology containing $\mathcal{A}$, and this was defined to be the topology generated by $\mathcal{A}$. □

As a corollary we have the following result for unions of topologies. This might seem like a very special case, but it is one we will need in a later section.

Corollary 3.5.7. *Suppose X is a set and $\mathbf{S} \subseteq \mathbf{T}(X)$ is non-empty. Suppose $\mathcal{A} = \bigcup_{\mathcal{T} \in \mathbf{S}} \mathcal{T}$ and $\mathcal{M}$ is the topology generated by $\mathcal{A}$. Then the elements of $\mathcal{M}$ are the unions of intersections of finitely many elements of $\mathcal{A}$.*

Proof. Since $\mathbf{S}$ is non-empty it contains some topology and that topology contains X, so $X \in \mathcal{A}$. □

If we already have intersections then we don't need to add them. That's the purpose of the following definition and proposition.

Definition 3.5.8. A set $\mathcal{B} \in \wp(\wp(X))$ is called a *base* for a topology on X if the following conditions are satisfied.

(a) For all $x \in X$ there is a $B \in \mathcal{B}$ such that $x \in B$.

(b) For all $A, B \in \mathcal{B}$ and all $x \in A \cap B$ there is $C \in \mathcal{B}$ such that $x \in C$ and $C \subseteq A \cap B$.

Proposition 3.5.9. *If $\mathcal{B}$ is a base then the topology $\mathcal{T}$ which it generates is the set of unions of elements of $\mathcal{B}$.*

Proof. Condition 3.5.8a and Proposition 3.5.4 show that every element of $\mathcal{T}$ is a union of intersections of finitely many elements of $\mathcal{B}$. Condition 3.5.8b was stated for the intersection of two elements of $\mathcal{B}$ but as usual we can prove a version for intersections finitely many sets by induction. Suppose $B_1, \ldots, B_k \in \mathcal{B}$. Then for all $x \in \bigcap_{j=1}^k B_k$ there is a $C \in \mathcal{B}$ such that $x \in C$ and $C \subseteq \bigcap_{j=1}^k B_k$. Let

$$U = \bigcup_{C \in \mathcal{B}, C \subseteq \bigcap_{j=1}^k B_k} C.$$

Each $x \in \bigcap_{j=1}^k B_k$ belongs to such a C so

$$\bigcap_{j=1}^k B_k \subseteq U.$$

On the other hand each of the sets C in the union which defines U is a subset of $\bigcap_{j=1}^k B_k$ so

$$U \subseteq \bigcap_{j=1}^k B_k$$

So

$$\bigcap_{j=1}^k B_k = U = \bigcup_{C \in \mathcal{B}, C \subseteq \bigcap_{j=1}^k B_k} C.$$

So any intersection of finitely many elements of $\mathcal{B}$ is a union of elements of $\mathcal{B}$. Therefore any union of such intersections is a union of unions of elements of $\mathcal{B}$, and hence a union of elements of $\mathcal{B}$. □

The following proposition illustrates the concept of the topology generated by a set.

Proposition 3.5.10. *Let $\mathcal{B}$ be the set of open balls in a metric space (X, d). Let $\mathcal{T}$ be the usual topology on X, i.e. the one consisting of open sets in the sense of Definition 1.10.1. The topology generated by $\mathcal{B}$ is $\mathcal{T}$.*

Proof. Suppose that $U \in \mathcal{T}$. Let

$$V = \bigcup_{B(x,r) \subseteq U} B(x,r).$$

Then $V \subseteq U$ because it's a union of subsets of U. On the other hand if $x \in U$ then there is an $r > 0$ such that $B(x,r) \subseteq U$ because U is open in the sense of Definition 1.10.1. $x \in B(x,r)$ and $B(x,r) \subseteq V$ so $x \in V$. Since $x \in U$ implies $x \in V$ we have $U \subseteq V$. Since $V \subseteq U$ and $U \subseteq V$ we have $U = V$ and hence

$$U = \bigcup_{B(x,r) \subseteq U} B(x,r).$$

Each $B(x,r)$ is in $\mathcal{B}$ by definition. $\mathcal{B} \subseteq \mathcal{M}$ so $B(x,r) \in \mathcal{M}$. $\mathcal{M}$ is a topology so any union elements of $\mathcal{M}$ is an element of $\mathcal{M}$. Therefore $U \in \mathcal{M}$. This holds for any $U \in \mathcal{T}$, so $\mathcal{T} \subseteq \mathcal{M}$. $\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{T}$ is a topology so by (b) from the proposition $\mathcal{M} \subseteq \mathcal{T}$. From $\mathcal{T} \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq \mathcal{T}$ we get $\mathcal{M} = \mathcal{T}$. □

In some sense this isn't a new example. This mirrors how the metric topology was originally constructed. In the case of $\mathbf{R}^n$ this proposition tells us that the balls $B(\mathbf{x}, r)$ with $\mathbf{x} \in \mathbf{R}^n$ and $r > 0$ generate the topology. It turns out that we don't need all the balls however. There is a countable collection of balls which suffices to generate the topology.

Proposition 3.5.11. *Let $\mathcal{C}$ be the set of balls $B(\mathbf{x}, r)$ in $\mathbf{R}^n$ with $\mathbf{x} \in \mathbf{Q}^n$ and $r \in \mathbf{Q}_+$. Then $\mathcal{C}$ generates the usual topology $\mathcal{T}$.*

Here $\mathbf{Q}_+$ denotes the positive rational numbers.

Proof. The beginning and end of the proof follows that of the previous proposition, but the middle is more complicated and uses the fact that every non-empty open interval in $\mathbf{R}$ contains a rational number. Suppose that $U \in \mathcal{T}$. Let

$$V = \bigcup_{B(\mathbf{x},r) \subseteq U, \mathbf{x} \in \mathbf{Q}^n, r \in \mathbf{Q}_+} B(\mathbf{x}, r).$$

Then $V \subseteq U$ because it's a union of subsets of U. Suppose $\mathbf{y} \in U$. U is open so there is some $s > 0$ such that $B(\mathbf{y}, s) \subseteq U$. Choose a rational number r such that

$$0 < r < s/2.$$

Choose $x_1, \ldots, x_n$ in $\mathbf{Q}$ such that

$$y_j - r/n < x_j < y_j + r/n$$

Then

$$\|\mathbf{x} - \mathbf{y}\| < r$$

and hence

$$B(\mathbf{x}, r) \subseteq B(\mathbf{y}, 2r) \subseteq B(\mathbf{y}, s) \subseteq U.$$

Also, $\mathbf{y} \in B(\mathbf{x}, r)$. Since $\mathbf{x} \in \mathbf{Q}^n$ and $r \in \mathbf{Q}_+$ we have $\mathbf{y} \in V$. So $U \subseteq V$. Since $V \subseteq U$ and $U \subseteq V$ we have $U = V$ and hence

$$U = \bigcup_{B(\mathbf{x},r) \subseteq U, \mathbf{x} \in \mathbf{Q}^n, r \in \mathbf{Q}_+} B(\mathbf{x}, r).$$

Each $B(\mathbf{x}, r)$ is in $\mathcal{C}$ by definition. $\mathcal{C} \subseteq \mathcal{M}$ so $B(\mathbf{x}, r) \in \mathcal{M}$. $\mathcal{M}$ is a topology so any union elements of $\mathcal{M}$ is an element of $\mathcal{M}$. Therefore $U \in \mathcal{M}$. This holds for any $U \in \mathcal{T}$, so $\mathcal{T} \subseteq \mathcal{T}$. $\mathcal{C} \subseteq \mathcal{T}$ and $\mathcal{T}$ is a topology so by (b) from the proposition $\mathcal{M} \subseteq \mathcal{T}$. From $\mathcal{T} \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq \mathcal{T}$ we get $\mathcal{M} = \mathcal{T}$.

Suppose $\mathcal{T}$ is a topology on X such that $\mathcal{A} \subseteq \mathcal{T}$. Then every union of intersections of finitely many elements of $\mathcal{A}$ is a union of intersections of finitely many elements of $\mathcal{T}$. $\mathcal{T}$ is a topology so any union of intersections of finitely many elements of $\mathcal{T}$ is an element of $\mathcal{M}$. So $\mathcal{F} \subseteq \mathcal{T}$. □

3.6 Continuous functions

Definition 3.6.1. Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f : X \to Y$ is a function. f is said to be *continuous* at $x \in X$ if the preimage of every neighbourhood of $f(x)$ is a neighbourhood of x. f is said to be continuous if the preimage of every open subset of Y is an open subset of X.

These are expressed at the level of sets but we could also express them at the level of sets of sets.

Proposition 3.6.2. *Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f : X \to Y$ is a function. f is continuous at x if and only if $\mathcal{N}(f(x)) \subseteq$*

$f^{**}(\mathcal{N}(x))$. *f is continuous if and only if $\mathcal{T}_Y \subseteq f^{**}(\mathcal{T}_X)$.*

Proof. By definition f is continuous at $x \in X$ if and only if the preimage of every neighbourhood of $f(x)$ is a neighbourhood of x, i.e. if and only if for all $Z \in \mathcal{N}(f(x))$ we have $f^*(Z) \in \mathcal{N}(x)$. By the definition of the preimage, applied to the function $f^* \colon \wp(Y) \to \wp(X)$, the last condition is equivalent to $Z \in f^{**}(\mathcal{N}(x))$. So f is continuous at x if and only if for all $Z \in \mathcal{N}(f(x))$ we have $Z \in f^{**}(\mathcal{N}(x))$, i.e. if and only if $\mathcal{N}(f(x)) \subseteq f^{**}(\mathcal{N}(x))$.

By definition f is continuous if and only if the preimage of every open subset of Y is an open subset of X, i.e. if and only if for every $W \in \mathcal{T}_Y$ we have $f^*(W) \in \mathcal{T}_X$. Using the definition of the preimage, applied to f^* again, this last condition is equivalent to $W \in f^{**}(\mathcal{T}_X)$. So f is continuous if and only if for every $W \in \mathcal{T}_Y$ we have $W \in f^{**}(\mathcal{T}_X)$, i.e. if and only if $\mathcal{T}_Y \subseteq f^{**}(\mathcal{T}_X)$. □

The terminology suggests the the following should be true.

Proposition 3.6.3. *Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f \colon X \to Y$ is a function. f is continuous if and only if f is continuous at each $x \in X$.*

Proof. Suppose f is continuous, $x \in X$ and $Z \in \mathcal{N}(f(x))$. By the definition of neighbourhood there is a $W \in \mathcal{T}_Y$ such that $f(x) \in W$ and $W \subseteq Z$. f is continuous so $f^*(W) \in \mathcal{T}_X$. $x \in f^*(W)$ because $f(x) \in W$. From $W \subseteq Z$ is follows that $f^*(W) \subseteq f^*(Z)$. So there is a $U \in \mathcal{T}_X$ such that $x \in U$ and $U \subseteq f^*(Z)$, namely $U = f^*(W)$. Therefore $f^*(Z) \in \mathcal{N}(x)$. We've just seen that for every $x \in X$ and every $Z \in \mathcal{N}(f(x))$ we have $f^*(Z) \in \mathcal{N}(x)$. In other words, for every $x \in X$ the function f is continuous at x.

Suppose, conversely, that f is continuous at x for every $x \in X$, and that $W \in \mathcal{T}_Y$. Let $U = f^*(W)$. If $x \in U$ then $f(x) \in W$. Since $W \in \mathcal{T}_Y$ it follows that $W \in \mathcal{O}(f(x))$ and hence $W \in \mathcal{N}(f(x))$. f is continuous at x and $U = f^*(W)$ so $U \in \mathcal{N}(x)$. In other words there is $V \in \mathcal{T}$ such that $x \in V$ and $V \subseteq U$. Then

$$U = \bigcup_{V \in \mathcal{T}, V \subseteq U} V$$

because every $x \in U$ is in at least one such V and each V is contained in U. U is therefore a union of elements of $\mathcal{T}_X$. Any such union is an element of $\mathcal{T}_X$ because $\mathcal{T}_X$ is a topology, so $f^*(W) = U \in \mathcal{T}_X$. We've now shown that for every $W \in \mathcal{T}_Y$ we have $f^*(W) \in \mathcal{T}_X$, so f is continuous. □

Continuity of compositions is described by the following proposition.

Proposition 3.6.4. *Suppose $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ are topological spaces and $f \colon X \to Y$ and $g \colon Y \to Z$ are functions. If f is continuous at x and g is continuous at $f(x)$ then $g \circ f$ is continuous at x. If f and g are continuous then $g \circ f$ is continuous.*

Proof. Suppose f is continuous at x and g is continuous at $f(x)$. By Proposition 3.6.2 then

$$\mathcal{N}(f(x)) \subseteq f^{**}(\mathcal{N}(x))$$

and

$$\mathcal{N}(g(f(x))) \subseteq g^{**}(\mathcal{N}(f(x))).$$

Preimages preserve inclusions so the first of these implies

$$g^{**}(\mathcal{N}(f(x))) \subseteq g^{**}(f^{**}(\mathcal{N}(x))).$$

Combining the previous two inclusions,

$$\mathcal{N}(g(f(x))) \subseteq g^{**}(f^{**}(\mathcal{N}(x))).$$

By the definition of composition then

$$\mathcal{N}((g \circ f)(x)) \subseteq (g^{**} \circ f^{**})(\mathcal{N}(x)).$$

Taking preimages reverses the order in a composition so taking preimages twice leaves the order unchanged, i.e.

$$g^{**} \circ f^{**} = (g \circ f)^{**}.$$

So

$$\mathcal{N}((g \circ f)(x)) \subseteq (g \circ f)^{**}(\mathcal{N}(x)).$$

By Proposition 3.6.2 this means that $g \circ f$ is continuous at x.

If f and g are continuous then f is continuous at x for all $x \in X$ and g is continuous at y for all $y \in Y$, and hence at $f(x)$ for all $x \in X$, by Proposition 3.6.3. So by what we've just proved $g \circ f$ is continuous at x for all $x \in X$. Using Proposition 3.6.3 again, $g \circ f$ is therefore continuous. □

There is a criterion for continuity in terms of subbases.

Proposition 3.6.5. *Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f\colon X \to Y$ is a function. Suppose further that $\mathcal{A}$ is subbase for $\mathcal{T}_Y$. Then f is continuous if and only if*

$$\mathcal{A} \subseteq f^{**}(\mathcal{T}_X).$$

Proof. Suppose f is continuous. Then $\mathcal{T}_Y \subseteq f^{**}(\mathcal{T}_X)$ by Proposition 3.6.2. $\mathcal{A}$ is a subbase for $\mathcal{T}_Y$ so $\mathcal{A} \subseteq \mathcal{T}_Y$ and hence $\mathcal{A} \subseteq f^{**}(\mathcal{T}_X)$.

Suppose, conversely, that $\mathcal{A} \subseteq f^{**}(\mathcal{T}_X)$. $f^{**}(\mathcal{T}_X)$ is a topology on Y by Lemma 3.1.1. $\mathcal{T}_Y$ is the weakest topology on Y containing $\mathcal{A}$ so it is weaker than $f^{**}(\mathcal{T}_X)$. In other words, $\mathcal{T}_Y \subseteq f^{**}(\mathcal{T}_X)$. Therefore f is continuous by Proposition 3.6.2. □

We can also describe continuity in terms of nets.

Proposition 3.6.6. *Suppose $(D, \preccurlyeq)$ is a directed set and $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces. Suppose $g\colon D \to X$ is a net and $f\colon X \to Y$ is a function. If $\lim g = z$ and f is continuous at z then $\lim f \circ g = f(z)$.*

Proof. Suppose that $W \in \mathcal{O}(f(z))$. Then $W \in \mathcal{N}(f(z))$. By the definition of continuity at x we then have $f^*(W) \in \mathcal{N}(z)$. By the definition of neighbourhoods there is a $Z \in \mathcal{T}_X$ such that $z \in Z$ and $Z \subseteq f^*(W)$. Then $Z \in \mathcal{O}(z)$. By the definition of limits of nets there is then an $a \in D$ such that if $a \preccurlyeq b$ then $g(b) \in Z$, and hence $g(b) \in f^*(W)$. The last statement is equivalent to $f(g(b)) \in W$ or $(f \circ g)(b) \in W$. So for every $W \in \mathcal{O}(f(z))$ there is an $a \in D$ such that if $a \preccurlyeq b$ then $(f \circ g)(b) \in W$. Using the definition of limits of nets again, this means $\lim f \circ g = f(z)$. □

This proposition has a sort of converse.

Proposition 3.6.7. *Suppose and $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f\colon X \to Y$ is a function. If f is not continuous at z then there is a directed set $(D, \preccurlyeq)$ and a net $g\colon D \to X$ such that $\lim g = z$ but $\lim f \circ g \neq f(z)$.*

Proof. If f is not continuous at z then there is a $V \in \mathcal{N}(f(x))$ such that $f^*(V) \notin \mathcal{N}(x)$. In other words, there is no $U \in \mathcal{O}(x)$ such that $U \in \mathcal{O}(x)$ and $U \subseteq f^*(V)$. Therefore for each $U \in \mathcal{O}(x)$ there is an $x \in U$ such that $x \notin f^*(V)$. By the Axiom of Choice we can find a function $g\colon \mathcal{O}(z) \to X$ such that $g(U) \in U$ and $g(U) \notin f^*(V)$. In other words, $g(U) \in U$ and $(f \circ g)(U) \notin V$. Let $\mathcal{D}$ be $\mathcal{O}(z)$, with the order relation $\supseteq$. This is directed set by Proposition 1.14.2m. f and $f \circ g$ are nets. $\lim f = z$ but $\lim f \circ g \neq f(z)$. □

3.7 Continuity and comparison

Next we examine the behaviour of continuity when we replace the topologies on the two spaces with stronger or weaker ones.

Proposition 3.7.1. *Suppose $\mathcal{W}_X$ and $\mathcal{S}_X$ are topologies on X and $\mathcal{W}_X$ is weaker than $\mathcal{S}_X$. Suppose $\mathcal{W}_Y$ and $\mathcal{S}_Y$ are topologies on Y and $\mathcal{W}_Y$ is weaker than $\mathcal{S}_Y$. Suppose that $f\colon X \to Y$ is continuous with respect to the topologies $\mathcal{W}_X$ and $\mathcal{S}_Y$. Then it's also continuous with respect to the topologies $\mathcal{S}_X$ and $\mathcal{W}_Y$.*

Proof. The conditions that $\mathcal{W}_X$ is weaker than $\mathcal{S}_X$ and $\mathcal{W}_Y$ is weaker than $\mathcal{S}_Y$ mean that

$$\mathcal{W}_X \subseteq \mathcal{S}_X$$

and

$$\mathcal{W}_Y \subseteq \mathcal{S}_Y.$$

The former implies

$$f^{**}(\mathcal{W}_X) \subseteq f^{**}(\mathcal{S}_X).$$

By Proposition 3.6.2 the condition that f is continuous with respect to the topologies $\mathcal{W}_X$ and $\mathcal{S}_Y$ implies

$$\mathcal{S}_Y \subseteq f^{**}(\mathcal{W}_X)$$

Combining the last three inclusions gives

$$\mathcal{W}_Y \subseteq f^{**}(\mathcal{S}_X).$$

Another application of Proposition 3.6.2 then shows that f is continuous with respect to the topologies $\mathcal{S}_X$ and $\mathcal{W}_Y$. □

Most of the time we only want to change the topology on one space or the other, but the proposition includes as special cases $\mathcal{W}_X = \mathcal{S}_X$ or $\mathcal{W}_Y = \mathcal{S}_Y$ since every topology is stronger or weaker than itself. Since the trivial topology is the weakest topology on any space and the discrete topology is the strongest the proposition implies that any function from a space equipped with the discrete topology or to a space equipped with the trivial topology is continuous. Both of these statements can of course also be proved directly, without using the proposition.

So far we've fixed the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$ on X and Y and examined which functions $f\colon X \to Y$ are continuous. It's also possible though to fix f and one of the two topologies and ask which topologies on the remaining space make f continuous. This question is answered by the following propositions.

Proposition 3.7.2. *Suppose $f\colon X \to Y$ is a function and $\mathcal{T}_Y$ is a topology on Y. There is a weakest topology $\mathcal{T}_X$ on X with the property that f is continuous with respect to the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$. This topology is $\mathcal{T}_X = (f^*)_*(\mathcal{T}_Y)$.*

Proof. f is continuous with respect to the topology $\mathcal{T}$ on X and $\mathcal{T}_Y$ on Y if and only if

$$\mathcal{T}_Y \subseteq f^{**}(\mathcal{T}).$$

Let $\mathbf{A}$ be the set of such topologies. In other words,

$$\mathbf{A} = \{\mathcal{T} \in \mathbf{T}(X) \colon \mathcal{T}_Y \subseteq f^{**}(\mathcal{T})\}.$$

By Proposition 3.5.3 the intersection of all the elements of $\mathcal{T}$ is a topology. Call this intersection $\mathcal{T}_X$:

$$\mathcal{T}_X = \bigcap_{\mathcal{T}\in\mathbf{A}} \mathcal{T}.$$

Preimages preserve intersections, so

$$f^{**}(\mathcal{T}_X) = f^{**}\left(\bigcap_{\mathcal{T}\in\mathbf{A}} \mathcal{T}\right) = \bigcap_{\mathcal{T}\in\mathbf{A}} f^{**}(\mathcal{T}).$$

Since $\mathcal{T}_Y \subseteq f^{**}(\mathcal{T})$ for each $\mathcal{T} \in \mathbf{A}$ we have

$$\mathcal{T}_Y \subseteq f^{**}(\mathcal{T}_X),$$

so f is continuous with respect to the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$. If f is continuous with respect to $\mathcal{T}$ and $\mathcal{T}_Y$ then $\mathcal{T} \in \mathbf{A}$ so $\mathcal{T}_X \subseteq \mathcal{T}$. In other words, $\mathcal{T}_X$ is weaker than $\mathcal{T}$. So $\mathcal{T}_X$ is the weakest topology with respect to which f is continuous.

Suppose $U \in (f^*)_*(\mathcal{T}_Y)$. Then $U = f^*(W)$ for some $W \in \mathcal{T}_Y$ f is continuous with respect to the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$. so $f^*(W) \in \mathcal{T}_X$. So $U \in (f^*)_*(\mathcal{T}_Y)$ implies $U \in \mathcal{T}_X$. In other words, $(f^*)_*(\mathcal{T}_Y) \subseteq \mathcal{T}_X$. Suppose $\mathcal{T}$ is a topology on X such that $(f^*)_*(\mathcal{T}_Y) \subseteq \mathcal{T}$. In other words, if $U = f^*(W)$ for some $W \in \mathcal{T}_Y$ then $U \in \mathcal{T}$. So $f^*(W) \in \mathcal{T}$ whenever $W \in \mathcal{T}_Y$. Therefore f is continuous with respect to the topologies $\mathcal{T}$ and $\mathcal{T}_Y$. $\mathcal{T}_X$ is the weakest topology with this property, so $\mathcal{T}_X$ contains $(f^*)_*(\mathcal{T}_Y)$ and is the weakest topology which does so. It was shown in Lemma 3.1.2 that $(f^*)_*(\mathcal{T}_Y)$ is a topology, so $\mathcal{T}_X = (f^*)_*(\mathcal{T}_Y)$. □

Proposition 3.7.3. *Suppose $f\colon X \to Y$ is a function and $\mathcal{T}_X$ is a topology on X. There is a strongest topology $\mathcal{T}_Y$ on Y with the property that f is continuous with respect to the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$. This topology is $\mathcal{T}_Y = f^{**}(\mathcal{T}_X)$.*

Proof. Define $\mathcal{T}_Y$ by

$$\mathcal{T}_Y = f^{**}(\mathcal{T}_X).$$

Then $\mathcal{T}_Y$ is a topology on Y by Lemma 3.1.1. f is continuous with respect to the topology $\mathcal{T}_X$ on X and $\mathcal{T}$ on Y if and only if

$$\mathcal{T} \subseteq f^{**}(\mathcal{T}_X),$$

or, equivalently, if and only if

$$\mathcal{T} \subseteq \mathcal{T}_Y$$

i.e. if and only if $\mathcal{T}_Y$ is stronger than $\mathcal{T}$. So $\mathcal{T}_Y$ is the strongest topology with respect to which f is continuous. □

It's possible, and useful, to replace the single function f in the propositions above with a collection of functions to or from an indexed collection of sets.

Proposition 3.7.4. *Suppose X is a set and $j\colon L \to \mathcal{Y}$ is an indexed collection of sets. Suppose for each $\lambda \in L$ that $\mathcal{T}_\lambda$ is a topology on $j(\lambda)$ and f_λ is a function from X to $j(\lambda)$. Then there is a weakest topology $\mathcal{T}_X$ on X with the property that f_λ is continuous with respect to the topologies $\mathcal{T}_X$ and $\mathcal{T}_\lambda$ for each $\lambda \in L$. $\bigcup_{\lambda\in L} (f_\lambda^*)_* (\mathcal{T}_\lambda)$ is a subbase for this topology.*

Proof. f is continuous with respect to the topology $\mathcal{T}$ on X and $\mathcal{T}_\lambda$ on $j(\lambda)$ if and only if

$$\mathcal{T}_\lambda \subseteq f_\lambda^{**}(\mathcal{T}).$$

Let $\mathbf{A}$ be the set of such topologies. In other words,

$$\mathbf{A} = \{\mathcal{T} \in \mathbf{T}(X)\colon \forall \lambda \in L\colon \mathcal{T}_\lambda \subseteq f_\lambda^{**}(\mathcal{T})\}.$$

By Proposition 3.5.3 the intersection of all the elements of $\mathcal{T}$ is a topology. Call this intersection $\mathcal{T}_X$:

$$\mathcal{T}_X = \bigcap_{\mathcal{T}\in\mathbf{A}} \mathcal{T}.$$

Preimages preserve intersections, so

$$f_\lambda^{**}(\mathcal{T}_X) = f_\lambda^{**}\left(\bigcap_{\mathcal{T}\in\mathbf{A}} \mathcal{T}\right) = \bigcap_{\mathcal{T}\in\mathbf{A}} f_\lambda^{**}(\mathcal{T}).$$

Since $\mathcal{T}_\lambda \subseteq f_\lambda^{**}(\mathcal{T})$ for each $\mathcal{T} \in \mathbf{A}$ we have

$$\mathcal{T}_\lambda \subseteq f_\lambda^{**}(\mathcal{T}_X),$$

so f_λ is continuous with respect to the topologies $\mathcal{T}_X$ and $\mathcal{T}_\lambda$. If f_λ is continuous with respect to $\mathcal{T}$ and $\mathcal{T}_\lambda$ for each $\lambda \in L$ then $\mathcal{T} \in \mathbf{A}$ so $\mathcal{T}_X \subseteq \mathcal{T}$. In other words, $\mathcal{T}_X$ is weaker than $\mathcal{T}$. So $\mathcal{T}_X$ is the weakest topology with respect to which each f_λ is continuous.

Suppose

$$U \in \bigcup_{\lambda\in L} (f^*)_* (\mathcal{T}_Y).$$

Then $U = f_\lambda^*(W)$ for some $\lambda \in L$ and $W \in \mathcal{T}_\lambda$. f_λ is continuous with respect to the topologies $\mathcal{T}_X$ and $\mathcal{T}_\lambda$ so $f_\lambda^*(W) \in \mathcal{T}_X$. So

$$U \in \bigcup_{\lambda\in L} (f_\lambda^*)_* (\mathcal{T}_\lambda)$$

implies $U \in \mathcal{T}_X$. In other words,

$$\bigcup_{\lambda\in L} (f^*)_* (\mathcal{T}_Y) \subseteq \mathcal{T}_X.$$

Suppose $\mathcal{T}$ is a topology on X such that

$$(f^*)_* (\mathcal{T}_Y) \subseteq \mathcal{T}.$$

In other words, if $U = f_\lambda^*(W)$ for some $\lambda \in L$ and $W \in \mathcal{T}_Y$ then $U \in \mathcal{T}$. So $f_\lambda^*(W) \in \mathcal{T}$ whenever $W \in \mathcal{T}_\lambda$. Therefore f_λ is continuous with respect to the topologies $\mathcal{T}$ and $\mathcal{T}_\lambda$ for each $\lambda \in L$. $\mathcal{T}_X$ is the weakest topology with this property, so $\mathcal{T}_X$ is contains $\bigcup (f_\lambda^*)_* (\mathcal{T}_\lambda)$ and is the weakest topology which does so. In other words, it is the topology generated by $\bigcup_{\lambda\in L} (f_\lambda^*)_* (\mathcal{T}_\lambda)$, or equivalently, $\bigcup_{\lambda\in L} (f_\lambda^*)_* (\mathcal{T}_\lambda)$ is a subbase for $\mathcal{T}_X$. $\square$

Proposition 3.7.5. *Suppose Y is a set and $i\colon K \to \mathcal{X}$ is an indexed collection of sets. Suppose for each $\kappa \in K$ that $\mathcal{T}_\kappa$ is a topology on $i(\kappa)$ and f_κ is a function from $i(\kappa)$ to Y. Then there is a strongest topology $\mathcal{T}_Y$ on Y with the property that f_κ is continuous for each $\kappa \in K$. This $\mathcal{T}_Y$ is $\bigcap_{\kappa\in K} f_\kappa^{**}(\mathcal{T}_\kappa)$.*

Proof. Define $\mathcal{T}_Y$ by

$$\mathcal{T}_Y = \bigcap_{\kappa\in K} f_\kappa^{**}(\mathcal{T}_\kappa).$$

Then $\mathcal{T}_Y$ is a topology on Y. To see this, note that for each $\kappa \in K$ $f_\kappa^{**}(\mathcal{T}_\kappa)$ is a topology by Lemma 3.1.1. and that the intersection of all of them is a topology by Proposition 3.5.3. f_κ is continuous with respect to the topology $\mathcal{T}_\kappa$ on $i(\kappa)$ and $\mathcal{T}$ on Y if and only if

$$\mathcal{T} \subseteq f_\kappa^{**}(\mathcal{T}_\kappa).$$

So f_κ is continuous for all $\kappa \in K$ if and only if

$$\mathcal{T} \subseteq \mathcal{T}_Y$$

i.e. if and only if $\mathcal{T}_Y$ is stronger than $\mathcal{T}$. So $\mathcal{T}_Y$ is the strongest topology with respect to which every f_κ is continuous. $\square$

3.8 Subspace topology

We can use Propositions 3.7.2 and 3.7.4 to define topologies on subsets and products. We'll consider subspaces in this section and products in the next.

Definition 3.8.1. Suppose $(X, \mathcal{T})$ is a topological space and $A \subseteq X$. The *subspace topology* on A is the weakest topology on A such that the inclusion function $i : A \to X$ is continuous.

The existence of such a weakest topology follows from Proposition 3.7.2.

It's possible to describe this topology more directly.

Proposition 3.8.2. *Suppose $(X, \mathcal{T}_X)$ is a topological space, $A \subseteq X$ and $\mathcal{T}_A$ is the subspace topology on A. Then $U \in \mathcal{T}_A$ if and only if there is a $V \in \mathcal{T}_X$ such that $U = A \cap V$.*

Proof. From Proposition 3.7.2 we know that is

$$\mathcal{T}_A = (i^*)_* (\mathcal{T}_X).$$

$U \in (i^*)_* (\mathcal{T}_X)$ if and only if there is a $V \in \mathcal{T}_X$ such that $i^*(V) = U$. $x \in U$ if and only if $x \in A$ and $i(x) = V$, i.e. if and only if $x \in A$ and $x \in V$, i.e. if and only if $x \in A \cap V$. So $i^*(V) = A \cap V$. Therefore $U \in \mathcal{T}_A$ if and only if $U = A \cap V$ for some $V \in \mathcal{T}_X$. $\square$

As an example, the subspace topology on $\mathbf{Z}$, considered as a subspace of $\mathbf{R}$ is the discrete topology. To see this, note that if $n \in \mathbf{N}$ then $\{n\} = \mathbf{Z} \cap (n-1, n+1)$ and $(n-1, n+1)$ is an open set in $\mathbf{R}$ so $\{n\}$ is an open set in $\mathbf{Z}$. Since every subset of $\mathbf{Z}$ is a union of such sets it follows that every subset is open, so the topology is the discrete topology.

Note that if $A \subseteq X$ and $U \in \mathcal{T}_A$ then U is an open subset of $\mathcal{A}$ and U is a subset of X but U needn't be an open subset of X. Indeed, in the example above every non-empty subset of $\mathbf{Z}$ is an open subset of $\mathbf{Z}$ but none of them are open subsets of $\mathbf{R}$.

The subspace topology on $\mathbf{Q}$, on the other hand, is not the discrete topology. There is no open subset V of $\mathbf{R}$ such that $\mathbf{Q} \cap V = \{0\}$, so $\{0\}$ is not open in the subspace topology. Since there are subsets which are not open the topology is not the discrete topology.

Here again the non-empty open subsets of $\mathbf{Q}$ are subsets of $\mathbf{R}$ but are not open subsets of $\mathbf{R}$ since every non-empty subset of $\mathbf{R}$ contains an irrational number.

Proposition 3.8.3. *Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $A \in \wp(Y)$. Then $f : X \to A$ is continuous if and only if $i \circ f$ is continuous, where $i : X \to A$ is the inclusion function.*

Proof. Suppose f is continuous. The inclusion i is continuous because of Definition 3.8.1. $i \circ f$ is then continuous by Proposition 3.6.4.

Suppose, conversely, that $i \circ f$ is continuous. In other words, if $W \in \mathcal{T}_Y$ then $(i \circ f)^*(W) \in \mathcal{T}_X$.

$$(i \circ f)^*(W) = (f^* \circ i^*)(W) = f^*(i^*(W)).$$

But $i^*(W) = A \cap W$ as we saw in the proof of Proposition 3.8.2. So if $W \in \mathcal{T}_Y$ then $f^*(A \cap W) \in \mathcal{T}_X$ for all $W \in \mathcal{T}_Y$. By Proposition 3.8.2 every element of $\mathcal{T}_A$ is of the form $A \cap W$ for some $W \in \mathcal{T}_Y$, so f is continuous. $\square$

One more useful observation is the following.

Proposition 3.8.4. *Suppose $(X, \mathcal{T}_X)$ is a topological space, $A \subseteq X$ and $\mathcal{T}_A$ is the subspace topology on A. If $\mathcal{T}_X$ is Hausdorff then so is $\mathcal{T}_A$.*

Proof. Suppose $x, y \in A$ are distinct. X is Hausdorff so there are $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $V \cap W = \varnothing$. $x, y \in A$ so $x \in A \cap V$ and $y \in A \cap W$. By Proposition 3.8.2 $A \cap V \in \mathcal{T}_A$ and $A \cap W \in \mathcal{T}_A$. Also

$$(A \cap V) \cap (A \cap W) = A \cap (V \cap W) = A \cap \varnothing = \varnothing.$$

$\square$

3.9 Quotient topology

Definition 3.9.1. Suppose $(X, \mathcal{T}_X)$ is a topological space, Y is a set and $f : X \to Y$ is a function. The *quotient topology* on Y is the strongest topology on Y such that f is continuous.

Note that there is such a topology by Proposition 3.7.3.

As with the subspace topology, it's possible to describe the quotient topology more directly.

Proposition 3.9.2. *Suppose $(X, \mathcal{T}_X)$ is a topological space, Y is a set and $f\colon X \to Y$ is a function. Let $\mathcal{T}_Y$ be the quotient topology on Y. Then $U \in \mathcal{T}_Y$ if and only if $f^*(U) \in \mathcal{T}_X$.*

Proof. By Proposition 3.7.3 $\mathcal{T}_Y = f^{**}(\mathcal{T}_X)$. $U \in f^{**}(\mathcal{T}_X)$ if and only if $f^*(U) \in \mathcal{T}_X$. □

Proposition 3.9.3. *Suppose $(X, \mathcal{T}_X)$ is a topological space, Y is a set, $f\colon X \to Y$ is a function and $\mathcal{T}_Y$ is the quotient topology on Y. If $(Z, \mathcal{T}_Z)$ is a topological space and $g\colon Y \to Z$ is a function then g is continuous if and only if $g \circ f$ is continuous.*

Proof. $\mathcal{T}_Y$ was chosen so that f is continuous. If g is continuous then $g \circ f$ is continuous by Proposition 3.6.4.

Suppose, conversely, that $g \circ f$ is continuous. In the course of the proof of Proposition 3.7.3 we saw that the quotient topology $\mathcal{T}_Y$ on Y is $f^{**}(\mathcal{T}_X)$. So if $V \in f^{**}(\mathcal{T}_X)$ then $V \in \mathcal{T}_Y$. In other words, if $f^*(V) \in \mathcal{T}_X$ then $V \in \mathcal{T}_Y$. We apply this to $V = g^*(W)$ where $W \in \mathcal{T}_Z$. We can do this since

$$f^*(V) = f^*(g^*(W)) = (g \circ f)^*(W) \in \mathcal{T}_X$$

and $g \circ f$ is continuous. Therefore $V \in \mathcal{T}_Y$. In other words, $g^*(W) \in \mathcal{T}_Y$. We've just seen that if $W \in \mathcal{T}_Z$ then $g^*(W) \in \mathcal{T}_Y$. In other words, g is continuous. □

The quotient topology can be rather badly behaved. For example, it needn't be Hausdorff even we start from a Hausdorff space, as the following example shows.

Proposition 3.9.4. *Define an equivalence relation $\sim$ on $\mathbf{R}$ by $x \sim y$ if and only if $x - y \in \mathbf{Q}$. Let $\mathcal{E}$ be the set of equivalence classes and $f\colon \mathbf{R} \to \mathcal{E}$ the function which assigns to each real number its equivalence class. Then the quotient topology on $\mathcal{E}$ is the trivial topology.*

Proof. Suppose V is a non-empty open set in the quotient topology. Then $f^*(V)$ is an open subset of $\mathbf{R}$ because f is continuous. $f^*(V)$ is non-empty because f is a surjection. We can therefore find an $x \in f^*(V)$. By the definition of the topology on $\mathbf{R}$ there is an $r > 0$ such that $B(x, r) \subseteq f^*(V)$. Suppose $z \in \mathcal{E}$. f is a surjection so there is a $y \in \mathbf{R}$ such that $f(y) = z$. Every real number is a limit of rationals. $y - x \in \mathbf{R}$ so there is a rational number q such that $q \in B(y - x, r)$. Then

$$d(y - q, x) = d(y - x, q) < r$$

so $y - q \in B(x, r)$ and hence $y - q \in f^*(V)$. In other words, $f(y - q) \in V$. But $y \sim y - q$ so $f(y) = f(y - q)$. Therefore $z = f(y) \in V$. We've just seen that for all $z \in \mathcal{E}$ we have $z \in V$, and therefore $V = \mathcal{E}$. V was an arbitrary non-empty open set so the only non-empty open set is $\mathcal{E}$. In other words, the topology on $\mathcal{E}$ is the trivial topology. □

3.10 Product topology

Definition 3.10.1. Suppose $j\colon L \to \mathcal{A}$ is an indexed collection of sets and P is its product. Suppose that for each $\lambda \in L$ $\mathcal{T}_\lambda$ is a topology on $j(\lambda)$. The *product topology* on P is the weakest topology on P such that the projection $\pi_\lambda\colon P \to j(\lambda)$ is continuous for each $\lambda \in L$.

The existence of such a topology was shown in Proposition 3.7.4.

Proposition 3.10.2. *Suppose $(X, \mathcal{T}_X)$ is a topological space, $j\colon L \to \mathcal{A}$ is an indexed collection of sets, P is its product and $\mathcal{T}_P$ is the product topology. Suppose further that for each $\lambda \in L$ $\mathcal{T}_\lambda$ is a topology on $j(\lambda)$ and $g_\lambda\colon X \to j(\lambda)$ is a continuous function with respect to the topologies $\mathcal{T}_X$ and $\mathcal{T}_\lambda$. Then there is a unique function $h\colon X \to P$ such that $\pi_\lambda \circ h = g_\lambda$ for each $\lambda \in L$ and this function is continuous.*

Proof. If $\pi_\lambda \circ h = g_\lambda$ and $h(x) = f$ then

$$f(\lambda) = \pi_\lambda(f) = (\pi_\lambda \circ h)(x) = g_\lambda(x).$$

There is only one f which satisfies this equation. Conversely, if we define h by $h(x) = f$ where f is the function $f(\lambda) = g_\lambda(x)$ then

$$(\pi_\lambda \circ h)(x) = \pi_\lambda(f) = f(\lambda) = g_\lambda(x)$$

for all $x \in X$ so $\pi_\lambda \circ h = g_\lambda$. We've now shown that there exists a unique $h\colon X \to P$ such that $\pi_\lambda \circ h =$

g_λ for all $\lambda \in L$. It remains to show that this h is continuous.

If

$$U \in \bigcup_{\lambda \in L} (\pi_\lambda^*)_* (\mathcal{T}_\lambda)$$

then there is a $\lambda \in L$ and a $W \in \mathcal{T}_\lambda$ such that $U = \pi_\lambda^*(W)$. Then

$$\begin{aligned} h^*(U) &= h^*(\pi_\lambda^*(W)) = (h^* \circ \pi_\lambda^*)(W) \\ &= (\pi_\lambda \circ h)^*(W) = g_\lambda^*(W). \end{aligned}$$

$g_\lambda^*(W) \in \mathcal{T}_X$ because $W \in \mathcal{T}_\lambda$ and g_λ was assumed to be continuous with respect to the topologies $\mathcal{T}_X$ and $\mathcal{T}_\lambda$. So $h^*(U) \in \mathcal{T}_X$. Proposition 3.7.4 shows that $\bigcup_{\lambda \in L} (\pi_\lambda^*)_* (\mathcal{T}_\lambda)$ is a subbase for $\mathcal{T}_P$. h is therefore continuous by Proposition 3.6.5. □

As with the subspace topology, we can describe the product topology more directly.

Proposition 3.10.3. *The elements of the product topology are the unions of sets of the for $U_{F,t}$ where F is a finite subset of L, $t\colon F \to \bigcup_{\lambda \in L} \mathcal{T}_\lambda$ is such that $t(\lambda) \in \mathcal{T}_\lambda$ for all $\lambda \in F$ and*

$$U_{F,t} = \{f \in P \colon \forall \lambda \in F \colon f(\lambda) \in t(\lambda)\}.$$

Proof. We begin by noting that the intersection of any two sets of this form is also of this form, since

$$U_{E,s} \cap U_{F,t} = U_{G,u}$$

where $G = E \cup F$ and

$$u(\lambda) = \begin{cases} s(\lambda) \cap t(\lambda) & \text{if } \lambda \in E \cap F, \\ s(\lambda) & \text{if } \lambda \in E \setminus F, \\ t(\lambda) & \text{if } \lambda \in F \setminus E. \end{cases}$$

By induction the union of finitely many such sets is also such a set.

Also, if $W \in \mathcal{T}_\lambda$ then

$$\pi_\lambda^*(W) = U_{F,t}$$

where $F = \{\lambda\}$ and $t(\lambda) = W$. The intersection of finitely many sets of the form $\pi_\lambda^*(W)$ is therefore a set of the form $U_{F,t}$. Conversely any set of the form $U_{F,t}$ is such a finite intersection since

$$U_{F,t} = \bigcap_{\lambda \in F} \pi_\lambda^*(t(\lambda)).$$

In other words, the set of sets of the form $U_{F,t}$ is the set of intersections finitely many elements of $\bigcup_{\lambda \in L} (\pi_\lambda^*)_* (\mathcal{T}_\lambda)$.

By Proposition 3.7.4 $\mathcal{T}_P$ is generated by $\bigcup_{\lambda \in L} (\pi_\lambda^*)_* (\mathcal{T}_\lambda)$. Each $(\pi_\lambda^*)_* (\mathcal{T}_\lambda)$ is a topology by Proposition 3.1.2. So by Corollary 3.5.7 $\mathcal{T}_P$ consists of unions of intersections of intersections of finitely many elements of $\bigcup_{\lambda \in L} (\pi_\lambda^*)_* (\mathcal{T}_\lambda)$. Therefore $\mathcal{T}_P$ consists of unions of sets of sets of the form $U_{F,t}$. □

We can be even more explicit in the case of the Cartesian product of two spaces.

Proposition 3.10.4. *Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces. Making the usual identification of $X \times Y$ with the product of $j\colon \{1, 2\} \to \{X, Y\}$, $j(1) = X$, $j(2) = Y$, the product topology on $X \times Y$ consists of the unions of sets of the form $V \times W$ where $V \in \mathcal{T}_X$ and $W \in \mathcal{T}_Y$.*

Proof. The usual identification identifies the ordered pair (x, y) in $X \times Y$ with the function $f\colon \{1, 2\} \to X \cup Y$ defined by $f(1) = x$, $f(2) = y$. If $F = \{1, 2\}$, $t(1) = V$ and $t(2) = W$ then $U_{F,t}$ is the set of functions f such that $f(1) \in V$ and $f(2) \in W$. This is identified with the set of ordered pairs (x, y) such that $x \in V$ and $y \in W$. In other words, $(x, y) \in V \times W$. If $F = \{1\}$ we just get $V \times Y$, while if $F = \{2\}$ we get $X \times W$ and if $F = \varnothing$ we get $X \times Y$. So the set of all $U_{F,t}$ is identified with the set of all $V \times W$ where $V \in \mathcal{T}_X$ and $W \in \mathcal{T}_W$. The product topology is the union of these. □

The projections from a product are continuous, i.e. the preimages of open sets are open, but they have the additional property that this images of open sets are open.

Definition 3.10.5. If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces then a function $f\colon X \to Y$ is called *open* if $f_*(U) \in \mathcal{T}_Y$ whenever $U \in \mathcal{T}_X$.

We could state this more compactly as $\mathcal{T}_X \subseteq (f_*)^* (\mathcal{T}_Y)$.

Proposition 3.10.6. *Suppose $j: L \to \mathcal{A}$ is an indexed collection of sets with $\mathcal{T}_\lambda$ being the topology on $j(\lambda)$, $(P, \mathcal{T}_P)$ is its product and $\pi_\lambda: P \to j(\lambda)$ is the projection at $\lambda \in L$. Then each π_λ is open.*

Proof.
$$\pi_\lambda(U_{F,t}) = \begin{cases} t(\lambda) & \text{if } \lambda \in F, \\ j(\lambda) & \text{if } \lambda \notin F, \end{cases}$$
so $\pi_\lambda(U_{F,t}) \in \mathcal{T}_\lambda$. The image of a union is the union of the images and $\mathcal{T}_\lambda$ is a topology so any union of elements of $\mathcal{T}_\lambda$ is an element of $\mathcal{T}_\lambda$ so the image of any element of $\mathcal{T}_P$ is an element of $\mathcal{T}_\lambda$. In other words, π_λ is open. □

Definition 3.10.7. Suppose $j: L \to \mathcal{A}$ is an indexed collection of sets and X is a set such that $j(\lambda) = X$ for all $\lambda \in L$. Let P be the product of j. The *diagonal function* is the function $h: X \to P$ defined for any $x \in X$ by $h(x) = f$ where $f(\lambda) = x$.

We're mostly interested in the case of the product of two copies of the same set. If we make the usual identification of P with the Cartesian product $X \times X$ then $h(x) = (x, x)$. The image
$$\Delta_X = h_*(X) = \{(x, y) \in X \times X : x = y\}$$
is called the diagonal subset of $X \times X$.

Proposition 3.10.8. *The diagonal function is continuous.*

Proof. For each $\lambda \in L$ we have $\pi_\lambda \circ h = g_\lambda$ where $g_\lambda: X \to X$ is the identity function $g_\lambda(x) = x$. These are all continuous. h is therefore continuous by Proposition 3.10.2. □

There are a number of important relations between the product construction and the Hausdorff property.

Proposition 3.10.9. *A topological space $(X, \mathcal{T})$ is Hausdorff if and only if Δ_X is closed.*

Proof. The product topology on the Cartesian product $X \times X$ is described by Proposition 3.10.4.

Suppose X is Hausdorff. If $(x, y) \in X \times X \setminus \Delta_X$ then $x \neq y$ so there are $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $V \cap W = \varnothing$. Then $(x, y) \in V \cap W$. From $V \cap W = \varnothing$ it follows that $V \times W \in X \times X \setminus \Delta_X$. So we've shown that for any $(x, y) \in X \times X \setminus \Delta_X$ there is a $Z \in \mathcal{T}_P$, namely $Z = V \times W$, such that $(x, y) \in Z$ and $Z \subseteq X \times X \setminus \Delta_X$. Therefore $X \times X \setminus \Delta_X$ is open and Δ_X is closed.

Suppose, conversely, that Δ_X is closed, i.e. that $X \times X \setminus \Delta_X$ is open. If $x \neq y$ then $(x, y) \in X \times X \setminus \Delta_X$. By Proposition 3.10.4 $X \times X \setminus \Delta_X$ is a union of sets of the form $V \times W$ with $V, W \in \mathcal{T}_X$. $(x, y) \in V \times W$ for some such V and W. Then $x \in V$ and $y \in W$.
$$\begin{aligned} V \cap W &= \{z \in X : z \in V, z \in W\} \\ &= \{z \in X : (z, z) \in V \times W\} = \varnothing \end{aligned}$$
since $(z, z) \in \Delta_X$. So $\mathcal{T}$ is a Hausdorff topology. □

Proposition 3.10.10. *Suppose $j: L \to \mathcal{A}$ is an indexed collection of sets and P is its product. Suppose that for each $\lambda \in L$ $\mathcal{T}_\lambda$ is a Hausdorff topology on $j(\lambda)$. Then the product topology is Hausdorff.*

Proof. Let $\mathcal{T}_P$ be the product topology. Suppose f, g are elements of the product and $f \neq g$. Then there is a $\lambda \in L$ such that $f(\lambda) \neq g(\lambda)$. $\mathcal{T}_\lambda$ is Hausdorff so there are $U, V \in \mathcal{T}_\lambda$ such that $f(\lambda) \in U$, $g(\lambda) \in V$ and $U \cap V \in \mathcal{T}_\lambda$. $\pi_\lambda(f) = f(\lambda)$ and $\pi_\lambda(g) = g(\lambda)$ so $\pi_\lambda(f) \in U$ and $\pi_\lambda(g) \in V$, or, equivalently, $f \in \pi_\lambda^*(U)$ and $g \in \pi_\lambda^*(V)$. $\pi_\lambda^*(U) \in \mathcal{T}_P$ and $\pi_\lambda^*(V) \in \mathcal{T}_P$ since π_λ is continuous. Also,
$$\pi_\lambda^*(U) \cap \pi_\lambda^*(V) = \pi_\lambda^*(U \cap V) = \pi_\lambda^*(\varnothing) = \varnothing.$$
So $\mathcal{T}_P$ is Hausdorff. □

The following is a sort of converse.

Proposition 3.10.11. *Suppose $j: L \to \mathcal{A}$ is an indexed collection of non-empty sets and P is its product. Suppose that for each $\lambda \in L$ $\mathcal{T}_\lambda$ is a topology on $j(\lambda)$ and $\mathcal{T}_P$ is the product topology on P. If $\mathcal{T}_P$ is a Hausdorff topology then each $\mathcal{T}_\mu$ is Hausdorff.*

Proof. By Proposition 2.11.5 there is an $f \in P$. Define a function $i: j(\mu) \to P$ by
$$i(x)(\lambda) = \begin{cases} x & \text{if } \lambda = \mu, \\ f(\lambda) & \text{if } \lambda \neq \mu, \end{cases}$$

Then $\pi_\lambda \circ i$ is the identity function on $j(\mu)$ if $\lambda = \mu$ and is the constant function $f(\lambda)$ if $\lambda \neq \mu$. In either case $\pi_\lambda \circ i$ is continuous, so i is continuous by Proposition 3.10.2.

Suppose $x, y \in j(\mu)$ and $x \neq y$. Then $i(x) \neq i(y)$ because $i(x)(\mu) \neq i(y)(\mu)$. By assumption $(P, \mathcal{T}_P)$ is Hausdorff, so there are $U, V \in \mathcal{T}_P$ such that $i(x) \in U$, $i(y) \in V$ and $U \cap V = \varnothing$. Define $\tilde{U} = i^*(U)$, $\tilde{V} = i^*(V)$. Then $\tilde{U}, \tilde{V} \in \mathcal{T}_\lambda$ because i is continuous. $x \in \tilde{U}$ and $y \in \tilde{V}$ since $i(x) \in U$ and $i(y) \in V$. Also

$$\tilde{U} \cap \tilde{V} = i^*(U) \cap i^*(V) = i^*(U \cap V) = i^*(\varnothing) = \varnothing.$$

So for any x, y in $j(\mu)$ such that $x \neq y$ there are sets $\tilde{U}, \tilde{V} \in \mathcal{T}_\mu$, such that $x \in \tilde{U}$, $y \in \tilde{V}$ and $\tilde{U} \cap \tilde{V} = \varnothing$. In other words, $\mathcal{T}_\mu$ is Hausdorff. ☐

3.11 Connectedness

Definition 3.11.1. A topological space $(X, \mathcal{T})$ is called *connected* if there are no non-empty $U, V \in \mathcal{T}$ such that $U \cap V = \varnothing$ and $U \cup V = X$. It is called *disconnected* if it is not connected. We say that a subset is connected or disconnected if it is connected or disconnected when considered as a topological space with the subspace topology.

The following propositions give simple examples of connected and disconnected sets.

Proposition 3.11.2. *Suppose $A \subseteq \mathbf{R}$ is an interval. Then A is connected.*

Proof. Let $\mathcal{T}_{\mathbf{R}}$ be the usual topology on $\mathbf{R}$ and let $\mathcal{T}_A$ be the subspace topology on A. Suppose A is disconnected, i.e. that there are $U, V \in \mathcal{T}_A$ such that $U \cap V = \varnothing$ and $U \cup V = A$. Choose $x \in U$ and $y \in V$. $U \cap V = \varnothing$ so $x \neq y$. For now we suppose $x < y$.

Define sequences u and v inductively as follows. $u_0 = x$ and $v_0 = y$. If $\frac{u_k + v_k}{2} \in U$ then set $u_{k+1} = \frac{u_k + v_k}{2}$ and $v_{k+1} = v_k$. If $\frac{u_k + v_k}{2} \in V$ we set $u_{k+1} = u_k$ and $v_{k+1} = \frac{u_k + v_k}{2}$. Then, by induction on k, $u_k \in U$ for all k and $v_k \in V$ for all k. Also by induction on k,

$$v_k = u_k + \frac{y - x}{2^k}$$

for all k,

$$u_k \leq u_{k+1} \leq u_k + \frac{y - x}{2^{k+1}},$$

and

$$v_{k+1} \leq v_k \leq v_{k+1} + \frac{y - x}{2^{k+1}}.$$

It follows the lemma below that $\lim_{k\to\infty} u_k$ and $\lim_{k\to\infty} v_k$ exist and are equal. Let z be their limit. $u_k \in U$ for all k so $z \in \overline{U}$ by Proposition 3.3.3. Similarly, $v_k \in V$ for all k so $z \in \overline{V}$. $U = A \setminus V$ and $V = A \setminus U$ are closed, so $\overline{U} = U$ and $\overline{V} = V$. So $s \in U \cap V$. But $U \cap V = \varnothing$, so we have a contradiction. The argument for the case $y < x$ is the same, except with x and y and U and V swapped everywhere. The assumption that A is disconnected therefore leads to a contradiction. ☐

Lemma 3.11.3. *Suppose $u, v \colon \mathbf{N} \to \mathbf{R}$ are sequences such that u is increasing, v is decreasing and $\lim_{k\to\infty}(v_k - u_k) = 0$. Then $\lim_{k\to\infty} u_k$ and $\lim_{k\to\infty} v_k$ exist and are equal.*

Proof. Choose an $\epsilon > 0$ and an N such that if $k \geq N$ then

$$|v_k - u_k| < \epsilon.$$

If $k \geq N$ then

$$u_k < v_k + \epsilon \leq v_N + \epsilon.$$

The second inequality holds because v is decreasing. If $k \leq N$ then

$$u_k \leq u_N < v_N + \epsilon.$$

The first inequality holds because u is increasing and the second is a special case of the inequality above. So

$$u_k < v_N + \epsilon$$

for all k. In other words, u is bounded from above. Every bounded increasing sequence has a limit.

Similarly, if $k \geq N$ then

$$u_N - \epsilon \leq u_k - \epsilon < v_k$$

while if $k \leq N$ then

$$u_N - \epsilon < v_N \leq v_k.$$

In each case

$$u_N - \epsilon < v_k$$

for all k. In other words, v is bounded from below. Every bounded decreasing sequence has a limit.

Then

$$\lim_{k\to\infty} v_k = \lim_{k\to\infty} u_k + \lim_{k\to\infty} (v_k - u_k) = \lim_{k\to\infty} u_k.$$

□

Proposition 3.11.4. *If $A \subseteq \mathbf{Q}$ has more than one element then A is disconnected.*

Proof. Suppose $x, y \in A$ and $x \neq y$. Then $I = (\min(x,y), \max(x,y))$ is an interval of positive length. Every interval of positive length contains an irrational number so choose a $s \in I$ such that $s \notin \mathbf{Q}$.

$$\min(x,y) < s < \max(x,y)$$

and $s \notin A$. Let $U = (-\infty, s)$ and $V = (s, +\infty)$ Then $U, V \in \mathcal{T}_{\mathbf{R}}$, hence $\mathbf{Q} \cap U, \mathbf{Q} \cap V \in \mathcal{T}_{\mathbf{Q}}$ and $A \cap U, A \cap V \in \mathcal{T}_A$. $A \cap U$ and $A \cap V$ are non-empty since $\min(x,y) \in A \cap U$ and $\max(x,y) \in A \cap V$. Also $(A \cap U) \cap (A \cap V) = \varnothing$ and $(A \cap U) \cup (A \cap V) = A$. So A is disconnected. □

One of the most useful properties of connected spaces is also one of the easiest to prove.

Proposition 3.11.5. *Suppose $(X, \mathcal{T}_Y)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f : X \to Y$ is a continuous surjection. If X is connected than so is Y.*

Proof. An equivalent statement is that if Y is disconnected then so is X. It's this equivalent statement that we'll prove. If Y is disconnected then there are non-empty $U, V \in \mathcal{T}_Y$ such that $U \cap V = \varnothing$ and $U \cup V = Y$. U and V are non-empty and f is a surjection so $f^*(U)$ and $f^*(V)$ are non-empty. f is continuous so $f^*(U), f^*(V) \in \mathcal{T}_X$.

$$f^*(U) \cap f^*(V) = f^*(U \cap V) = f^*(\varnothing) = \varnothing$$

and

$$f^*(U) \cup f^*(V) = f^*(U \cup V) = f^*(Y) = X.$$

So X is disconnected. □

Proposition 3.11.2 gives us an important sufficient condition for a set to be connected.

Definition 3.11.6. A topological space $(X, \mathcal{T}_X)$ is called *path connected* if for every $x, y \in X$ there is a continuous function $p : [0,1] \to X$ such that $p(0) = x$ and $p(1) = y$.

Proposition 3.11.7. *If $(X, \mathcal{T}_X)$ is path connected then it is connected.*

Proof. We prove the equivalent statement that if $(X, \mathcal{T}_X)$ then it is not path connected. Suppose $(X, \mathcal{T}_X)$ is disconnected, i.e there are non-empty $U, V \in \mathcal{T}_X$ such that $U \cap V = \varnothing$ and $U \cup V = X$. U and V are non-empty so there is an $x \in U$ and a $y \in V$. If X were path connected then there would be a continuous function $p : [0,1] \to X$ such that $p(0) = x$ and $p(1) = y$. Set $\tilde{U} = p^*(U)$ and $\tilde{V} = p^*(V)$. These are non-empty because $0 \in \tilde{U}$ and $1 \in \tilde{V}$. They are open subsets of $[0,1]$ because p is continuous. Also

$$\tilde{U} \cap \tilde{V} = p^*(U) \cap p^*(V) = p^*(U \cap V) = p^*(\varnothing) = \varnothing$$

and

$$\tilde{U} \cup \tilde{V} = p^*(U) \cup p^*(V) = p^*(U \cup V) = p^*(X) = [0,1].$$

But this would imply that $[0,1]$ is disconnected, contrary to Proposition 3.11.2. So X is not path connected. □

Proposition 3.11.8. *Suppose $(X, \mathcal{T}_X)$ is a topological space, $A, B \in \wp(X)$, $A \cap B \neq \varnothing$ and $A \cup B = X$. If A and B are connected then so is X.*

Proof. By assumption $A \cap B \neq \varnothing$ so there is an $x \in A \cap B$. Suppose there are $V, W \in \mathcal{T}_X$ such that $V \cap W = \varnothing$ and $V \cup W = X$. $x \in X$ so either $x \in V$ or $x \in W$. Suppose $x \in V$. By Proposition 3.8.2 $A \cap V \in \mathcal{T}_A$ and $A \cap W \in \mathcal{T}_A$.

$$(A \cap V) \cap (A \cap W) = A \cap (V \cap W) = A \cap \varnothing = \varnothing$$

and

$$(A \cap V) \cup (A \cap W) = A \cap (V \cup W) = A \cap X = A.$$

$x \in A \cap V$ so $A \cap V$ is non-empty. A is connected so $A \cap W$ must be empty. Similarly, B is connected so $B \cap W$ must be empty. Then

$$W = X \cap W = (A \cup B) \cap W = (A \cap W) \cup (B \cap W) = \varnothing.$$

Similarly, if $x \in W$ then V is empty. In other words, if $V, W \in \mathcal{T}_X$ are such that $V \cap W = \varnothing$ and $V \cup W = X$ then either V is empty or W is empty. Therefore X is connected. $\square$

Proposition 3.11.9. *Suppose $j \colon L \to \mathcal{A}$ is an indexed collection of sets, $\mathcal{T}_\lambda$ is a topology on $j(\lambda)$ for each $\lambda \in L$, P is its product, and $\mathcal{T}_P$ is the product topology on P. Then $(P, \mathcal{T}_P)$ is connected if and only if $(j(\lambda), \mathcal{T}_\lambda)$ is connected for each $\lambda \in L$.*

Proof. The projections π_λ are continuous surjections so if $(P, \mathcal{T}_P)$ is connected then $(j(\lambda), \mathcal{T}_\lambda)$ is connected for each $\lambda \in L$ by Proposition 3.11.5. This is establishes the "only if" part of the statement of the proposition.

For the "if" part we begin by assuming that there are $V, W \in \mathcal{T}_P$, with V non-empty, such that $V \cup W = P$ and $V \cap W = \varnothing$ and choose $f \in V$ and $g \in P$. By Proposition 3.10.3 there is a $U_{F,t}$ such that $f \in U_{F,t}$ and $U_{F,s} \subseteq V$. F is finite, i.e.

$$F = \{\lambda_1, \ldots, \lambda_m\}$$

for some $\lambda_1, \ldots, \lambda_m \in L$. For $1 \leq k \leq m$ define $s_k \colon j(\lambda_k) \to P$ by

$$s_k(x)(\lambda) = \begin{cases} f(\lambda) & \text{if } \lambda = \lambda_j, j < k, \\ x & \text{if } \lambda = \lambda_k, \\ g(\lambda) & \text{if } \lambda = \lambda_j, j > k, \\ g(\lambda) & \text{if } \lambda \notin F. \end{cases}$$

Note that

$$s_k(g(\lambda_k)) = s_{k+1} f(\lambda_{k+1}).$$

Also $s_1(f(\lambda_1))(\lambda) = f(\lambda)$ for all $\lambda \in F$ and $f(\lambda) \in t(\lambda)$ for all $\lambda \in F$ so $s_1(f(\lambda_1))(\lambda) \in t(\lambda)$ for all $\lambda \in F$. In other words, $s_1(f(\lambda_1)) \in U_{F,t}$. Therefore $s_1(f(\lambda_1)) \in V$. Let $\tilde{V}_k = s_k^*(V)$ and Let $\tilde{W}_k = s_k^*(W)$. $V_k, W_k \in \mathcal{T}_{\lambda_k}$ because s_k is continuous by Proposition 3.10.2. Also,

$$\tilde{V}_k \cap \tilde{W}_k = s_k^*(V) \cap s_k^*(W) = s_k^*(V \cap W) = \varnothing$$

and

$$\begin{aligned} \tilde{V}_k \cup \tilde{W}_k &= s_k^*(V) \cup s_k^*(W) = s_k^*(V \cup W) \\ &= s_k^*(P) = j(\lambda_k). \end{aligned}$$

$(j(\lambda_k), \mathcal{T}_{\lambda_k})$ is connected by hypothesis, so if $s_k(f(\lambda_k)) \in V$ then $f(\lambda_k) \in \tilde{V}_k$, so $\tilde{V}_k$ is non-empty and therefore $\tilde{W}_k$ is empty, i.e. $\tilde{V}_k = j(\lambda_k)$. It follows that $g(\lambda_k) \in \tilde{V}_k$ and hence $s_k(g(\lambda_k)) \in V$. So if

$$s_k(f(\lambda_k)) \in V$$

then

$$s_k(g(\lambda_k)) \in V.$$

We've already seen that

$$s_1(f(\lambda_1)) \in V$$

and

$$s_k(g(\lambda_k)) = s_{k+1} f(\lambda_{k+1}).$$

So

$$s_m(g(\lambda_k)) \in V.$$

But

$$s_m(g(\lambda_k)) = g,$$

so $g \in V$. g was an arbitrary element of P though so $V = P$ and $W = \varnothing$. So if $V, W \in \mathcal{T}_P$, with V non-empty, are such that $V \cup W = P$ and $V \cap W = \varnothing$ then W is empty. In other words, P is connected. $\square$

Proposition 3.11.10. *Suppose that $(X, \mathcal{T}_X)$ is a topological space and $A \subseteq X$. If A is connected then so is its closure $\overline{A}$.*

Proof. As usual, we prove the corresponding statement for disconnected spaces: If $\overline{A}$ is disconnected then A is disconnected. Let $\mathcal{T}_A$ and $\mathcal{T}_{\overline{A}}$ be the subspace topologies on A and $\overline{A}$ respectively. By Proposition 3.8.2 the open sets in $\mathcal{T}_{\overline{A}}$ are the sets of the form $\overline{A} \cap U$ where $U \in \mathcal{T}_X$. The assumption that $(\overline{A}, \mathcal{T}_{\overline{A}})$ is disconnected therefore means that there are $U, V \in \mathcal{T}_X$ such that $\overline{A} \cap U$ and $\overline{A} \cap V$ are non-empty,

$$\left(\overline{A} \cap U\right) \cap \left(\overline{A} \cap V\right) = \varnothing$$

and

$$\left(\overline{A} \cap U\right) \cup \left(\overline{A} \cap V\right) = \overline{A}.$$

But $A \cap U \subseteq \overline{A} \cap U$ and $A \cap V \subseteq \overline{A} \cap V$ so

$$(A \cap U) \cap (A \cap V) \subseteq \left(\overline{A} \cap U\right) \cap \left(\overline{A} \cap V\right)$$

and hence

$$(A \cap U) \cap (A \cap V) = \varnothing$$

since $\varnothing$ is the only subset of $\varnothing$. Also

$$\begin{aligned}
(A \cap U) \cup (A \cap V) &= A \cap (U \cup V) \\
&= (A \cap \overline{A}) \cap (U \cup V) \\
&= A \cap (\overline{A} \cap (U \cup V)) \\
&= A \cap \big((\overline{A} \cap U) \cup (\overline{A} \cap V)\big) \\
&= A \cap \overline{A} = A.
\end{aligned}$$

To show that $(A, \mathcal{T}_A)$ is disconnected it therefore suffices to show that $A \cap U$ and $A \cap V$ are non-empty. $\overline{A} \cap U$ and $\overline{A} \cap V$ are non-empty. There is then $x \in \overline{A} \cap U$. $x \in \overline{A}$, $x \in U$ and $U \in \mathcal{T}_X$. In other words, $x \in \overline{A}$ and $U \in \mathcal{O}(x)$. By Proposition 3.2.2l then $A \cap U \neq \varnothing$. Similarly the fact that $\overline{A} \cap V$ is non-empty implies $A \cap V \neq \varnothing$. ☐

Note that the corresponding statements for interiors is false, as the following example shows. Let

$$C = \{(x, y) \in \mathbf{R}^2 \colon xy \geq 0\}.$$

C is connected. We can see this in either of two ways. We can write $C = A \cup B$ where $A = [0, +\infty) \times [0, +\infty)$ and $B = (-\infty, 0] \times (-\infty, 0]$. $[0, +\infty)$ and $(-\infty, 0]$ are intervals and hence are connected by Proposition 3.11.2. A and B then connected by Proposition 3.11.9. $A \cap B = \{(0, 0)\}$ is non-empty so C is connected by Proposition 3.11.8.

Alternatively we can observe that if $(x_1, y_1), (x_2, y_2) \in R$ then p, defined by

$$p(t) = \begin{cases} ((1 - 2t)x_1, (1 - 2t)y_1) & \text{if } 0 \leq t \leq 1/2, \\ ((2t - 1)x_2, (2t - 1)y_2) & \text{if } 1/2 < t \leq 1, \end{cases}$$

is a continuous function from $[0, 1]$ to C, so C is path connected and therefore, by Proposition 3.11.7, is connected.

The interior

$$C^\circ = \{(x, y) \in \mathbf{R}^2 \colon xy > 0\}$$

of C is disconnected, since it can be written as the union of the disjoint non-empty open sets

$$U = \{(x, y) \in \mathbf{R}^2 \colon x > 0, y > 0\}$$

and

$$V = \{(x, y) \in \mathbf{R}^2 \colon x < 0, y < 0\}.$$

3.12 Compactness

Definition 3.12.1. An *open cover* of a topological space $(X, \mathcal{T}_X)$ is a set $\mathcal{G} \subseteq \mathcal{T}$ such that

$$X = \cup_{U \in \mathcal{G}} U.$$

An open cover of a subset $A \in \wp(X)$ with respect to $(X, \mathcal{T}_X)$ is a set $\mathcal{G} \subseteq \mathcal{T}_X$ such that

$$A \subseteq \cup_{U \in \mathcal{G}} U.$$

$\mathcal{F}$ is said to be a *subcover* of $\mathcal{G}$ if $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{F}$ is an open cover. $(X, \mathcal{T}_X)$ is said to be *compact* if every open cover of $(X, \mathcal{T}_X)$ has a finite subcover. A subset is called compact if it is compact when considered as a topological space with the subspace topology. A subset is called *relatively compact* if its closure is compact when considered as a topological space with the subspace topology. A topological space is called *σ-compact* if it is a union of countably many compact subsets. A topological space is called *locally compact* if every point has a compact neighbourhood.

The following propositions give some examples of compact sets.

Proposition 3.12.2. *Suppose $(X, \mathcal{T})$ is a topological space. If X is finite then $(X, \mathcal{T})$ is compact. If $(X, \mathcal{T})$ is compact and $\mathcal{T}$ is the discrete topology then then X is finite.*

Proof. If X is finite then so is $\wp(X)$. $\mathcal{T}$ is a subset of $\wp(X)$ and so is also finite. Every open cover is a subset of $\mathcal{T}$ and so is finite. Therefore every open cover has a finite subcover, namely itself. So if X is finite then $(X, \mathcal{T})$ is compact.

Suppose $\mathcal{T}$ is the discrete topology on X. The set of sets of the form $\{x\}$ for $x \in X$ is an open cover of $(X, \mathcal{T})$. No proper subset of it is an open cover so the only way it can have a finite subcover is if it is finite, which means X is finite. So if $(X, \mathcal{T})$ is finite. ☐

Proposition 3.12.3. *An interval $I \subseteq \mathbf{R}$ is compact if and only if it is either empty or of the form $[a, b]$ for some $a \leq b$.*

Proof. The empty interval is compact because any open cover of it has an empty subcover.

We prove that $[a,b]$ is compact by contradiction. Suppose then that $[a,b]$ is not compact, i.e. that there is an open cover $\mathcal{G}$ of $[a,b]$ which has no finite subcover, and let

$$\alpha_{j,k} = a + (j-1)(b-a)/2^k, \quad \beta_{j,k} = a + j(b-a)/2^k,$$

for $1 \le j \le 2^k$. Then

$$[a,b] = \bigcup_{j=1}^{2^k} [\alpha_{j,k}, \beta_{j,k}].$$

$\mathcal{G}$ is an open cover of $[\alpha_{j,k}, \beta_{j,k}]$ for each j and k. Let S_k be the set of j for which this cover has a finite subcover $\mathcal{F}_{j,k}$. If $S_k = \{1, \ldots, 2^k\}$ then $\bigcup_{j=1}^{2^k} \mathcal{F}_{j,k}$ is an open cover of $[a,b]$, and is in fact a finite subcover, but there is no such finite subcover. So for each k there is at least one j such that $\mathcal{G}$, considered as an open cover of $[\alpha_{j,k}, \beta_{j,k}]$, has no finite subcover. Let i_k be the first element of $\{1, \ldots, 2^k\}$ which is not in S_k. $\alpha_{i_k,k}$ is an increasing sequence because any subinterval of an interval with a finite subcover also has a finite subcover and $\beta_{i_k,k}$ is a decreasing sequence for the same reason.

$$\lim_{k\to\infty} (\beta_{i_k,k} - \alpha_{k,i_k}) = \lim_{k\to\infty} \frac{b-a}{2^k} = 0$$

so $\lim_{k\to\infty} \alpha_{i_k,k}$ and $\lim_{k\to\infty} \beta_{i_k,k}$ exist and are equal by Lemma 3.11.3. Let

$$x = \lim_{k\to\infty} \alpha_k.$$

Then $x \in [a,b]$ so there is a $U \in \mathcal{G}$ such that $x \in U$. This U is open so there is an $\delta > 0$ such that

$$[a,b] \cap (x-\delta, x+\delta) \subseteq U.$$

If k is chosen large enough that $(b-a)/2^k < \delta$ then $[\alpha_{i_k,k}, \beta_{i_k,k}] \subseteq U$. So $\{U\}$ is a finite subcover of $\mathcal{G}$. But i_k was chosen such that there is no finite subcover. So our assumption that $[a,b]$ is not compact leads to a contradiction.

If I is of any of the forms $(-\infty,+\infty)$, $(a,+\infty)$, $[a,+\infty)$, $(-\infty,b)$ or $(-\infty,b]$) then let $\mathcal{G}$ be the set of sets of the form $I \cap (-r,r)$ for $r > 0$. This is an open cover of I. It has no finite subcover $\{(-r_1,r_1), \ldots, (-r_m,r_m)\}$ because I contains elements of absolute value at least $\max_{1\le j\le m} r_j$. So I is not compact. Similarly $[a,b)$ and (a,b) are not compact because the set of sets of the form $I \cap (-\infty, x)$ for $x < b$ are an open cover without a finite subcover and $(a,b]$ is not compact because the sets of the form $(y,+\infty)$ for $y > a$ form an open cover with no finite subcover. □

There's an alternate characterisation of compact spaces in terms of closed sets.

Proposition 3.12.4. *The following two conditions are equivalent:*

(a) $(X,\mathcal{T})$ *is compact*

(b) For every set $\mathcal{C}$ *of closed subsets of* X *with the property that*

$$\bigcap_{V\in\mathcal{E}} V \neq \varnothing$$

for all finite $\mathcal{E} \subseteq \mathcal{C}$ *we have*

$$\bigcap_{V\in\mathcal{C}} V \neq \varnothing.$$

Proof. Suppose $(X,\mathcal{T})$ is compact and $\mathcal{C}$ is a set of closed subsets of X with the property that

$$\bigcap_{V\in\mathcal{E}} V \neq \varnothing$$

for all finite $\mathcal{E} \subseteq \mathcal{C}$. Define $\mathcal{G}$ to be the set of subsets of X of the form $X \setminus V$ for some $V \in \mathcal{G}$. The elements of $\mathcal{G}$ are open subsets. Suppose $\mathcal{E} \subseteq \mathcal{C}$ is finite and define $\mathcal{F}$ to be the set of subsets of X of the form $X \setminus V$ for some $V \in \mathcal{E}$. Then $\mathcal{F} \subseteq \mathcal{G}$.

$$\bigcup_{U\in\mathcal{F}} U = \bigcup_{V\in\mathcal{E}} (X \setminus V) = X \setminus \left(\bigcap_{V\in\mathcal{E}} V \right) \neq X$$

because $\bigcap_{V\in\mathcal{E}} V \neq \varnothing$. If $\mathcal{G}$ were an open cover of $(X,\mathcal{T})$ then it would have no finite subcover by what we've just proved. So $\mathcal{G}$ is not an open cover of X. Its elements are open sets so

$$\bigcup_{U\in\mathcal{G}} U \neq X.$$

Therefore

$$\bigcap_{V\in\mathcal{C}} V = \bigcap_{U\in\mathcal{G}} (X \setminus U) = X \setminus \left(\bigcap_{U\in\mathcal{G}} U\right) \neq \varnothing.$$

$\mathcal{C}$ was an arbitrary set of closed subsets of X with the property that

$$\bigcap_{V\in\mathcal{F}} V \neq \varnothing$$

for all finite $\mathcal{E} \subseteq \mathcal{C}$ and we've shown that

$$\bigcap_{V\in\mathcal{C}} V \neq \varnothing.$$

So the first of the conditions in the statement of the proposition implies the second.

Suppose, conversely, that for every set $\mathcal{C}$ of closed subsets of X with the property that

$$\bigcap_{V\in\mathcal{F}} V \neq \varnothing$$

for all finite $\mathcal{E} \subseteq \mathcal{C}$ we have

$$\bigcap_{V\in\mathcal{C}} V \neq \varnothing.$$

Suppose $\mathcal{G}$ is an open cover of $(X, \mathcal{T})$. Define $\mathcal{C}$ to be the set of subsets of X of the form $X \setminus U$ for some $U \in \mathcal{G}$. The elements of $\mathcal{C}$ are closed subsets of X. Also,

$$\bigcap_{V\in\mathcal{C}} V = \bigcap_{U\in\mathcal{G}} (X \setminus U) = X \setminus \left(\bigcap_{U\in\mathcal{G}} U\right) = X \setminus X = \varnothing$$

since $\mathcal{G}$ is an open cover of $(X, \mathcal{T})$. So there must be a finite $\mathcal{E} \subseteq C$ such that

$$\bigcap_{V\in\mathcal{E}} V = \varnothing.$$

Define $\mathcal{F}$ to be the set of subsets of X of the form $X \setminus V$ for some $V \in \mathcal{E}$. Then $\mathcal{F} \subseteq \mathcal{G}$ and

$$\bigcup_{U\in\mathcal{F}} U = \bigcup_{V\in\mathcal{E}} (X \setminus V) = X \setminus \left(\bigcup_{V\in\mathcal{E}} V\right) = X \setminus \varnothing = X.$$

So $\mathcal{F}$ is an open cover and hence a subcover of $\mathcal{G}$. We've found a finite subcover for the arbitrary open cover $\mathcal{G}$ of $(X, \mathcal{T})$, so $(X, \mathcal{T})$ is compact. This shows that the first condition in the statement of the proposition follows from the second. □

The proposition is most often applied to $\mathcal{C} = \{K_1, K_2, \ldots\}$ where the K's are a nested sequence of non-empty compact subsets, i.e. $K_1 \supseteq K_2 \supseteq \cdots$.

The following lemma is often useful for proving the compactness of subsets.

Lemma 3.12.5. *Suppose $(X, \mathcal{T}_X)$ is a topological space and $(A, \mathcal{T}_A)$ is a subspace. Then $(A, \mathcal{T}_A)$ is compact if and only if every $A \subseteq X$. Then A is compact if and only if every open cover of A with respect to $(X, \mathcal{T}_X)$ has a finite subcover.*

Proof. Suppose A is compact and $\mathcal{G}$ is an open cover of A with respect to $(X, \mathcal{T}_X)$. Let

$$\mathcal{H} = (i^*)_* (\mathcal{G}).$$

We note that $\mathcal{G} \subseteq \mathcal{T}_X$ so

$$\mathcal{H} = (i^*)_* (\mathcal{G}) \subseteq (i^*)_* (\mathcal{T}_X) = \mathcal{T}_A.$$

Also,

$$\bigcup_{V\in\mathcal{H}} V = \bigcup_{V\in(i^*)_*(\mathcal{G})} V = \bigcup_{U\in\mathcal{G}, V=i^*(U)} V$$

$$= \bigcup_{U\in\mathcal{G}} i^*(U) = i^*\left(\bigcup_{U\in\mathcal{G}} U\right)$$

and $\mathcal{G}$ is a subset of $A \subseteq \bigcup_{U\in\mathcal{G}} U$ so

$$A = i^*(A) \subseteq i^*\left(\bigcup_{U\in\mathcal{G}} U\right) = \bigcup_{V\in\mathcal{H}} V.$$

On the other hand, $V \subseteq A$ for all $V \in \mathcal{H}$ so

$$\bigcup_{V\in\mathcal{H}} V \subseteq A$$

and hence

$$A = \bigcup_{V\in\mathcal{H}} V.$$

So $\mathcal{H}$ is an open cover of $(A, \mathcal{T}_A)$. It therefore has a finite subcover, which we call $\mathcal{E}$. The restriction of i^* to $\mathcal{G}$ is a surjection from $\mathcal{G}$ to $\mathcal{H}$. By Proposition 2.7.2 there is a function $s\colon H \to \mathcal{G}$ such that $i^* \circ s$ is the identity on $\mathcal{H}$. Let $\mathcal{F} = s_*(\mathcal{E})$. Then $\mathcal{F}$ is the image of a finite set and hence is finite. Note that

$$A \cap \left(\bigcup_{U \in \mathcal{F}} U \right) = i^* \left(\bigcup_{U \in \mathcal{F}} U \right) = i^* \left(\bigcup_{V \in \mathcal{E}} s(V) \right)$$
$$= \bigcup_{V \in \mathcal{E}} i^*(s(V)) = \bigcup_{V \in \mathcal{E}} V = A$$

so

$$A \subseteq \bigcup_{U \in \mathcal{F}} U.$$

So $\mathcal{F}$ is a cover of A with respect to $(X, \mathcal{T}_X)$. Also $\mathcal{F} \subseteq \mathcal{G}$, so it is finite subcover. This shows that if A is compact then every open cover with respect to $(X, \mathcal{T}_X)$ has a finite subcover.

Suppose now that every open cover with respect to $(X, \mathcal{T}_X)$ has a finite subcover and that $\mathcal{H}$ is an open cover of $(A, \mathcal{T}_A)$. Using Proposition 2.7.2 again there is a function $s\colon \mathcal{T}_A \to \mathcal{T}_X$ such that $i^* \circ s$ is the identity on $\mathcal{T}_A$. Let

$$\mathcal{G} = s_*(\mathcal{H}).$$

Then

$$A \cap \left(\bigcup_{U \in \mathcal{G}} U \right) = i^* \left(\bigcup_{U \in \mathcal{G}} U \right) = i^* \left(\bigcup_{V \in \mathcal{H}} s(V) \right)$$
$$= \bigcup_{V \in \mathcal{H}} i^*(s(V)) = \bigcup_{V \in \mathcal{H}} V = A$$

so

$$A \subseteq \bigcup_{U \in \mathcal{G}} U.$$

So $\mathcal{G}$ is an open cover of A with respect to $(X, \mathcal{T}_X)$. By assumption it has a finite subcover, which we'll call $\mathcal{F}$. Let

$$\mathcal{E} = (i^*)_* (\mathcal{F}).$$

$\mathcal{E}$ is the image of a finite set and so is finite.

$$\mathcal{E} \subseteq (i^*)_* (\mathcal{G}) = (i^*)_* (s_*(\mathcal{H})) = (i^* \circ s)_* \mathcal{H} = \mathcal{H}.$$

Also,

$$\bigcup_{V \in \mathcal{E}} V = \bigcup_{U \in \mathcal{F}} i^*(V) = i^* \left(\bigcup_{U \in \mathcal{F}} V \right)$$
$$= A \cap \left(\bigcup_{U \in \mathcal{F}} V \right) = A.$$

So $\mathcal{E}$ is a finite subcover of $\mathcal{H}$. □

Compact sets have many interesting and useful properties.

Proposition 3.12.6. *If $(X, \mathcal{T})$ is compact and $K \in \wp(X)$ is closed then K is compact.*

Proof. We use Lemma 3.12.5, which says that K is compact if and only if every open cover of K with respect to $(X, \mathcal{T})$ has a finite subcover.

Suppose $\mathcal{G}$ is an open cover of K with respect to $(X, \mathcal{T})$. In other words

$$K \subseteq \bigcup_{U \in \mathcal{G}} U.$$

Let

$$\mathcal{H} = \mathcal{G} \cup \{X \setminus K\}.$$

$X \setminus K$ is open and

$$X = K \cup (X \setminus K) \subseteq \left(\bigcup_{U \in \mathcal{G}} U \right) \cup (X \setminus K) = \bigcup_{U \in \mathcal{H}} U$$

so $\mathcal{H}$ is an open cover of X. $(X, \mathcal{T})$ is compact so $\mathcal{H}$ has a finite subcover. This finite subcover is also an open cover of K with respect to $(X, \mathcal{T})$. This cover may or may not include $X \setminus K$ but if it does we can remove $X \setminus K$ and the result will still be a cover of K. This new open cover is a finite subcover of $\mathcal{G}$. So every open cover of K with respect to $(X, \mathcal{T})$ has a finite subcover. In other words, K is compact. □

Proposition 3.12.7. *Suppose that $(X, \mathcal{T})$ is a topological space and $K_1, \ldots, K_m \in \wp(X)$ are compact. Then so is $\bigcup_{j=1}^m K_j$.*

Proof. Again we use Lemma 3.12.5. Suppose $\mathcal{G}$ is an open cover of $\bigcup_{j=1}^m K_j$ with respect to $(X, \mathcal{T})$. Then it's also an open cover of each K_j with respect to $(X, \mathcal{T})$. K_j is compact so there's a finite subcover $\mathcal{F}_j$. Then $\bigcup_{j=1}^m \mathcal{F}_j$ is an open cover of $\bigcup_{j=1}^m K_j$. As a finite union of finite sets it's finite and so is a finite subcover of $\mathcal{G}$. □

Proposition 3.12.8. *Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f\colon X \to Y$ is a continuous function. If $K \in \wp(X)$ is compact then so is $f_*(K)$.*

Proof. As usual we use Lemma 3.12.5. Suppose $\mathcal{G}$ is an open cover of $f_*(K)$ with respect to $(Y, \mathcal{T}_Y)$. Let

$$\mathcal{H} = (f^*)_* (\mathcal{G}).$$

Then $\mathcal{H} \subseteq \mathcal{T}_X$ by Lemma 3.1.2. If $x \in K$ then $f(x) \in f_*(K)$ so $f(x) \in V$ for some $V \in \mathcal{G}$. But then $x \in f^*(V)$ and $f^*(V) \in \mathcal{H}$. So $\mathcal{H}$ is an open cover of K. By assumption K is compact so there's a finite subcover $\mathcal{E}$ of $\mathcal{H}$. Each element $U \in \mathcal{E}$ is $f^*(V)$ for some $V \in \mathcal{G}$. There might be more than one such V for a given U but choose one and let $\mathcal{F}$ be the set of those V. There are then at most as many elements of $\mathcal{F}$ as of $\mathcal{E}$ so $\mathcal{F}$ is a finite subset of $\mathcal{G}$. If $y \in f_*(K)$ then $y = f(x)$ for some $x \in K$. Then $x \in U$ for some $U \in \mathcal{E}$ and this U is $f^*(V)$ for some $V \in \mathcal{F}$. Then $x \in f^*(V)$ so $f(x) \in V$, i.e. $y \in V$. So every $y \in f_*(K)$ is an element of some $V \in \mathcal{F}$. So $\mathcal{F}$ is a finite subcover of $\mathcal{G}$. □

Lemma 3.12.9. *Suppose $(X, \mathcal{T})$ is a Hausdorff space $A \in \wp(X)$ is compact and $y \notin A$. Then there are $V, W \in \mathcal{T}$ such that $A \subseteq V$, $y \in W$ and $V \cap W = \varnothing$.*

Proof. For each $x \in A$ we have $x \neq y$ so by the definition of Hausdorff there are $V_x, W_x \in \mathcal{T}$ such that $x \in V_x$, $y \in W_x$ and $V_x \cap W_x = \varnothing$. Then

$$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} V_x$$

so this is an open cover of A. A is compact so there is a finite subcover. In other words, there is a finite $F \in \wp(A)$ such that

$$A \subseteq \bigcup_{x \in F} V_x.$$

Let

$$V = \bigcup_{x \in F} V_x$$

and

$$W = \bigcap_{x \in F} W_x.$$

Then $A \subseteq V$. Finite unions or intersections of open sets are open so $V \in \mathcal{T}$ and $W \in \mathcal{T}$. Also, $y \in W_x$ for each x so $y \in W$. If $z \in V \cap W$ then $z \in V_x$ for some $x \in F$. But $W_x \subseteq W$ so $z \in W_x$ as well. $V_x \cap W_x = \varnothing$, so there is no $z \in V \cap W$. In other words, $V \cap W = \varnothing$. □

Proposition 3.12.10. *If $(X, \mathcal{T})$ is a Hausdorff topological space and $A \in \wp(X)$ is compact then A is closed.*

Proof. The lemma implies that for each $y \in X \setminus A$ there is a $V_y \in \mathcal{O}(y)$ such that $V_y \subseteq X \setminus A$. But then

$$A = \bigcup_{y \in X \setminus A} \{y\} \subseteq \bigcup_{y \in X \setminus A} V_y$$

and

$$\bigcup_{y \in X \setminus A} V_y \subseteq X \setminus A$$

so

$$X \setminus A = \bigcup_{y \in X \setminus A} V_y$$

is a union of open sets and hence open. A is therefore closed. □

Proposition 3.12.11. *Suppose $(X, \mathcal{T})$ is a Hausdorff topological space and $\mathcal{K}$ is a set of compact subsets of X. Then $\bigcap_{K \in \mathcal{K}} K$ is compact.*

Proof. Each $K \in \mathcal{K}$ is closed by Proposition 3.12.10. Any intersection of closed sets is closed so $\bigcap_{K \in \mathcal{K}} K$ is closed. It is a closed subset of any particular $K \in \mathcal{K}$ and closed subsets of a compact space are compact by Proposition 3.12.6, so $\bigcap_{K \in \mathcal{K}} K$ is compact. □

Proposition 3.12.12. *Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are compact spaces. Then $(P, \mathcal{T}_P)$ is compact, where $P = X \times Y$ and $\mathcal{T}_P$ is the product topology.*

Proof. Suppose $\mathcal{G}$ is an open cover of P. For each $x \in X$ and $y \in Y$ there is a $Z \in \mathcal{T}_P$ such that $(x, y) \in Z$ and $Z \in \mathcal{G}$. By Proposition 3.10.4 there are $V \in \mathcal{T}_X$ and $W \in \mathcal{T}_Y$ such that $x \in V$, $y \in W$ and $V \times W \subseteq Z$. Let $Z_{x,y}$, $V_{x,y}$ and $W_{x,y}$ be such Z, V and W. For given $x \in X$ the sets $W_{x,y}$ are an open cover of Y. Y is compact so there's a finite subcover. In other words, there is a finite subset F_x of Y such that the set of $W_{x,y}$ with $y \in F_x$ cover Y, i.e.,

$$Y = \bigcup_{y \in F_x} W_{x,y}.$$

Define

$$U_x = \bigcap_{y \in F_x} V_{x,y}.$$

Then $x \in U_x$ and $U_x \in \mathcal{T}_X$, since it's a finite intersection of open sets. These sets therefore form an open cover of X. X is compact so there's a finite subcover. In other words, there's a finite subset E of X such that the set of U_x with $x \in E$ cover X, i.e.

$$X = \bigcup_{x \in E} U_x.$$

Suppose $(s, t) \in P$. Then $s \in X$ so $s \in U_x$ for some $x \in E$. Then $t \in W_{x,y}$ for some $y \in F_x$. Also, $s \in V_{x,y}$, since $U_x \subseteq V_{x,y}$ for all $y \in F_x$. So $(s, t) \in V_{x,y} \times W_{x,y}$ and therefore $(s, t) \in Z_{x,y}$. So the sets $Z_{x,y}$ where $x \in E$ and $y \in F_x$ cover P. This is a finite union of finite sets and so is finite, so it is a finite subcover of $\mathcal{G}$. □

The proposition was stated for the product of two sets but can easily be extended to the product of finitely many sets by induction. Tychonoff's Theorem is the corresponding statement for all products. This is considerably more difficult to prove, so we will skip it for now.

Proposition 3.12.13. *Suppose $(X, \mathcal{T})$ is a compact Hausdorff space and $A, B \in \wp(X)$ are closed subsets such that $A \cap B = \varnothing$. Then there are $V, W \in \mathcal{T}$ such that $A \subseteq V$, $B \subseteq W$ and $V \cap W = \varnothing$.*

Proof. A and B are closed subsets of a compact Hausdorff space and so are compact by Proposition 3.12.10. By Lemma 3.12.9 there are, for each $y \in B$, open V_y and W_y such that $A \subseteq V_y$, $y \in W_y$ and $V_y \cap W_y = \varnothing$.

$$B = \bigcup_{y \in B} \{y\} \subseteq \bigcup_{y \in B} W_y$$

so these form an open cover of B. B is compact so there is a finite subcover. In other words, there's a finite $F \in \wp(Y)$ such that

$$B \subseteq \bigcup_{y \in F} W_y.$$

Let

$$V = \bigcap_{y \in F} V_y$$

and

$$W = \bigcup_{y \in F} W_y.$$

Finite intersections or unions of open sets are open so V and W are open. $A \subseteq V_y$ for each Y so $A \subseteq V$. We've already seen that $B \subseteq W_y$. If $z \in V \cap W$ then $z \in W_y$ for some $y \in F$. But $V \subseteq V_y$ so $z \in V_y$. But $V_y \cap W_y = \varnothing$, so there is not $z \in V \cap W$ and therefore $V \cap W = \varnothing$. □

The next two theorems are particularly important. The following theorem is known as the Heine-Borel Theorem.

Theorem 3.12.14. *A subset $S \in \wp(\mathbf{R}^n)$ is compact if and only if it is closed and bounded.*

Proof. Suppose S is compact. $\mathbf{R}^n$ is Hausdorff so S must be closed by Proposition 3.12.10. The open sets $B(0, r)$ for $r > 0$ cover S and therefore must have a finite subcover $\{B(0, r_1), \ldots, B(0, r_m)\}$. So

$$S \subseteq \bigcup_{j=1}^{m} B(0, r_j) = B\left(0, \max_{1 \leq j \leq m} r_j\right)$$

and S is therefore bounded.

Suppose conversely that S is closed and bounded. There is then an $r > 0$ such that $S \subseteq B(0, r)$. S is then a subset of the product of n copies of the interval $[-r, r]$. This interval is compact by Proposition 3.12.3 and the product is compact by Proposition 3.12.12. S is thus a closed subset of a compact set. By Proposition 3.12.6 it is compact. □

The following theorem is known as the Extreme Value Theorem.

Theorem 3.12.15. *Suppose $(X, \mathcal{T})$ is a compact topological space and $f\colon X \to \mathbf{R}$ is a continuous function. Then f attains a maximum and minimum value. In other words, there are $w, z \in X$ such that $f(w) \leq f(x) \leq f(z)$ for all $x \in X$.*

Proof. $f_*(X)$ is compact by Proposition 3.12.8 and therefore is closed and bounded by the Heine-Borel Theorem. The fact that it's bounded implies that it has an upper bound and hence a supremum. Call this supremum b. Then $y \leq b$ for all $y \in f_*(X)$. If b were not in $f_*(X)$ then there would be a $\delta > 0$ such that $B(b, \delta) \subseteq \mathbf{R} \setminus f_*(X)$, since $f_*(X)$ is closed. But then $b - \delta/2$ would be an upper bound for $f_*(X)$ lower than the supremum. This is impossible so $b \in f_*(X)$. In other words, there is a $z \in X$ such that $f(z) = b$. So $y \leq f(z)$ for all $y \in f_*(X)$ or, equivalently, $f(x) \leq f(z)$ for all $x \in X$. The proof that there is a $w \in X$ such that $f(w) \leq f(x)$ for all $x \in X$ is similar. □

The following theorem is known as the Bolzano-Weierstrass Theorem.

Theorem 3.12.16. *Suppose $K \subseteq \mathbf{R}^n$ is compact. $\alpha\colon \mathbf{N} \to X$ is a sequence such that $\alpha_n \in K$ for all n. Then α has a convergent subsequence.*

Proof. For $m \in \mathbf{N}$ set

$$T_m = \{n \in \mathbf{N} \colon m \leq n\}$$

and

$$C_m = \overline{\alpha_*(T_m)}.$$

Let $\mathcal{C}$ be the set of all C_m from $m \in \mathcal{N}$. Then $\mathcal{C}$ satisfies the hypotheses of Proposition 3.12.4 because for any finite subset $\mathcal{E} = \{C_{m_1}, \ldots, C_{m_k}\}$ we have $\alpha_n \in \bigcap_{V \in \mathcal{E}} V$ where $n = \max(m_1, \ldots, m_k)$, so $\bigcap_{V \in \mathcal{E}} V \neq \varnothing$. It then follows from Proposition 3.12.4 that

$$\bigcap_{V \in \mathcal{C}} V = \bigcap_{m \in \mathbf{N}} C_m \neq \varnothing.$$

There is therefore a $z \in \bigcap_{m \in \mathbf{N}} C_m$. Suppose $Z \in \mathcal{O}(z)$. By Proposition 3.2.2l $\alpha_*(T_m) \cap Z \neq \varnothing$. In other words, there is an $n \geq m$ such that $\alpha_n \in Z$. This holds in particular for $Z = B(z, 1/2^k)$ for $k \in \mathbf{N}$. So for each $k \in \mathbf{N}$ there is an $n_k \in \mathbf{N}$ such that

$$\alpha_{n_k} \in B(z, 1/2^k).$$

Then

$$\lim_{k \to \infty} \alpha_{n_k} = z.$$

□

3.13 Normal spaces

Proposition 3.12.13 motivates the following definition.

Definition 3.13.1. A topological space $(X, \mathcal{T})$ is called *normal* if for any closed $A, B \in \wp(X)$ that $A \cap B = \varnothing$ there are $V, W \in \mathcal{T}$ such that $A \subseteq V$, $B \subseteq W$ and $V \cap W = \varnothing$.

Proposition 3.12.13 shows that compact Hausdorff spaces are normal. We'll see later that metric spaces are also normal.

For an example of a topological space which is not normal, consider an infinite set X with the cofinite topology $\mathcal{T}$. If $a, b \in X$ and $a \neq b$ then $A = \{a\}$ and $B = \{b\}$ are closed sets and $A \cap B = \varnothing$. If V and W are open and $A \subseteq V$ and $B \subseteq W$ then V and W are non-empty so $X \setminus V$ and $X \setminus W$ are finite. But then

$$X \setminus (V \cap W) = (X \setminus V) \cup (X \setminus W)$$

is finite. Therefore $X \setminus (V \cap W) \neq X$ and so $V \cap W \neq \varnothing$.

The following result is known as Urysohn's Lemma:

Lemma 3.13.2. *Suppose $(X, \mathcal{T})$ is a normal topological space and A and B are closed subsets of X such that $A \cap B = \varnothing$. Then there is a continuous function $f\colon X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.*

Proof. Note that the following three statements are equivalent for and $P, Q \in \wp(X)$:

- $P \cap Q = \varnothing$.
- $P \subseteq X \setminus Q$.
- $Q \subseteq X \setminus P$.

Also, the following three are equivalent:

- $P \cup Q = X$.
- $X \setminus P \subseteq Q$.
- $X \setminus Q \subseteq P$.

We'll need these facts repeatedly below with various choices of P and Q.

We begin the proof by constructing some set valued functions. Let D_k be the set of rational numbers of the form $j/2^k$ where j and k are integers such that $0 < j < 2^k$. By induction on k we show that there are functions $V\colon D_k \cup \{1\} \to \wp(X)$ and $W\colon D_k \cup \{0\} \to \wp(X)$ with the following properties:

(a) $V(y), W(y) \in \mathcal{T}$,

(b) $A \subseteq V(y)$ and $B \subseteq W(y)$,

(c) If $y_1 \leq y_2$ then $V(y_1) \cap W(y_2) = \varnothing$, and

(d) If $y_1 < y_2$ then $W(y_1) \cup V(y_2) = X$.

These are to hold for any y, y_1 and y_2 for which the functions are defined.

We start from $k = 0$, $W(0) = X \setminus A$ and $V(1) = X \setminus B$. (a) is satisfied because A and B are closed. (b) is satisfied because $A \cap B = \varnothing$, so $A \subseteq X \setminus B$ and $B \subseteq X \setminus A$. (c) is vacuously true for $k = 0$ because there are no $y_1 \leq y_2$ for which $V(y_1)$ and $W(y_2)$ are both defined. (d) is satisfied because

$$\begin{aligned} W(1) \cup V(0) &= (X \setminus A) \cup (X \setminus B) = X \setminus (A \cap B) \\ &= X \setminus \varnothing = X. \end{aligned}$$

For the inductive step we assume V and W have been defined on $D_k \cup \{1\}$ and $D_k \cup \{0\}$ respectively in a way which satisfies all the conditions above and then define them on $D_{k+1} \cup \{1\}$ and $D_{k+1} \cup \{0\}$ in such a way that the conditions are still satisfied. If $y \in D_k \cup \{1\}$ or $y \in D_k \cup \{0\}$ we leave $V(y)$ and $W(y)$ unchanged. We therefore only need to define it at the points $j/2^{k+1}$ where $0 < j < 2^{k+1}$ is an odd integer. We can write this as $j = 2i + 1$, where $0 \leq i < 2^k$. By (a) from the previous stage of the induction. $W(i/2^k)$ and $V((i+1)/2^k)$ are open, and hence their complements are closed.

$$\begin{aligned} &\left(X \setminus W\left(\frac{i}{2^k}\right)\right) \cap \left(X \setminus V\left(\frac{i+1}{2^k}\right)\right) \\ &\quad = X \setminus \left(W\left(\frac{i}{2^k}\right) \cup V\left(\frac{i+1}{2^k}\right)\right) \\ &\quad = X \setminus X = \varnothing \end{aligned}$$

by (d) of the previous stage. $(X, \mathcal{T})$ is normal so there are $V, W \in \mathcal{T}$ such that

$$X \setminus W\left(\frac{i}{2^k}\right) \subseteq V,$$

$$X \setminus V\left(\frac{i+1}{2^k}\right) \subseteq W,$$

and

$$V \cap W = \varnothing.$$

We defined $V(j/2^{k+1})$ and $W(j/2^{k+1})$ to be these V and W. In other words,

$$V\left(\frac{j}{2^{k+1}}\right), W\left(\frac{j}{2^{k+1}}\right) \in \mathcal{T},$$

$$X \setminus W\left(\frac{i}{2^k}\right) \subseteq V\left(\frac{j}{2^{k+1}}\right),$$

$$X \setminus V\left(\frac{i+1}{2^k}\right) \subseteq W\left(\frac{j}{2^{k+1}}\right),$$

and

$$V\left(\frac{j}{2^{k+1}}\right) \cap W\left(\frac{j}{2^{k+1}}\right) = \varnothing.$$

The first of these statements is (a). To show (b) we note that

$$A \subseteq V\left(\frac{i}{2^k}\right) \subseteq X \setminus W\left(\frac{i}{2^k}\right) \subseteq V\left(\frac{j}{2^{k+1}}\right).$$

The first of these inclusions was (b) from the previous stage of the induction while the second one follows from (c) of the previous stage and the third inclusion was already established above. Similarly,

$$B \subseteq W\left(\frac{i+1}{2^k}\right) \subseteq X \setminus V\left(\frac{i}{2^k}\right) \subseteq W\left(\frac{j}{2^{k+1}}\right).$$

This establishes (c). Suppose $j_1 \leq j_2$. If they are even then we already have

$$V\left(\frac{j_1}{2^{k+1}}\right) \cap W\left(\frac{j_2}{2^{k+1}}\right) = \varnothing$$

from the previous stage of the induction. If they are odd and equal then it was proved above. If they are odd and unequal then $j_1 = 2i_1 + 1$ and $j_2 = 2i_2 + 1$ where $i_1 + 1 \leq i_2$.

$$\begin{aligned} V\left(\frac{j_1}{2^{k+1}}\right) &\subseteq X \setminus W\left(\frac{j_1}{2^{k+1}}\right) \subseteq V\left(\frac{i_1+1}{2^k}\right) \\ &\subseteq X \setminus W\left(\frac{i_2}{2^k}\right) \subseteq V\left(\frac{j_2}{2^{k+1}}\right) \\ &\subseteq X \setminus W\left(\frac{j_2}{2^{k+1}}\right) \end{aligned}$$

The first of these and fifth of these inclusions follow from the statement

$$V\left(\frac{j}{2^{k+1}}\right) \cap W\left(\frac{j}{2^{k+1}}\right) = \varnothing$$

proved above, applied to $j = j_1$ and $j = j_2$. Similarly, the second and fourth inclusions follow from

$$X \setminus V\left(\frac{i+1}{2^k}\right) \subseteq W\left(\frac{j}{2^{k+1}}\right)$$

and

$$X \setminus W\left(\frac{i}{2^k}\right) \subseteq V\left(\frac{j}{2^{k+1}}\right)$$

with $j = j_1$ and $j = j_2$ respectively. The third inclusion follows from (c) from the previous stage of the induction. So we have

$$V\left(\frac{j_1}{2^{k+1}}\right) \subseteq X \setminus W\left(\frac{j_2}{2^{k+1}}\right),$$

which is equivalent to

$$V\left(\frac{j_1}{2^{k+1}}\right) \cap W\left(\frac{j_2}{2^{k+1}}\right) = \varnothing,$$

i.e. (c). The cases where one of j_1 or j_2 is odd and the other is even are similar, but we only need three inclusions instead of five. Suppose now $j_1 < j_2$. Again we already have

$$W\left(\frac{j_1}{2^{k+1}}\right) \cup V\left(\frac{j_2}{2^{k+1}}\right) = X$$

if j_1 and j_2 are both even. If they are both odd then we proceed as follows.

$$\begin{aligned} X \setminus W\left(\frac{j_1}{2^{k+1}}\right) \subseteq V\left(\frac{i_1+1}{2^k}\right) \subseteq X \setminus W\left(\frac{i_2}{2^k}\right) \\ \subseteq V\left(\frac{j_1}{2^{k+1}}\right) \end{aligned}$$

and hence

$$W\left(\frac{j_1}{2^{k+1}}\right) \cup V\left(\frac{j_1}{2^{k+1}}\right) = X.$$

Again, the cases where one j is even and one is odd are simpler and will be skipped. This establishes (d) and completes the induction.

Having defined V on $D_k \cup \{1\}$ and W on $D_k \cup \{0\}$ for all k we can regard them as defined on the $D \cup \{1\}$ and $D \cup \{0\}$ respectively, where

$$D = \bigcup_{k=0}^{\infty} D_k.$$

These V and W still satisfy the four properties (a) through (d) because for any y, y_1 and y_2 in D there is a k such that they are all in D_k.

We now define

$$S(x) = \{y \in D \colon x \in V(y)\}$$

and

$$T(x) = \{y \in D \colon x \in W(y)\}$$

From the properties (b) through (d) it follows that

(i) If $x \in A$ then $S(x) = D \cup \{1\}$.

(ii) If $x \in B$ then $T(x) = D \cup \{0\}$.

(iii) For all $x \in X$ if $y_1 \in T(x)$ and $y_2 \in S(x)$ then $y_1 < y_2$.

(iv) For all $x \in X$ if $y_1 \notin S(x)$ and $y_2 \notin T(x)$ then $y_1 \leq y_2$.

If both $S(x)$ and $T(x)$ are non-empty then it follows from (iii) that

$$\inf S(x) \geq \sup T(x)$$

while it follows from (iv) that

$$\inf S(x) \leq \sup T(x).$$

It therefore makes sense to define

$$f(x) = \inf S(x) = \sup T(x)$$

for such x. We define $f(x) = 0$ if $T(x) = \varnothing$ and $f(x) = 1$ if $S(x) = \varnothing$. We then have $y \in T(x)$ if $y < f(x)$ and $y \in S(x)$ if $y > f(x)$. $f(x) = 0$ if $x \in A$ by (i), while $f(x) = 1$ if $x \in B$ by (ii).

The only thing remaining to be proved is that f is continuous. If $f(x) > a$ then $a < \sup T(x)$ so a is not an upper bound for $T(x)$ and there is a $q \in T(x)$ with $q > a$. Therefore $x \in W(q)$ for some $q > a$. In other words,

$$x \in \bigcup_{q>a} W_q.$$

Conversely, if $x \in \bigcup_{q>a} W_q$ then $q \in T(x)$ for some $q > a$ so a is not an upper bound for $T(x)$ and therefore $a < \sup T(x)$. Then $f(x) > a$. So $f(x) > a$ if and only if $x \in \bigcup_{q>a} W_q$. In other words,

$$f^*((a, +\infty)) = \bigcup_{q>a} W_q.$$

Similarly, if $f(x) < b$ then $b > \inf S(x)$ so b is not a lower bound for $S(x)$ and there is a $q \in S(x)$ with $q < b$. Therefore $x \in V(q)$ for some $q < b$. In other words,

$$x \in \bigcup_{q<b} V_q.$$

Conversely, if $x \in \bigcup_{q<b} V_q$ then $q \in S(x)$ for some $q < b$ so b is not an upper bound for $S(x)$ and therefore $b > \inf S(x)$. Then $f(x) < b$. So $f(x) < b$ if and only if $x \in \bigcup_{q<b} V_q$. In other words,

$$f^*((-\infty, b)) = \bigcup_{q<b} V_q.$$

Since the preimage of an intersection is the intersection of the preimages we have that

$$f^*((a,b)) = \left(\bigcup_{q\in D, q>a} W(q) \right) \cap \left(\bigcup_{q\in D, q<b} V(q) \right),$$

which is an open subset of X. Every open set in $\mathbf{R}$ is a union of open intervals and the preimage of a union is the union of the preimages so the preimage of any open set in $\mathbf{R}$ is an open set in X. Therefore f is continuous. $\square$

The following theorem is known as the Tietze Extension Theorem.

Theorem 3.13.3. *Suppose $(X, \mathcal{T})$ is a normal topological space, $A \in \wp(X)$ is closed and $f: A \to [a, b]$ is continuous. Then there is a continuous $g: X \to [a, b]$ such that $g(x) = f(x)$ for all $x \in A$.*

Proof. It's easier to work with the midpoint $m = \frac{a+b}{2}$ and length $l = b - a$ of the interval than with its endpoints a and b.

We prove by induction that there is a sequence h of continuous functions such that

(a)
$$|f(x) - h_k(x)| \le \frac{2^{k-1}}{3^k} l$$
for all $x \in A$.

(b)
$$|h_k(x) - h_j(x)| \le \left(\frac{2^{j-1}}{3^j} - \frac{2^{k-1}}{3^k} \right) l$$
for all $x \in X$ and $j \le k$.

The base case is easy since

$$h_0(x) = m$$

satisfies both conditions. Assume therefore that h_k has been constructed in such a way that both conditions are satisfied. Let

$$P_k = \left\{ x \in A : f(x) - h_k(x) \le -\frac{2^{k-1}}{3^{k+1}} l \right\}$$

and

$$Q_k = \left\{ x \in A : f(x) - h_k(x) \ge \frac{2^{k-1}}{3^{k+1}} l \right\}.$$

These are closed sets because $f - h_k$ is continuous. By the Tietze Extension theorem there is therefore a continuous function $e_k: X \to [0, 1]$ such that $e_k(x) = 0$ for $x \in P_k$ and $e_k(x) = 1$ for $x \in Q_k$. We then define h_{k+1} by

$$h_{k+1}(x) = h_k(x) + \frac{2^{k-1}}{3^{k+1}} l (2e_k(x) - 1).$$

If $x \in P_k$ then

$$-\frac{2^{k-1}}{3^k}l \leq f(x) - h_k(x) \leq -\frac{2^{k-1}}{3^{k+1}}l.$$

The inequality on the left is the part of the inductive hypothesis while the one on the right is the definition of P_k. We also have

$$h_{k+1}(x) - h_k(x) = -\frac{2^{k-1}}{3^{k+1}}l$$

by the definition of h_{k+1} so

$$\frac{2^k}{3^{k+1}}l \leq f(x) - h_{k+1}(x) \leq 0.$$

If $x \in Q_k$ then

$$\frac{2^{k-1}}{3^{k+1}}l \leq f(x) - h_k(x) \leq \frac{2^{k-1}}{3^k}l.$$

The inequality on the left is the definition of Q_k while the one on the right is part of the inductive hypothesis. We also have

$$h_{k+1}(x) - h_k(x) = \frac{2^{k-1}}{3^{k+1}}l$$

by the definition of h_{k+1} so

$$0 \leq f(x) - h_{k+1}(x) \leq \frac{2^k}{3^{k+1}}l.$$

If $x \in A \setminus (P_k \cup Q_k)$ then

$$-\frac{2^{k-1}}{3^{k+1}}l < f(x) - h_k(x) < \frac{2^{k-1}}{3^{k+1}}l$$

by the definitions of P_k and Q_k while

$$-\frac{2^{k-1}}{3^{k+1}}l < h_{k+1}(x) - h_k(x) < \frac{2^{k-1}}{3^{k+1}}l$$

by the definition of h_{k+1} so

$$-\frac{2^k}{3^{k+1}}l < f(x) - h_{k+1}(x) < \frac{2^k}{3^{k+1}}l.$$

So if $x \in A$ then

$$-\frac{2^k}{3^{k+1}}l \leq f(x) - h_{k+1}(x) \leq \frac{2^k}{3^{k+1}}l.$$

In other words,

$$|f(x) - h_{k+1}(x)| \leq \frac{2^k}{3^{k+1}}l.$$

This is (a) with k replaced by $k+1$. We have

$$|h_{k+1}(x) - h_j(x)| \leq \left(\frac{2^{j-1}}{3^j} - \frac{2^k}{3^{k+1}}\right)l$$

trivially if $j = k+1$. Otherwise we can use the inductive hypothesis

$$|h_k(x) - h_j(x)| \leq \left(\frac{2^{j-1}}{3^j} - \frac{2^{k-1}}{3^k}\right)l$$

together with

$$|h_{k+1}(x) - h_k(x)| \leq \frac{2^{k-1}}{3^{k+1}}l = \frac{2^{k-1}}{3^k}l - \frac{2^k}{3^{k+1}}l,$$

which follows from the definition of h_{k+1} to get

$$|h_{k+1}(x) - h_j(x)| \leq \left(\frac{2^{j-1}}{3^j} - \frac{2^k}{3^{k+1}}\right)l$$

This is (b) with k replaced by $k+1$. This completes the inductive construction of the sequence h.

For any $\epsilon > 0$ there is an m such that

$$\frac{2^{m-1}}{3^m}l < \epsilon$$

and hence if $n \geq m$ then

$$\frac{2^{n-1}}{3^n}l < \epsilon$$

and therefore

$$|h_k(x) - f(x)| < \epsilon$$

for all $x \in A$. Therefore

$$\lim_{k\to\infty} h_k(x) = f(x)$$

for all $x \in A$. We can also define

$$g(x) = \lim_{k\to\infty} h_k(x)$$

for all $x \in X$. The convergence follows from Cauchy's criterion and the inequality

$$|h_k(x) - h_j(x)| \leq \left(\frac{2^{j-1}}{3^j} - \frac{2^{k-1}}{3^k}\right) l$$

If we take m large enough that

$$\frac{2^{m-1}}{3^m} < \epsilon$$

then for all $x \in X$ and $j, k \geq m$ we have

$$|h_k(x) - h_j(x)| < \epsilon.$$

The fact that this m is independent of x means the Cauchy criterion is satisfied uniformly, so the sequence converges uniformly. Since each h_k is continuous the limiting g is also continuous. It clearly satisfies $g(x) = f(x)$ for $x \in A$. It satisfies $g(x) \in [a, b]$ because applying

$$|h_k(x) - h_j(x)| \leq \left(\frac{2^{j-1}}{3^j} - \frac{2^{k-1}}{3^k}\right) l$$

with $j = 0$ gives

$$|h_k(x) - m| \leq \left(\frac{1}{2} - \frac{2^{k-1}}{3^k}\right) l \leq \frac{1}{2} l$$

and taking the limit as k tends to infinity gives

$$|g(x) - m| \leq \frac{1}{2} l.$$

□

The following proposition will be useful when we define integrals.

Proposition 3.13.4. *Suppose $(X, \mathcal{T})$ is a normal topological space and let $\mathcal{C}$ be the set of closed subsets of X. that $K \colon L \to \mathcal{C}$ and $U \colon L \to \mathcal{T}$ are indexed collections of sets with the following properties.*

(a) $X = \bigcup_{\lambda \in L} K_\lambda$.

(b) $K(\lambda) \subseteq U(\lambda)$ *for all* $\lambda \in L$.

(c) For all $x \in X$ there is $V \in \mathcal{O}(x)$ such that the set $\{\lambda \in L \colon V \cap U(\lambda) \neq \varnothing\}$ is finite.

Then there is a function $g \colon L \times X \to [0, 1]$, continuous in its second argument, satisfying the following conditions.

(i) For each $\lambda \in L$ the set $\{x \colon g(\lambda, x) > 0\}$ is a subset of $U(\lambda)$.

(ii) For each $x \in X$ we have $\sum_{\lambda \in L} g(\lambda, x) = 1$.

Note that the Condition 3.13.4i ensures that the sum in Condition 3.13.4ii has only finitely many nonzero terms. A function g satisfying the conditions above is called a *partition of unity*.

Proof. Urysohn's Lemma, applied to $X \setminus U(\lambda)$ and $K(\lambda)$ gives a continuous function $f \colon X \to [0, 1]$ such that $f(x) = 0$ for $x \notin U(\lambda)$ and $f(x) = 1$ for $x \in K(\lambda)$. This function depends on λ, so we write it as $f(\lambda, x)$.

Let $h(x) = \sum_{\lambda \in L} f(\lambda, x)$. For each $x \in X$ there is, by Condition 3.13.4c, a $V \in \mathcal{T}$ and a finite $F \in \wp(L)$ such that $V \cap U(\lambda) = \varnothing$ if $\lambda \notin F$. $f(\lambda, y) = 0$ if $y \in V$ and $V \cap U(\lambda) = \varnothing$ so

$$h(y) = \sum_{\lambda \in F} f(\lambda, y)$$

for $y \in V$. A finite sum of continuous functions is continuous so h is continuous in V. By the lemma below h is therefore continuous in X. If $x \in X$ then $x \in K(\lambda)$ for some $\lambda \in L$ by Condition 3.13.4a. $f(\lambda, x) = 1$ for this x and all the other summands are non-negative so $h(x) \geq 1$. In particular, the function h has no zeroes so the quotient

$$g(\lambda, x) = \frac{f(\lambda, x)}{h(x)}$$

is a continuous function of its second argument. It follows from $f(\lambda, x) \in [0, 1]$ and $h(x) \geq 1$ that $g(\lambda, x) \in [0, 1]$. Also,

$$\sum_{\lambda \in L} g(\lambda, x) = \sum_{\lambda \in L} f(\lambda, x)/h(x) = h(x)/h(x) = 1.$$

□

Lemma 3.13.5. *Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, $\mathcal{G}$ is an open cover of X and*

$f\colon X \to Y$ is a function such that for all $V \in \mathcal{G}$ the restriction of f to V is a continuous function from V, with the subspace topology, to Y. Then f is continuous.

Proof. Suppose $W \in \mathcal{T}_Y$. Then

$$f^*(W) = X \cap f^*(W) = \left(\bigcup_{V \in \mathcal{G}} V\right) \cap f^*(W)$$
$$= \bigcup_{V \in \mathcal{G}} \left(V \cap f^*(W)\right).$$

$V \cap f^*(W)$ is the preimage of W under the restriction of f to V and so, by the continuity hypothesis, is an open subset of V in the subspace topology. In other words,

$$V \cap f^*(W) = V \cap U$$

for some $U \in \mathcal{T}_X$. Finite intersections of open sets are open so $V \cap U \in \mathcal{T}_X$ and hence $V \cap f^*(W) \in \mathcal{T}_X$. Unions of open sets are open so

$$\bigcup_{V \in \mathcal{G}} \left(V \cap f^*(W)\right) \in \mathcal{T}_X$$

and hence $f^*(W) \in \mathcal{T}_X$. So f is continuous. □

4 Metric spaces

4.1 Review and elementary properties

Metrics were given in defined in Definition 1.6.1 and their elementary properties were given in the sections that followed. One of the most important is Proposition 1.10.3, which tells us that metric spaces are Hausdorff topological spaces, with the topology being that of open sets, defined in Definition 1.8.1. Unless otherwise specified, when talk about metric spaces we always consider them with this topology. All the notions from the last chapter therefore apply to metric spaces. In some cases though general definitions which apply to all topological spaces can be replaced by simpler criteria in the case of metric spaces.

Proposition 4.1.1. *Suppose (X, d) is a metric space and $A \in \wp(X)$.*

(a) $x \in A^\circ$ if and only if there is an $r > 0$ such that $B(x, r) \subseteq A$.

(b) $x \in \overline{A}$ if and only if for all $r > 0$ we have $B(x, r) \cap A \neq \varnothing$.

(c) $x \in \partial A$ if and only if for all $r > 0$ we have $B(x, r) \cap A \neq \varnothing$ and $B(x, r) \cap (X \setminus A) \neq \varnothing$.

Proof. We use the criteria from the last three parts of Proposition 3.2.2.

If $x \in A^\circ$ then there is a $W \in \mathcal{O}(x)$ such that $W \subseteq A$. W is open and $x \in W$ by the definition of $\mathcal{O}(x)$. By Definition 1.8.1 this means there is an $r > 0$ such that $B(x, r) \subseteq W$. But then $B(x, r) \subseteq A$. If, conversely, there is an $r > 0$ such that $B(x, r) \subseteq A$ then there is a $W \in \mathcal{O}(x)$ such that $W \subseteq A$, namely $W = B(x, r)$. So $x \in A^\circ$.

If $x \in \overline{A}$ then for all $W \in \mathcal{O}(x)$ we have $W \cap A \neq \varnothing$. $B(x, r) \in \mathcal{O}(x)$ so

$$B(x, r) \cap A \neq \varnothing.$$

If, conversely, $B(x, r) \cap A \neq \varnothing$ for all $r > 0$ and $W \in \mathcal{O}(x)$ then there is, by Definition 1.8.1, an $r > 0$ such that $B(x, r) \subseteq W$. Therefore $B(x, r) \cap A \neq \varnothing$. But

$$B(x, r) \cap A \subseteq W \cap A$$

and supersets of non-empty sets are non-empty so $W \cap A \neq \varnothing$. So $x \in \overline{A}$.

If $x \in \partial A$ then for all $W \in \mathcal{O}(x)$ we have $W \cap A \neq \varnothing$ and

$$W \cap (X \setminus A) \neq \varnothing.$$

$B(x, r) \in \mathcal{O}(x)$ so $B(x, r) \cap A \neq \varnothing$ and so

$$B(x, r) \cap (X \setminus A) \neq \varnothing.$$

If, conversely, $B(x, r) \cap A \neq \varnothing$ and

$$B(x, r) \cap (X \setminus A) \neq \varnothing$$

for all $r > 0$ and $W \in \mathcal{O}(x)$ then there is, by Definition 1.8.1, an $r > 0$ such that $B(x, r) \subseteq W$. Therefore $B(x, r) \cap A \neq \varnothing$ and

$$B(x, r) \cap (X \setminus A) \neq \varnothing.$$

But $B(x,r) \cap A \subseteq W \cap A$ and

$$B(x,r) \cap (X \setminus A) \subseteq W \cap (X \setminus A)$$

and supersets of non-empty sets are non-empty so $W \cap A \neq \varnothing$ and

$$W \cap (X \setminus A) \neq \varnothing.$$

So $x \in \partial A$. □

Proposition 4.1.2. *Suppose (X, d) is a metric space, $\alpha\colon \mathbf{N} \to X$ is a sequence, and $z \in X$. Then $\lim_{n\to\infty} \alpha_n = z$ if and only if for all $\epsilon > 0$ there is an $m \in \mathbf{N}$ such that $\alpha_n \in B(z, \epsilon)$ for all $n \geq m$.*

This is a proposition, not a definition. Limits of sequences have already been defined as the special case $U = \mathbf{N}$ of Definition 1.18.1.

$$\lim_{n\to\infty} \alpha_n = z$$

therefore means that for all $Z \in \mathcal{O}(z)$ there is an $a \in \mathbf{R}$ such that if $n \in \mathbf{N}$ and $n \geq a$ then $\alpha_n \in Z$.

Proof. Suppose that

$$\lim_{n\to\infty} \alpha_n = z$$

If $r > 0$ then $B(z, r) \in \mathcal{O}(z)$ so there is an $a \in \mathbf{R}$ such that if $n \geq a$ then $\alpha_n \in B(z, r)$. Choose any non-negative integer m such that $m \geq a$. If $n \geq m$ then $n \geq a$ so $\alpha_n \in B(z, r)$. So for every $r > 0$ there is an $m \in \mathbf{N}$ such that if $n \geq m$ then $\alpha_n \in B(z, r)$.

Suppose, conversely, that $r > 0$ there is an $m \in \mathbf{N}$ such that if $n \geq m$ then $\alpha_n \in B(z, r)$. If $Z \in \mathcal{O}(z)$ then there is an $r > 0$ such that $B(z, r) \subseteq Z$. There is an $m \in \mathbf{N}$ such that if $n \geq m$ then $\alpha_n \in B(z, r)$ and hence $\alpha_n \in Z$. Let $a = m$. Then if $a \geq x$ then $\alpha_n \in B(z, r)$. So for every $Z \in \mathcal{O}(z)$ there is an $a \in \mathbf{R}$ such that if $a \geq x$ then $\alpha_n \in B(z, r)$. In other words,

$$\lim_{n\to\infty} \alpha_n = z.$$

□

Proposition 4.1.3. *Suppose (X, d) is a metric space and $A \in \wp(X)$. Then $z \in \overline{A}$ if and only if there is a sequence $\alpha\colon \mathbf{N} \to X$ such that $\alpha_n \in A$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} \alpha_n = z$.*

This holds also with nets in place of sequences by Propositions 3.3.1 and 3.3.2. Also, the "if" part was Proposition 3.3.3. None of those propositions required the topology to be metrisable.

Proof. As noted above, we only need to prove the "only if" part. Suppose that $z \in \overline{A}$. For each $n \in \mathbf{N}$ we have $B(z, 1/2^n) \in \mathcal{O}(z)$ so

$$B(z, 1/2^n) \cap A \neq \varnothing$$

by Proposition 4.1.1. Choose

$$\alpha_n \in B(z, 1/2^n) \cap A.$$

If $\epsilon > 0$ then there is an m such that $1/2^m < \epsilon$. Then also $1/2^n < \epsilon$ for all $n \geq m$. So $\alpha_n \in B(z, \epsilon)$ for all $n \geq m$. Since we've just shown that for any $\epsilon > 0$ there is an $m \in \mathbf{N}$ such that if $n \geq m$ then $\alpha_n \in B(z, \epsilon)$ it follows from Proposition 4.1.2 that

$$\lim_{n\to\infty} \alpha_n = z.$$

□

Proposition 4.1.4. *Suppose (X, d) is a metric space and $A \in \wp(X)$. Then A is dense if and only if for every $x \in X$ and $r > 0$ we have $B(x, r) \cap A \neq \varnothing$.*

Proof. By definition, A is dense if and only if $\overline{A} = X$, i.e. if and only if $x \in \overline{A}$ for all $x \in X$. By Proposition 4.1.1 $x \in \overline{A}$ if and only if for all $r > 0$ we have $B(x, r) \cap A \neq \varnothing$. □

Proposition 4.1.5. *Suppose (X, d) is a metric space and $A \in \wp(X)$. Then A is dense if and only if for every $z \in X$ there is a sequence $\alpha\colon \mathbf{N} \to X$ such that $\alpha_n \in A$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} \alpha_n = z$.*

The corresponding statement for nets was Proposition 3.4.3. It didn't require the topology to be metrisable. Every sequence is a net so the "if" part of this proposition follows from the "if" part of that proposition.

Proof. By the remarks above it suffices to prove the "only if" part of the statement. Suppose A is dense and $z \in X$. Then $z \in \overline{A}$. By Proposition 4.1.3 there is a sequence $\alpha\colon \mathbf{N} \to X$ such that $\alpha_n \in A$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} \alpha_n = z$. □

Proposition 4.1.6. *(a) Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f\colon X \to Y$ is a function. Then f is continuous at $x \in X$ if and only if for all $V \in \mathcal{O}(f(x))$ there is a $U \in \mathcal{O}(x)$ such that $U \subseteq f^*(V)$.*

(b) Suppose $(X, \mathcal{T}_X)$ is a topological space, (Y, d_Y) is a metric space and $f\colon X \to Y$ is a function. Then f is continuous at $x \in X$ if and only if for all $\epsilon > 0$ there is a $U \in \mathcal{O}(x)$ such that $U \subseteq f^*(B(f(x), \epsilon))$.

(c) Suppose (X, d_X) is a metric space, $(Y, \mathcal{T}_Y)$ is a topological space and $f\colon X \to Y$ is a function. Then f is continuous at $x \in X$ if and only if for all $V \in \mathcal{O}(f(x))$ there is a $\delta > 0$ such that $B(x, \delta) \subseteq f^*(V)$.

(d) Suppose (X, d_X) and (Y, d_Y) are metric spaces and $f\colon X \to Y$ is a function. Then f is continuous at $x \in X$ if and only if for all $\epsilon > 0$ there is a $\delta > 0$ such that $B(x, \delta) \subseteq f^*(B(f(x), \epsilon))$.

Proof. We prove these in turn.

(a) This is nearly the definition. Definition 3.6.1 says that f is continuous at x if and only if $f^*(W) \in \mathcal{N}(x)$ whenever $W \in \mathcal{N}(f(x))$. Suppose f is continuous at x and $V \in \mathcal{O}(f(x))$. Then $V \in \mathcal{N}(f(x))$ so $f^*(V) \in \mathcal{N}(x)$. By the definitions of neighbourhoods and open neighbourhoods there is then a $U \in \mathcal{O}(x)$ such that $U \subseteq f^*(V)$. So for all $V \in \mathcal{O}(f(x))$ there is a $U \in \mathcal{O}(x)$ such that $U \subseteq f^*(V)$.

Suppose, conversely, that for all $V \in \mathcal{O}(f(x))$ there is a $U \in \mathcal{O}(x)$ such that $U \subseteq f^*(V)$. Suppose $W \in \mathcal{N}(f(x))$. By the definitions of neighbourhoods and open neighbourhoods there is a $V \in \mathcal{O}(f(x))$ such that $V \subseteq W$. There is then a $U \in \mathcal{O}(x)$ such that $U \subseteq f^*(V)$. But $V \subseteq W$ implies $f^*(V) \subseteq f^*(W)$ so $U \subseteq f^*(W)$. Since $f^*(W)$ contains an open neighbourhood of x it is a neighbourhood of x, i.e. $f^*(W) \in \mathcal{N}(x)$. Therefore f is continuous at x.

(b) In this and the remaining parts we use the previous part as our criterion for continuity. Suppose f is continuous at x.

$$B(f(x), \epsilon) \in \mathcal{O}(f(x))$$

so there is a $U \in \mathcal{O}(x)$ such that

$$U \subseteq f^*(B(f(x), \epsilon)).$$

Suppose, conversely, that for every $\epsilon > 0$ there is a $U \in \mathcal{O}(x)$ such that

$$U \subseteq f^*(B(f(x), \epsilon)).$$

Suppose $V \in \mathcal{O}(f(x))$. By the definition of open sets in a metric space there is an $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq V$. There is a $U \in \mathcal{O}(x)$ such that

$$U \subseteq f^*(B(f(x), \epsilon)).$$

From $B(f(x), \epsilon) \subseteq V$ it follows that

$$f^*(B(f(x), \epsilon)) \subseteq f^*(V)$$

so $U \subseteq f^*(V)$. For every $V \in \mathcal{O}(f(x))$ we have a $U \in \mathcal{O}(x)$ such that $U \subseteq f^*(V)$, so f is continuous at x.

(c) Suppose f is continuous at x. Suppose $V \in \mathcal{O}(f(x))$. Then there is a $U \in \mathcal{O}(x)$ such that $U \subseteq f^*(V)$. By the definition of open neighbourhoods in a metric space there is then a $\delta > 0$ such that $B(x, \delta) \subseteq U$. So $B(x, \delta) \subseteq f^*(V)$.

Suppose, conversely, that for every $V \in \mathcal{O}(f(x))$ there is a $\delta > 0$ such that

$$B(x, \delta) \subseteq f^*(V).$$

$B(x, \delta) \in \mathcal{O}(x)$, so there is a $U \in \mathcal{O}(x)$ such that $U \subseteq f^*(V)$. Therefore f is continuous at x.

(d) Suppose f is continuous at x and $\epsilon > 0$.

$$B(f(x), \epsilon) \in \mathcal{O}(f(x))$$

so there is a $U \in \mathcal{O}(x)$ such that

$$U \subseteq f^*(B(f(x), \epsilon)).$$

There is a $\delta > 0$ such that $B(x, \delta) \subseteq U$ so

$$B(x, \delta) \subseteq f^*(B(f(x), \epsilon)).$$

Suppose, conversely, that for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$B(x, \delta) \subseteq f^*(B(f(x), \epsilon)).$$

Suppose $V \in \mathcal{O}(f(x))$. Then there is an $\epsilon > 0$ such that

$$B(f(x), \epsilon) \subseteq V.$$

Then

$$f^*(B(f(x), \epsilon)) \subseteq f^*(V).$$

There is a $\delta > 0$ such that

$$B(x, \delta) \subseteq f^*(B(f(x), \epsilon)).$$

Taking $U = B(x, \delta)$ we have $U \in \mathcal{O}(x)$ and $U \subseteq f^*(V)$. So f is continuous at x.

□

All of the various theorems which were proved for continuous functions between topological spaces continue to hold when one or both spaces are metric spaces, because metric spaces are topological spaces. For example, a function is continuous if and only if it is continuous at each point in its domain and the composition of continuous functions is continuous. It's only worth revisiting theorems on topological spaces when their statements can be either simplified or improved in the case of metric spaces, as below, where we can replace nets by sequences.

Proposition 4.1.7. *Suppose (X, d_X) is a metric space and (Y, d_Y) is a topological space. A function $f \colon X \to Y$ is continuous at x if and only if for every sequence $\alpha \colon \mathbf{N} \to X$ such that $\lim_{n\to\infty} \alpha_n = x$ we have $\lim_{n\to\infty} f(\alpha_n) = f(x)$.*

The corresponding statement for nets is Propositions 3.6.6 and 3.6.7. It doesn't require the topology on Y to be metrisable. Every sequence is a net, so the "only if" part follows from Proposition 3.6.6.

Proof. By the remarks above it suffices to prove the "if" part. Suppose f is not continuous at x. By Proposition 4.1.6c there is a $V \in \mathcal{T}_Y$ such that no $U \in \mathcal{O}(x)$ is a subset of $f^*(V)$. In particular, $B(x, 1/2^n)$ is not a subset of f for any $n \in \mathbf{N}$. So for each such n there is an α_n such that $\alpha_n \in B(x, 1/2^n)$ but $\alpha_n \notin V$. So

$$\lim_{n\to\infty} \alpha_n = x$$

but

$$\lim_{n\to\infty} f(\alpha_n) \neq f(x).$$

□

Suppose (X, d_X) is a metric space and $A \in \wp(X)$. There are at least two ways to get a topology on A. There is a topology $\mathcal{T}_X$ consisting of the open sets with respect to the metric d_X. We can then take the subspace topology on A. Or we can restrict the metric d_X from $X \times X$ to $A \times A$. The result, as shown in Lemma 1.6.3, is a metric. The set of open sets with respect to this metric is a topology on A. Fortunately these two topologies are the same.

Proposition 4.1.8. *Suppose (X, d_X) is a metric space and $A \in \wp(X)$. Let $\mathcal{T}_X$ be the topology of open sets with respect to the metric d_X and let $\mathcal{T}_1$ be the subspace topology on A. Let $d_A \colon A \times A \to \mathbf{R}$ be the restriction of d_X and let $\mathcal{T}_2$ be the topology of open sets with respect to the metric d_A. Then $\mathcal{T}_1 = \mathcal{T}_2$.*

Proof. We need to distinguish balls in X from balls in A. For purposes of this proof we'll therefore write

$$B_X(x, r) = \{y \in X \colon d_X(x, y) < r\}$$

for $x \in X$ and and

$$B_A(x, r) = \{y \in A \colon d_A(x, y) < r\}$$

for $x \in A$. The second of these could equally well be written with d_X in place of d_A because the two agree whenever both are defined. Suppose $U \in \mathcal{T}_1$. Then $U = A \cap V$ for some $V \in \mathcal{T}_X$. For each $x \in U$ we have $x \in V$ and V is an open set in $\mathcal{T}_X$ so there is an $r > 0$ such that $B_X(x, r) \subseteq V$. The fact that there is such an r for each $x \in A$ means that

$$U \subseteq \bigcup_{\substack{x \in U, r > 0 \\ B_X(x,r) \subseteq V}} B_X(x, r).$$

On the other hand, each element of the union is a subset of V so

$$\bigcup_{\substack{x \in U, r > 0 \\ B_X(x,r) \subseteq V}} B_X(x, r) \subseteq V.$$

Therefore

$$A \cap U \subseteq A \cap \left(\bigcup_{\substack{x \in U, r>0 \\ B_X(x,r) \subseteq V}} B_X(x,r) \right) \subseteq A \cap V.$$

But $A \cap U$ and $A \cap V$ are both U, so

$$\begin{aligned} U &= A \cap \left(\bigcup_{\substack{x \in U, r>0 \\ B_X(x,r) \subseteq V}} B_X(x,r) \right) \\ &= \bigcup_{\substack{x \in U, r>0 \\ B_X(x,r) \subseteq V}} \left(A \cap B_X(x,r) \right). \end{aligned}$$

But

$$\begin{aligned} A \cap B_X(x,r) &= \{y \in X \colon x \in A, d_X(x,y) < r\} \\ &= \{y \in A \colon d_X(x,y) < r\} = B_A(x,r). \end{aligned}$$

So

$$U = \bigcup_{\substack{x \in U, r>0 \\ B_X(x,r) \subseteq V}} B_A(x,r).$$

Each $B_A(x,r)$ is an open set in $\mathcal{T}_2$ so their union is in $\mathcal{T}_2$. In other words $U \in \mathcal{T}_2$. So

$$\mathcal{T}_1 \subseteq \mathcal{T}_2.$$

Suppose, conversely, that $U \in \mathcal{T}_2$. For each $x \in U$ there is an $r > 0$ such that $B_A(x,r) \subseteq U$, so

$$U = \bigcup_{\substack{x \in U, r>0 \\ B_A(x,r) \subseteq U}} B_A(x,r).$$

Using again the fact that $A \cap B_X(x,r) = B_A(x,r)$ we find

$$\begin{aligned} U &= \bigcup_{\substack{x \in U, r>0 \\ B_A(x,r) \subseteq U}} \left(A \cap B_X(x,r) \right) \\ &= A \cap \left(\bigcup_{\substack{x \in U, r>0 \\ B_A(x,r) \subseteq U}} B_X(x,r) \right). \end{aligned}$$

Let

$$V = \bigcup_{\substack{x \in U, r>0 \\ B_A(x,r) \subseteq U}} B_X(x,r).$$

Then $U = A \cap V$. Each element of the union is an element of $\mathcal{T}_X$ so $V \in \mathcal{T}_X$. Therefore $A \in \mathcal{T}_1$. So

$$\mathcal{T}_2 \subseteq \mathcal{T}_1.$$

Since we already have the reverse inclusion we get

$$\mathcal{T}_1 = \mathcal{T}_2.$$

□

4.2 Boundedness

Definition 4.2.1. A metric space (X,d) is called *bounded* if there is an $r > 0$ such that $d(x,y) \leq r$ for all $x, y \in X$. A subset $A \in \wp(X)$ is called bounded if it is bounded as a metric space with the restriction of d as its metric.

As simple examples, the intervals (a,b), $[a,b)$, $(a,b]$ and $[a,b]$ are all bounded, as is the empty interval. The intervals $[a,+\infty)$, $(a,+\infty)$, $(-\infty,b]$, $(-\infty,b)$ and $(-\infty,+\infty) = \mathbf{R}$ are not bounded.

The image or preimage of a bounded set under a continuous function needn't be bounded. The function $f \colon (0,+\infty) \to (0,+\infty)$ defined by $f(x) = 1/x$ provides counter-examples. $(0,1)$ is bounded but $(1,+\infty)$, which is both its image and preimage under f, is not bounded.

The following lemma gives various conditions equivalent to boundedness.

Lemma 4.2.2. *Suppose (X,d) is a metric space and X is non-empty. The following conditions are equivalent.*

(a) X is bounded, i.e. there is an $r > 0$ such that $d(x,y) \leq r$ for all $x, y \in X$.

(b) There is an $r > 0$ such that $d(x,y) < r$ for all $x, y \in X$.

(c) There is an $r > 0$ such that $X = B(x,r)$ for all $x \in X$.

(d) There is an $r > 0$ such that $X = \bar{B}(x,r)$ for all $x \in X$.

(e) For all $x \in X$ there is an $r > 0$ such that $X = \bar{B}(x,r)$.

(f) *For all $x \in X$ there is an $r > 0$ such that $X = B(x,r)$.*

(g) *There is an $x \in X$ and an $r > 0$ such that $X = B(x,r)$.*

(h) *There is an $x \in X$ and an $r > 0$ such that $X = \bar{B}(x,r)$.*

Proof. (a) implies (b): If $r > 0$ then $r/2 > 0$. X is bounded so $d(x,y) \leq r/2$ for each $x, y \in X$. If $d(x,y) \leq r/2$ then $d(x,y) < r$.

(b) implies (c): If $d(x,y) < r$ for all $y \in X$ then $X \subseteq B(x,r)$. The reverse inclusion is trivial so $X = B(x,r)$.

(c) implies (d): $B(x,r) \subseteq \bar{B}(x,r) \subseteq X$ so if $B(x,r) = X$ then $\bar{B}(x,r) = X$.

(d) implies (e): Since there's an $r > 0$ which works for all $x \in X$ there's one which works for each $x \in X$.

(e) implies (f): There is an s such that $X = \bar{B}(x,s)$. Let $r = 2s$. Then

$$\bar{(}X,s) \subseteq B(x,r) \subseteq X.$$

$\bar{B}(x,s) = X$ so $B(x,r) = X$.

(f) implies (g): X is non-empty, so if a statement holds for all x in X then there is an $x \in X$ for which it holds.

(g) implies (h): $B(x,r) \subseteq \bar{B}(x,r) \subseteq X$ so if $B(x,r) = X$ then $\bar{B}(x,r) = X$.

(h) implies (a): If $X = \bar{B}(x,r)$ and $y, z \in X$ then

$$d(y,z) \leq d(x,z) + d(y,z) \leq 2r.$$

Let $s = 2r$. Then $s > 0$ and $d(y,z) \leq s$ for all $y, z \in X$. So there is an $s > 0$ such that $d(y,z) \leq s$ for all $y, z \in X$. So X is bounded. ☐

The empty set is bounded.

There are corresponding statements for subsets.

Lemma 4.2.3. *Suppose (X,d) is a metric space and $A \in \wp(X)$ is non-empty. The following conditions are equivalent.*

(a) *A is bounded, i.e. t here is an $r > 0$ such that $d(x,y) \leq r$ for all $x, y \in A$.*

(b) *There is an $r > 0$ such that $d(x,y) < r$ for all $x, y \in A$.*

(c) *There is an $r > 0$ such that $A \subseteq B(x,r)$ for all $x \in A$.*

(d) *There is an $r > 0$ such that $A \subseteq \bar{B}(x,r)$ for all $x \in A$.*

(e) *For all $x \in A$ there is an $r > 0$ such that $A \subseteq \bar{B}(x,r)$.*

(f) *For all $x \in A$ there is an $r > 0$ such that $A \subseteq B(x,r)$.*

(g) *There is an $x \in A$ and an $r > 0$ such that $A \subseteq B(x,r)$.*

(h) *There is an $x \in A$ and an $r > 0$ such that $A \subseteq \bar{B}(x,r)$.*

The balls are meant to be balls in X, although the lemma would still hold if they were interpreted as balls in A.

Proof. This follows from the preceding lemma and the fact that $B_A(x,r) = A \cap B_X(x,r)$ and $\bar{B}_A(x,r) = A \cap \bar{B}_X(x,r)$. ☐

The following properties are straightforward to prove.

Proposition 4.2.4. *Any subset of a bounded set is bounded.*

Proof. Suppose (X,d) is a metric space and $A, B \in \wp(X)$. If $A \subseteq B$ and $d(x,y) \leq r$ for all $x, y \in B$ then $d(x,y) \leq r$ for all $x, y \in A$. ☐

Proposition 4.2.5. *Suppose (X,d) is a metric space. If $A_1, \ldots A_m$ are bounded subsets of X then $\bigcup_{j=1}^{m} A_j$ is bounded.*

Proof. Any empty A's don't contribute to the union and so can be ignored. We can therefore assume that all A's are non-empty. We can also assume that $m > 0$ for the same reason.

For each j there is, by Lemma 4.2.2, an $x_j \in A_j$ and an $r_j > 0$ such that such that

$$A_j \subseteq B(x_j, r_j).$$

Let

$$r = \max_{1 \leq j \leq m} \left(d(x_1, x_j) + r_j \right).$$

If $y \in A_j$ then

$$d(x_1, y) \le d(x_1, x_j) + d(x_j, y) < d(x_1, x_j) + r_j \le r$$

so

$$A_j \subseteq B(x_1, r)$$

and

$$\bigcup_{j=1}^{m} A_j \subseteq B(x_1, r).$$

By Proposition 4.2.2 it follows that $\bigcup_{j=1}^{m} A_j$ is bounded. □

There is also a notion of total boundedness, which is stronger than boundedness.

Definition 4.2.6. A metric space (X, d) is called *totally bounded* if for every $r > 0$ there is a finite $F \in \wp(X)$ such that

$$X = \bigcup_{x \in F} B(x, r).$$

Proposition 4.2.7. *If (X, d) is totally bounded then it is bounded.*

Proof. Each $B(x, r)$ is bounded so this follows from Proposition 4.2.5. □

For an example of a metric space which is bounded but not totally bounded, consider the discrete metric d on an infinite space X. X is bounded because $d(x, y) \le 1$ for all $x, y \in X$. It is not totally bounded because $1/2 > 0$ and

$$\bigcup_{x \in F} B(x, 1/2) = F.$$

Proposition 4.2.8. *Suppose (X, d) is a compact metric space. Then X is totally bounded.*

Proof. The set of balls $B(x, r)$ for $x \in X$ is, for each $r > 0$, an open cover of X and therefore has a finite subcover. □

4.3 Lipschitz and uniform continuity

The following two definitions introduce notions which are stronger than continuity for functions between metric spaces.

Definition 4.3.1. Suppose (X, d_X) and (Y, d_Y) are metric spaces. Then $f : X \to Y$ is called *Lipschitz continuous* if there is a $K \ge 0$ such that

$$d_Y(f(s), f(t)) \le K d_X(s, t)$$

for all $s, t \in X$.

Note that if X contains more than one point then the restriction to $K \ge 0$ is redundant, since there are then $s, t \in X$ such that $d_X(s, t) > 0$ while $d_Y(f(s), f(t)) \ge 0$. If K were less than zero we would have $K d_X(s, t) < d_Y(f(s), f(t))$.

Definition 4.3.2. Suppose (X, d_X) and (Y, d_Y) are metric spaces. Then $f : X \to Y$ is called *uniformly continuous* if for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $x \in X$

$$B(x, \delta) \subseteq f^*(B(f(x), \epsilon))$$

Proposition 4.3.3. *Suppose (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$ is a function.*

(a) If f is Lipschitz continuous then it is uniformly continuous.

(b) If f is uniformly continuous then it is continuous.

Proof. Suppose f is Lipschitz continuous. In other words, there is a $K \ge 0$ such that

$$d_Y(f(s), f(t)) \le K d_X(s, t)$$

for all $s, t \in X$. For $\epsilon > 0$ let

$$\delta = \frac{\epsilon}{K + 1}.$$

If

$$d_X(s, t) < \delta$$

then

$$d_Y(f(s), f(t)) \le K\delta < (K + 1)\delta = \epsilon.$$

So f is uniformly continuous.

Suppose f is uniformly continuous. In other words for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $x \in X$

$$B(x, \delta) \subseteq f^*(B(f(x), \epsilon)).$$

Then for every $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that

$$B(x, \delta) \subseteq f^*(B(f(x), \epsilon)).$$

So f is continuous. □

Neither of the converses hold in general. If $f\colon [0,1] \to [0,1]$ is defined by

$$f(x) = \sqrt{x}$$

then f is uniformly continuous but not Lipschitz continuous. It's uniformly continuous because if

$$\delta = \epsilon^2$$

then

$$B(x, \delta) \subseteq f^*(B(f(x), \epsilon))$$

for all $x \in [0, 1]$. To prove this suppose $y \in B(x, \delta)$ and let $w = \min(x, y)$ and $z = \max(x, y)$. Then $w \leq z$ and $f(w) \leq f(z)$. Then

$$\begin{aligned} d_{\mathbf{R}}(f(x), f(y))^2 &= d_{\mathbf{R}}(f(w), f(z))^2 = |f(z) - f(w)|^2 \\ &= (f(z) - f(w))^2 \\ &\leq (f(z) - f(w))(f(z) + f(w)) \\ &= f(z)^2 - f(w)^2 = z - w = |z - w| \\ &= d_{\mathbf{R}}(w, z) = d_{\mathbf{R}}(x, y) < \delta = \epsilon^2 \end{aligned}$$

Then

$$d_{\mathbf{R}}(f(x), f(y)) < \epsilon.$$

In other words, $f(y) \in B(f(x), \epsilon)$ or $y \in f^*(B(f(x), \epsilon))$. So

$$B(x, \delta) \subseteq f^*(B(f(x), \epsilon)).$$

It's not Lipschitz continuous because if there were a $K \geq 0$ such that

$$d_{\mathbf{R}}(f(s), f(t)) \leq K d_{\mathbf{R}}(s, t)$$

for all $s, t \in [0, 1]$ then taking $s = 0$ we would have

$$\sqrt{t} \leq Kt$$

for all $t \in [0, 1]$. Squaring this would give $t \leq K^2 t^2$. For $t \in (0, 1]$ we can divide by t to get $1 \leq K^2 t$ for all such t. This implies $K^2 > 0$ so we the have

$$t \geq \frac{1}{K^2}$$

for all $t \in (0, 1]$. But this isn't true, no matter which K we choose, so f is not Lipschitz continuous.

If $f\colon \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2$ then f is continuous but not uniformly continuous. Suppose that there were for each $\epsilon > 0$ a $\delta > 0$ such that

$$B(x, \delta) \subseteq f^*(B(f(x), \epsilon))$$

for all $x \in \mathbf{R}$.

$$x + \delta/2 \in B(x, \delta)$$

so

$$f(x + \delta/2) \in B(f(x), \epsilon).$$

So

$$\begin{aligned} d_{\mathbf{R}}(f(x + \delta/2), f(x)) &= |f(x + \delta/2) - f(x)| \\ &= \left| \delta x + \frac{1}{4}\delta^2 \right| < \epsilon. \end{aligned}$$

for all $x \in \mathbf{R}$. Taking $x = \frac{\epsilon}{\delta}$ gives

$$\left| \epsilon + \frac{1}{2}\delta^2 \right| < \epsilon,$$

which is false, so f is not uniformly continuous.

There is however one important case in which we can deduce uniform continuity from continuity.

Proposition 4.3.4. *Suppose (X, d_X) and (Y, d_Y) are metric spaces and X is compact. If $f\colon X \to Y$ is continuous then it is uniformly continuous.*

Proof. Suppose $\epsilon > 0$. Then $\epsilon/2 > 0$ as well, so for every $x \in X$ there is a $\delta > 0$ such that

$$B(x, \delta) \subseteq f^*(B(f(x), \epsilon/2)).$$

For any such δ we have $x \in B(x, \delta/2)$, so

$$X = \bigcup_{x \in X, \delta > 0, B(x,\delta) \subseteq f^*(B(f(x),\epsilon/2))} B(x, \delta/2).$$

The balls $B(x, \delta/2)$ for such x and δ therefore form an open cover of X. X is compact, so there is a finite subcover. In other words, there are $x_1, \ldots, x_m \in X$ and $\delta_1, \ldots, \delta_m > 0$ such that

$$B(x_j, \delta_j) \subseteq f^*(B(f(x_j), \epsilon/2))$$

and

$$X = \bigcup_{j=1}^{m} B(x_j, \delta_j/2).$$

Let

$$\delta = \frac{1}{2} \min_{1 \leq j \leq m} \delta_j.$$

Suppose $x \in X$ and $y \in B(x, \delta)$, i.e.

$$d(x, y) < \delta.$$

$x \in B(x_j, \delta_j/2)$ for some j because these sets cover X. In other words,

$$d(x, x_j) < \delta_j/2.$$

$\delta \leq \delta_j/2$ so

$$d(x, y) < \delta_j/2.$$

It follows that

$$d(x_j, y) < \delta_j.$$

From this and $d(x, x_j) < \delta_j/2 < \delta$ we get

$$d(f(x_j), f(y)) < \epsilon/2$$

and

$$d(f(x_j), f(x)) < \epsilon/2.$$

Therefore

$$d(f(x), f(y)) < \epsilon.$$

So $f(y) \in B(f(x), \epsilon)$ or, equivalently,

$$y \in f^*(B(f(x), \epsilon)).$$

For all $x \in X$ we've shown that if $y \in B(x, \delta)$ then

$$y \in f^*(B(f(x), \epsilon)).$$

In other words,

$$B(x, \delta) \subseteq f^*(B(f(x), \epsilon)).$$

So f is uniformly continuous. □

This can't be strengthened to obtain Lipschitz continuity, as the example of $f(x) = \sqrt{x}$ on $[0, 1]$ shows.

The following properties are all straightforward consequences of the definitions.

Proposition 4.3.5. *Suppose (X, d_X) and (Y, d_Y) are metric spaces and $A \in \wp(X)$. Let d_A be the restriction of d_X to $A \times A$. Suppose $f\colon X \to Y$ is Lipschitz continuous and g is the restriction of f to A. Then g is Lipschitz continuous.*

Proof. If

$$d_Y(f(s), f(t)) \leq K d_X(s, t)$$

for all $s, t \in X$ then

$$d_Y(f(s), f(t)) \leq K d_X(s, t)$$

for all $s, t \in A$ and therefore

$$d_Y(g(s), g(t)) \leq K d_A(s, t)$$

since $g(s) = f(s)$, $g(t) = f(x)$ and

$$d_A(s, t) = d_X(s, t)$$

for all such s, t. □

Proposition 4.3.6. *Suppose (X, d_X) and (Y, d_Y) are metric spaces and $A \in \wp(X)$. Let d_A be the restriction of d_X to $A \times A$. Suppose $f\colon X \to Y$ is uniformly continuous and g is the restriction of f to A. Then g is uniformly continuous.*

Proof. For each $\epsilon > 0$ there is a $\delta > 0$ such that

$$B_X(x, \delta) \subseteq f^*(B_Y(f(x), \epsilon))$$

for all $x \in X$.

$$g^*(B_Y(f(x), \epsilon)) = A \cap f^*(B_Y(f(x), \epsilon))$$

since

$$g(w) \in B_Y(f(x), \epsilon)$$

if and only if $w \in A$ and

$$f(w) \in B_Y(f(x), \epsilon).$$

If $x \in A$ then we can write this as

$$g^*(B_Y(g(x), \epsilon)) = A \cap f^*(B_Y(f(x), \epsilon)).$$

Also,

$$B_A(x, \delta) = A \cap B_X(x, \delta).$$

f is uniformly continuous, so for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$B_X(x, \delta) \subseteq f^*(B_Y(f(x), \epsilon))$$

for all $x \in X$.

$$A \cap B_X(x, \delta) \subseteq A \cap f^*(B_Y(f(x), \epsilon))$$

or

$$B_A(x, \delta) \subseteq g^*(B_Y(g(x), \epsilon))$$

for all $x \in A$. In other words, g is absolutely continuous. □

Proposition 4.3.7. *Suppose (X, d_X), (Y, d_Y) and (Z, d_Z) are metric spaces and $f: X \to Y$ and $g: Y \to Z$ are Lipschitz continuous. Then $g \circ f$ is Lipschitz continuous.*

Proof. g is Lipschitz continuous so there is an $L \geq 0$ such that

$$d_Z(g(p), g(q)) \leq L d_Y(p, q)$$

for all $p, q \in Y$. This holds in particular for $p = f(s)$ and $q = f(t)$ where $s, t \in X$, so

$$d_Z(g(f(s)), g(f(s))) \leq L d_Y(f(s), f(t))$$

for all $s, t \in X$. f is uniformly continuous, so there is a $K \geq 0$ such that

$$d_Y(f(s), f(t)) \leq K d_X(s, t)$$

for all $s, t \in X$. Therefore

$$d_Z(g(f(s)), g(f(s))) \leq KL d_X(s, t)$$

for all $s, t \in X$. There is therefore an $M \geq 0$, namely $M = KL$, such that

$$d_Z((g \circ f)(s), (g \circ f)(s)) \leq M d_X(s, t)$$

for all $s, t \in X$. In other words $g \circ f$ is Lipschitz continuous. □

Proposition 4.3.8. *Suppose (X, d_X), (Y, d_Y) and (Z, d_Z) are metric spaces and $f: X \to Y$ and $g: Y \to Z$ are uniformly continuous. Then $g \circ f$ is uniformly continuous.*

Proof. Suppose $\epsilon > 0$. g is uniformly continuous, so there is a $\theta > 0$ such that for each $y \in Y$

$$B_Y(y, \theta) \subseteq g^*(B(g(y), \epsilon)).$$

This holds in particular for $y = f(x)$ where $x \in X$, so

$$B_Y(f(x), \theta) \subseteq g^*(B(g(f(x)), \epsilon))$$

for all $x \in X$. It follows that

$$\begin{aligned} f^*(B_Y(f(x), \theta)) &\subseteq f^*(g^*(B(g(f(x)), \epsilon))) \\ &= (f^* \circ g^*)(B((g \circ f)(x), \epsilon)) \\ &= (g \circ f)^*(B((g \circ f)(x), \epsilon)) \end{aligned}$$

for all $x \in X$. f is uniformly continuous and $\theta > 0$ so there is a $\delta > 0$ such that

$$B_X(x, \delta) \subseteq f^*(B_Y(f(x), \theta))$$

for all $x \in X$. So

$$B_X(x, \delta) \subseteq (g \circ f)^*(B((g \circ f)(x), \epsilon))$$

for all $x \in X$. So for each $\epsilon > 0$ there is a $\delta > 0$ such that the inclusion above holds for all $x \in X$. In other words, $g \circ f$ is uniformly continuous. □

Proposition 4.3.9. *Suppose (X, d_X) and (Y, d_Y) are metric spaces, (X, d_X) is bounded and $f: X \to Y$ is Lipschitz continuous then $f_*(X)$ is bounded.*

Proof. (X, d_X) is bounded so there is an $r > 0$ such that

$$d_X(s, t) \leq r$$

for all $s, t \in X$. f is Lipschitz continuous so there is a $K \geq 0$ such that

$$d_Y(f(s), f(t)) \leq K d_X(s, t)$$

for all $s, t \in X$. Suppose $w, z \in f_*(X)$, i.e. that $w = f(s)$ and $z = f(t)$ for some $s, t \in X$. For these s and t we have

$$d_Y(w, z) \leq K d_X(s, t).$$

Let
$$q = Kr + 1.$$
Then
$$d_Y(w, z) < q$$
This holds for all $w, z \in f_*(X)$. Since there is $q > 0$ such that $d_Y(w, z) < q$ for all $w, z \in f_*(X)$ we conclude that $f_*(X)$ is bounded. □

This proposition wouldn't be true if we replaced Lipschitz continuity with uniform continuity. Consider the inclusion $f\colon \mathbf{Z} \to \mathbf{R}$, with the discrete metric on Z and the usual metric on $\mathbf{R}$. This is uniformly continuous because
$$B_{\mathbf{Z}}(n, 1/2) = \{n\} \subseteq B_{\mathbf{R}}(n, \epsilon) = B_{\mathbf{R}}(f(n), \epsilon)$$
for all $\epsilon > 0$. $\mathbf{Z}$ is, like any set, bounded with respect to the discrete metric. $\mathbf{R}$ is, of course, not bounded.

Proposition 4.3.10. *Suppose (X, d_X) and (Y, d_Y) are metric spaces, (X, d_X) is totally bounded and $f\colon X \to Y$ is uniformly continuous then $f_*(X)$ is totally bounded.*

Proof. Suppose $r > 0$. f is uniformly continuous so there is a $\delta > 0$ such that for all $x \in X$
$$B_X(x, \delta) \subseteq f^*(B_Y(f(x), r)).$$
(X, d_X) is totally bounded so there are $x_1, \ldots, x_m \in X$ such that
$$X = \bigcup_{j=1}^{m} B_X(x_j, \delta).$$
Then
$$f_*(X) = f_*\left(\bigcup_{j=1}^{m} B_X(x_j, \delta)\right) = \bigcup_{j=1}^{m} f_*(B_X(x_j, \delta)).$$
From
$$B_X(x_j, \delta) \subseteq f^*(B_Y(f(x_j), r)).$$
it follows that
$$f_*(B_X(x_j, \delta)) \subseteq B_Y(f(x_j), r).$$
Therefore
$$f_*(X) \subseteq \bigcup_{j=1}^{m} B_Y(y_j, r)$$
where $y_j = f(x_j)$. So for any $r > 0$ there are $y_1, \ldots, y_m \in Y$ such that
$$f_*(X) \subseteq \bigcup_{j=1}^{m} B_Y(y_j, r).$$
In other words, $f_*(X)$ is totally bounded. □

This proposition would not be true if we replaced uniform continuity by continuity.

One important source of Lipschitz functions is the following proposition.

Proposition 4.3.11. *Suppose (X, d) is a metric space and $A \in \wp(X)$ is non-empty. Define $r\colon X \to \mathbf{R}$ by*
$$r(x) = \inf_{y \in A} d(x, y).$$
Then $r(x) \geq 0$ for all x, $r(x) = 0$ if and only if $x \in \overline{A}$ and r is Lipschitz continuous.

It then follows from Proposition 4.3.3 that r is uniformly continuous and continuous.

Proof. $d(x, y) \geq 0$ for all $y \in A$ and there is at least one $y \in A$ so the infimum exists and is non-negative.

Suppose $x \in \overline{A}$. By Proposition 4.1.1 for each $\delta > 0$ we have
$$A \cap B(x, \delta) \neq \varnothing.$$
In other words, there is a $y \in A$ such that $d(x, y) < \delta$ and therefore $r(x) < \delta$. Since this holds for all $\delta > 0$ we have $r(x) \leq 0$. Combined with the inequality $r(x) \geq 0$ which we already have this gives $r(x) = 0$.

Suppose, conversely, that $r(x) = 0$. For each $\delta > 0$ we have $r(x) < \delta$ and hence δ is not a lower bound for $d(x, y)$. So there is a $y \in A$ with $d(x, y) < \delta$. Therefore $A \cap B(x, \delta) \neq \varnothing$. This holds for all $\delta > 0$ so $x \in \overline{A}$ by Proposition 4.1.1.

Suppose $s, t \in X$ and $\delta > 0$.
$$r(t) + \delta > \inf_{y \in A} d(t, y)$$
so $r(t) + \delta$ is not a lower bound $d(t, y)$ there is therefore a $y \in A$ with
$$d(t, y) < r(t) + \delta.$$

But then

$$d(s,y) \le d(s,t) + d(t,y) < d(s,t) + r(t) + \delta.$$

$y \in A$ so

$$r(s) < d(s,t) + r(t) + \delta.$$

This holds for all $\delta > 0$ so

$$r(s) \le d(s,t) + r(t)$$

or

$$r(s) - r(t) \le d(s,t).$$

The same argument with the roles of s and t reversed gives

$$r(t) - r(s) \le d(t,s) = d(s,t).$$

So

$$\begin{aligned} d_{\mathbf{R}}(r(s), r(t)) &= |r(s) - r(t)| \\ &= \max(r(s) - r(t), r(t) - r(s)) \\ &\le d(s,t). \end{aligned}$$

This is the Lipschitz condition with $K = 1$. $\square$

The special case $X = \mathbf{R}$, $A = \{0\}$ shows that the absolute value function on $\mathbf{R}$ is Lipschitz continuous. This can, of course, also be proved directly.

The following corollary to Proposition 4.3.11 will be needed in the proof that metric spaces are normal.

Corollary 4.3.12. *Suppose (X, d) is a metric space, $A, B \in \wp(X)$ are closed and non-empty, and $A \cap B = \varnothing$. Then there is a continuous function $f \colon X \to [0,1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.*

Proof. Let

$$r_A(x) = \inf_{y \in A} d(x,y),$$

$$r_B(x) = \inf_{y \in B} d(x,y),$$

and

$$f(x) = \frac{r_A(x)}{r_A(x) + r_B(x)}.$$

r_A and r_B are continuous by Proposition 4.3.11 and $r_A(x) = 0$ for $x \in A$ and $r_B(x) = 0$ for $x \in B$. $r_A(x) + r_B(x) = 0$ if and only if

$$x \in \overline{A} \cap \overline{B} = A \cap B,$$

but this set is empty, so

$$r_A(x) + r_B(x) > 0.$$

f is therefore continuous and $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$. $\square$

We've already seen that compact Hausdorff spaces are normal. The following proposition gives us many more normal spaces.

Proposition 4.3.13. *Metric spaces are normal.*

Proof. Suppose (X, d) is a metric space, $A, B \in \wp(X)$ are closed, and $A \cap B = \varnothing$. If A is empty let $V = \varnothing$ and $W = X$. Then V and W are open, $V \subseteq A$, $B \subseteq W$ and $V \cap W = \varnothing$. If A is non-empty but B is empty then let $V = X$ and $W = \varnothing$. Again, V and W are open, $V \subseteq A$, $B \subseteq W$ and $V \cap W = \varnothing$. If A and B are non-empty then Corollary 4.3.12 guarantees the existence of a continuous function $f \colon X \to [0,1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$. Let

$$V = f^*([0, 1/3)), \qquad W = f^*((2/3, 1]).$$

$[0, 1/3)$ and $(2/3, 1]$ are open in the subspace topology on $[0,1]$ so V and W are open. $A \subseteq V$, $B \subseteq W$ and $V \cap W = \varnothing$, so (X, d) is normal. $\square$

4.4 Filters and convergence

A number of useful constructions are most easily described in terms of filters. These were discussed briefly in Chapter 1 but we will need more.

Proposition 4.4.1. *Suppose X and Y are sets, $f \colon X \to Y$ is a function and $\mathcal{F}$ is a filter on X. Then $f^{**}(\mathcal{F})$ is a filter on Y.*

Proof. We check conditions 1.15.1a through 1.15.1d.

$X = f^*(Y)$ and $X \in \mathcal{F}$ so $Y \in f^{**}(\mathcal{F})$. This establishes 1.15.1a. If $\varnothing \in f^{**}(\mathcal{F})$ then $f^*(\varnothing) \in \mathcal{F}$.

$f^*(\varnothing) = \varnothing$ and $\varnothing \notin \mathcal{F}$ so $f^*(\varnothing) \notin \mathcal{F}$ and hence $\varnothing \notin f^{**}(\mathcal{F})$. This establishes 1.15.1b.

If $A, B \in f^{**}(\mathcal{F})$ then $f^*(A) \in \mathcal{F}$ and $f^*(B) \in \mathcal{F}$. By Lemma 1.15.4 then

$$f^*(A) \cap f^*(B) \in \mathcal{F}.$$

But

$$f^*(A) \cap f^*(B) = f^*(A \cap B).$$

So $f^*(C) \in \mathcal{F}$, where $C = A \cap B$. But then $C \in f^{**}(\mathcal{F})$ and $C \subseteq A \cap B$. This establishes 1.15.1c.

Suppose $A \in f^{**}(\mathcal{F})$ and $A \subseteq B$. Then $f^*(A) \in \mathcal{F}$ and $f^*(A) \subseteq f^*(B)$. $\mathcal{F}$ is upward closed so $f^*(B) \in \mathcal{F}$. Therefore $B \in f^{**}(\mathcal{F})$. This establishes 1.15.1d. □

Proposition 4.4.2. *Suppose $\mathcal{F}_0$, $\mathcal{F}_1$, ... are a sequence of filters on a set X such that*

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots .$$

Let

$$\mathcal{G} = \bigcup_{j=0}^{\infty} \mathcal{F}_j.$$

Then $\mathcal{G}$ is a filter on X.

Proof. We check that $\mathcal{G}$ satisfies the four conditions 1.15.1a through 1.15.1d. Each $\mathcal{F}_j$ is non-empty so their union is also non-empty. The empty set belongs to none of them, so it also doesn't belong to their union.

If $A \in \mathcal{G}$ and $B \in \mathcal{G}$ then $A \in \mathcal{F}_j$ and $B \in \mathcal{F}_k$ for some j and k. Let $m = \max(j, k)$. Then $\mathcal{F}_j \subseteq \mathcal{F}_m$ and $\mathcal{F}_k \subseteq \mathcal{F}_m$. $\mathcal{F}_m$ is a filter so there is a $X \in \mathcal{F}_m$ such that $C \subseteq A \cap B$. But $\mathcal{F}_m \subseteq \mathcal{G}$, so there is a $C \in \mathcal{G}$ such that $C \subseteq A \cap B$.

Suppose $A \in \mathcal{G}$ and $A \subseteq B \subseteq X$. There is a j such that $A \in \mathcal{F}_j$. $\mathcal{F}_j$ is a filter so $B \in \mathcal{F}_j$. But $\mathcal{F}_j \subseteq \mathcal{G}$ so $B \in \mathcal{G}$. □

The following definitions will be used to relate nets to filters.

Definition 4.4.3. Suppose $(D, \preccurlyeq)$ is a non-empty directed set. Define a function $\tau \colon D \to \wp(D)$ by

$$\tau(a) = \{b \in D \colon a \preccurlyeq b\}.$$

The *eventuality filter* of $(D, \preccurlyeq)$ is a the upward closure of $\tau_*(D)$. If X is a set and $f \colon D \to X$ is a net then the *tail filter* of f is the upward closure of $f^{**}(\mathcal{F})$ where $\mathcal{F}$ is the eventuality filter of $(D, \preccurlyeq)$.

The name "eventuality filter" wouldn't make much sense if it weren't a filter. Fortunately it is. Note that $(\tau_*(D), \supseteq)$ is a directed set by Proposition 1.14.4c and so $\tau_*(D)$ is a prefilter by Proposition 1.15.2. The eventuality filter is then a filter by Proposition 1.15.7d. Similarly, the tail filter is a filter by the following proposition.

Proposition 4.4.4. *Suppose $(D, \preccurlyeq)$ is a non-empty directed set, X is a set and $f \colon D \to X$ is a net. Let $\mathcal{G}$ be the tail filter of f. Then $W \in \mathcal{G}$ if and only if there is an $a \in D$ such that $f(b) \in W$ for all $b \in D$ such that $a \preccurlyeq b$.*

Proof. Let $\mathcal{F}$ be the eventuality filter of $(D, \preccurlyeq)$, so $\mathcal{G} = f^{**}(\mathcal{F})$.

Each of the following statements is equivalent to the preceding one:

(a) $W \in \mathcal{G}$.

(b) $f^*(W) \in \mathcal{F}$.

(c) There is an $a \in D$ such that $\tau(a) \subseteq f^*(W)$.

(d) There is an $a \in \mathcal{D}$ such that if $b \in \tau(a)$ then $b \in f^*(W)$.

(e) There is an $a \in \mathcal{D}$ such that if $a \preccurlyeq b$ then $f(b) \in W$.

□

The following definition gives a different point of view on limits.

Definition 4.4.5. Suppose $(X, \mathcal{T})$ is a topological space. A filter $\mathcal{F}$ on X is said to *converge* to $z \in X$ if $\mathcal{N}(z) \subseteq \mathcal{F}$. A prefilter $\mathcal{E}$ is said to converge to z if its upward closure converges to z. Suppose $(D, \preccurlyeq)$ is a directed set and $f \colon D \to X$ is a net. Then f is said to converge to $z \in X$ if its tail filter converges to z. A filter, prefilter or net on X is said to be *convergent* if there is some $z \in X$ such that it converges to z.

The definition of convergence may look unfamiliar, but convergence of nets is the same as the notion of limits we met earlier. In the case where the directed set is $(\mathbf{N}, \leq)$, i.e. when the net is a sequence, this is the usual notion of limits of sequences.

Proposition 4.4.6. *Suppose $(X, \mathcal{T})$ is a topological space, $(D, \preccurlyeq)$ is a directed set and $f\colon D \to X$ is a net. Then f converges to $z \in X$ if and only if $\lim f = z$ in the sense of Definition 1.18.3.*

Proof. Suppose f converges to z. Then $\mathcal{N}(z) \subseteq \mathcal{F}$, where $\mathcal{F}$ is the tail filter of f. $\mathcal{O}(z) \subseteq \mathcal{N}(z)$ so $\mathcal{O}(z) \subseteq \mathcal{F}$. In other words, if $U \in \mathcal{O}(z)$ then $U \in \mathcal{F}$. By Proposition 4.4.4 this is equivalent to the statement that if $U \in \mathcal{O}(z)$ then there is an $a \in D$ such that $f(b) \in U$ for all $b \in D$ such that $a \preccurlyeq b$. In other words, $\lim f = z$.

Suppose, conversely, that $\lim f = z$, i.e. that if $U \in \mathcal{O}(z)$ then there is an $a \in D$ such that if $U \in \mathcal{O}(z)$ then there is an $a \in D$ such that $f(b) \in U$ for all $b \in D$ such that $a \preccurlyeq b$. By Proposition 4.4.4 this is equivalent to the statement that $\mathcal{O}(z) \subseteq \mathcal{F}$. But $\mathcal{F}$ is a filter and $\mathcal{N}(z)$ is the upward closure of $\mathcal{O}(z)$ so it follows from Proposition 1.15.7e that $\mathcal{O}(z) \subseteq \mathcal{F}$. In other words, $\mathcal{F}$ converges to z, so f converges to z. □

We now have a long series of lemmas, most of which will be needed in the next section.

Lemma 4.4.7. *Suppose X is a set and $\mathcal{F}$ is a filter on X. If $\mathcal{B} \subset \mathcal{F}$ is finite then*

$$\bigcap_{B \in \mathcal{B}} B \neq \varnothing.$$

Proof. The intersection of a pair of elements of $\mathcal{F}$ is an element of $\mathcal{F}$ by condition 1.15.1c from the definition of a filter. We can extend this to the intersection of finitely many elements by induction, so

$$\bigcap_{B \in \mathcal{B}} B \in \mathcal{F}.$$

But $\varnothing \notin \mathcal{F}$ by the condition 1.15.1b from the definition. □

Lemma 4.4.8. *Suppose (X, d) is a metric space, $z \in X$ and $\mathcal{F}$ is a filter on X. Then $\mathcal{F}$ converges to z if and only if there is an $n > 0$ such that*

$$B(z, nr) \in \mathcal{F}$$

for all $r > 0$.

Proof. If $\mathcal{N}(z) \subseteq \mathcal{F}$ then

$$B(z, nr) \in \mathcal{F}$$

because $B(z, nr) \in \mathcal{N}(z)$. This proves the "only if" part of the lemma.

To prove the "if" part, suppose that there is an $n > 0$ such that

$$B(z, nr) \in \mathcal{F}$$

for all $r > 0$. If $V \in \mathcal{N}(z)$ then there is a $\delta > 0$ such that $B(z, \delta) \subseteq V$. Let $r = \delta/n$. Then

$$B(z, nr) \subseteq V.$$

It follows from 1.15.1d that

$$V \in \mathcal{F}.$$

This holds for all $V \in \mathcal{N}(z)$ so

$$\mathcal{N}(z) \subseteq \mathcal{F}.$$

In other words, $\mathcal{F}$ converges to z. □

Lemma 4.4.9. *Suppose (X, d) is a metric space, $\mathcal{F}$ and $\mathcal{G}$ are filters on X, $x, y \in X$ and $r > 0$. If $B(x, r) \in \mathcal{F}$ and $B(y, r) \in \mathcal{F} \cap \mathcal{G}$ then $B(x, 3r) \in \mathcal{G}$.*

Proof. $B(x, r) \in \mathcal{F}$ and $B(y, r) \in \mathcal{F}$ so

$$B(x, r) \cap B(y, r) \neq \varnothing$$

by Lemma 4.4.7. There is therefore a

$$z \in B(x, r) \cap B(y, r).$$

Suppose $w \in B(y, r)$. Then

$$d(x, w) \leq d(x, z) + d(z, y) + d(y, w) < r + r + r$$

so

$$w \in B(x, 3r).$$

Since this holds for all $w \in B(y, r)$ we have

$$B(y, r) \subseteq B(x, 3r).$$

But $B(y, r) \in \mathcal{G}$ so

$$B(x, 3r) \in \mathcal{G}$$

by 1.15.1d. □

Lemma 4.4.10. *Suppose $\mathcal{F}$ is a filter on a set X and $U_1, \ldots, U_m \in \wp(X)$ are such that*

$$X = \bigcup_{j=1}^{m} U_j.$$

Then there a j such that for all $V \in \mathcal{F}$ we have

$$U_j \cap V \neq \varnothing$$

Proof. Otherwise there is for each j a V such that $U_j \cap V = \varnothing$. This V will in general depend on j so call it V_j. Then

$$U_j \cap V_j = \varnothing.$$

Let

$$W = \bigcap_{j=1}^{m} V_j.$$

Then $V_j \subseteq W$ for each j so

$$U_j \cap W = \varnothing.$$

Therefore

$$W = X \cap W = \left(\bigcup_{j=1}^{n} U_j\right) \cap W = \bigcup_{j=1}^{n} U_j \cap W = \varnothing.$$

But

$$W \neq \varnothing$$

by Lemma 4.4.7. □

Lemma 4.4.11. *Suppose X is a set, $U \in \wp(X)$, and $\mathcal{F}$ is a filter on X. Let $\mathcal{G}$ be the set of $W \in \wp(X)$ such that there is a $V \in \mathcal{F}$ with $U \cap V \subseteq W$. Then $\mathcal{G}$ is a filter if and only if*

$$U \cap V \neq \varnothing$$

for all $V \in \mathcal{F}$.

Proof. $\mathcal{F} \neq \varnothing$ so there is a $V \in \mathcal{F}$. Then $U \cap V \subseteq U \cap V$ so $U \cap V \in \mathcal{G}$ and hence

$$\mathcal{G} \neq \varnothing.$$

So $\mathcal{G}$ satisfies the condition 1.15.1a.

Suppose $W_1, W_2 \in \mathcal{G}$. Then there are $V_1, V_2 \in \mathcal{F}$ such that $U \cap V_1 \subseteq W_1$ and $U \cap V_2 \subseteq W_2$. $\mathcal{F}$ is a filter so there is a $T \in \mathcal{F}$ such that $T \subseteq V_1 \cap V_2$. But then

$$U \cap T \subseteq U \cap (V_1 \cap V_2) = (U \cap V_1) \cap (U \cap V_2) \subseteq W_1 \cap W_2.$$

Since there is a T such that

$$U \cap T \subseteq W_1 \cap W_2$$

we conclude that

$$W_1 \cap W_2 \in \mathcal{G}.$$

So $\mathcal{G}$ satisfies the condition 1.15.1c.

If $W \in \mathcal{G}$ and $W \subseteq Z$ then there is a $V \in \mathcal{F}$ such that $U \cap V \subseteq W$ and hence $U \cap V \subseteq Z$. So $Z \in \mathcal{G}$. So if $W \in \mathcal{G}$ and $W \subseteq Z$ then $Z \in \mathcal{G}$. So $\mathcal{G}$ satisfies the condition 1.15.1c.

The only remaining condition is 1.15.1b. $\mathcal{G}$ is therefore a filter if and only if 1.15.1b is satisfied. If 1.15.1b is satisfied and $V \in \mathcal{F}$ then $U \cap V \in \mathcal{G}$ and hence $U \cap V \neq \varnothing$. Conversely, if $U \cap V \neq \varnothing$ then every $W \in \mathcal{G}$ contains a non-empty set and so is non-empty. Therefore $\mathcal{G}$ satisfies the condition 1.15.1b. □

Lemma 4.4.12. *Suppose $\mathcal{F}$ is a filter on a set X and $U_1, \ldots, U_m \in \wp(X)$ are such that*

$$X = \bigcup_{j=1}^{m} U_j.$$

Then there is a filter $\mathcal{G}$ and a j such that $\mathcal{F} \subseteq \mathcal{G}$ and $U_j \in \mathcal{G}$.

Proof. By Lemma 4.4.10 there is a j such that $U_j \cap V \neq \varnothing$ for all $V \in \mathcal{F}$. Let $\mathcal{G}$ be the set of $W \in \wp(X)$ such that there is a $V \in \mathcal{F}$ with $U \cap V \subseteq W$. Then $\mathcal{G}$ is a filter by Lemma 4.4.11. $\mathcal{F}$ is a filter and hence non-empty so there is a $V \in \mathcal{F}$. Then $U \cap V \in \mathcal{G}$. But $U \cap V \subseteq U$ and $\mathcal{G}$ is upward closed so $U \in \mathcal{G}$. □

Proposition 4.4.13. *Suppose $(X, \mathcal{T})$ is a topological space. Suppose that for every filter $\mathcal{F}$ on X there is a convergent filter $\mathcal{G}$ such that $\mathcal{F} \subseteq \mathcal{G}$. Then $(X, \mathcal{T})$ is compact.*

Proof. Suppose $\mathcal{C}$ is a non-empty collection of closed subsets of X such that any intersection of finitely many elements of $\mathcal{C}$ is non-empty. Let $\mathcal{E}$ be the set of such finite intersections. It's non-empty because $\mathcal{C} \subseteq \mathcal{E}$ and $\mathcal{C} \neq \varnothing$. Also $\varnothing \notin \mathcal{C}$ by the finite intersection assumption above. The intersection of two elements of $\mathcal{E}$ is again an element of $\mathcal{E}$. Therefore $\mathcal{E}$ is a prefilter. Let $\mathcal{F}$ be its upward closure. Then $\mathcal{F}$ is a filter on X so there is, by the hypotheses of the proposition, a convergent filter $\mathcal{G}$ on X such that $\mathcal{F} \subseteq \mathcal{G}$. $\mathcal{G}$ is convergent so there is a $z \in X$ such that $\mathcal{N}(z) \subseteq \mathcal{G}$. We have

$$\mathcal{C} \subseteq \mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{G}$$

so $\mathcal{C} \subseteq \mathcal{G}$. Suppose $W \in \mathcal{C}$ and $U \in \mathcal{N}(z)$. Then U and W are both elements of $\mathcal{G}$ so

$$U \cap W \neq \varnothing$$

by Lemma 4.4.7. For a given $W \in \mathcal{C}$ this holds for all $U \in \mathcal{N}(z)$. By Proposition 3.2.2l then $z \in \overline{W}$. But W is closed so $z \in W$. This holds for all $W \in \mathcal{C}$ so

$$z \in \bigcap_{W \in \mathcal{C}} W.$$

Therefore

$$\bigcap_{W \in \mathcal{C}} W \neq \varnothing.$$

We've just seen that if $\mathcal{C}$ is a non-empty set of closed subsets of X such that the intersection of finitely many elements of $\mathcal{C}$ is non-empty then the intersection of all of them is non-empty. By Proposition 3.12.4 the space $(X, \mathcal{T})$ is compact. $\square$

4.5 Cauchy filters and completeness

The following definition will play a large role in the remainder of this chapter.

Definition 4.5.1. Suppose (X, d) is a metric space. A filter $\mathcal{F}$ is said to be *Cauchy* if for all $r > 0$ there is an $x \in X$ such that $B(x, r) \in \mathcal{F}$. A prefilter $\mathcal{E}$ is said to be Cauchy if its upward closure is Cauchy. A net $f : D \to X$ is said to be Cauchy if its tail filter is Cauchy.

Note that we need X to have a metric, not just a topology, for the definition to make sense.

Proposition 4.5.2. *If (X, d) is a metric space and $\mathcal{F}$ is a convergent filter on X then $\mathcal{F}$ is a Cauchy filter. If it's a convergent prefilter then it's a Cauchy prefilter. If $(D, \preccurlyeq)$ is a directed set then any net $f : D \to X$ which is convergent is Cauchy.*

Proof. The statements about prefilters and nets follow from those on filters, so it suffices to prove that every convergent filter is a Cauchy filter. If $\mathcal{F}$ converges to z then for each $r > 0$ we have $B(z, r) \in \mathcal{N}(z)$ and hence $B(z, r) \in \mathcal{F}$. $\square$

Proposition 4.5.3. *Suppose $(D, \preccurlyeq)$ is a directed set and (X, d) is a metric space. Then a net $f : D \to X$ is Cauchy if and only if for every $\epsilon > 0$ there is an $a \in D$ such that $d(f(b), f(c)) < \epsilon$ for all $b, c \in D$ such that $a \preccurlyeq b$ and $a \preccurlyeq c$.*

Proof. Suppose f is a Cauchy net. Let $\mathcal{F}$ be its tail filter. Then $\mathcal{F}$ is a Cauchy filter, so there is an $x \in X$ such that $B(x, \epsilon/2) \in \mathcal{F}$. By Proposition 4.4.4 this means that there is an a such that

$$f(b) \in B(x, \epsilon/2)$$

if $a \preccurlyeq b$, i.e that

$$d(f(b), x) < \frac{\epsilon}{2}$$

if $a \preccurlyeq b$. Then of course

$$d(f(c), x) < \frac{\epsilon}{2}$$

if $a \preccurlyeq c$. But d is a metric, so

$$d(f(b), f(c)) < \epsilon$$

if $a \preccurlyeq b$ and $a \preccurlyeq c$.

Suppose, conversely, that there is an a such that

$$d(f(b), f(c)) < \epsilon$$

if $a \preccurlyeq b$ and $a \preccurlyeq c$. Apply this with $c = a$ to get

$$d(f(b), f(a)) < \epsilon.$$

So there is an $a \in D$ such that

$$f(b) \in B(f(a), \epsilon)$$

for all $b \in D$ such that $a \preccurlyeq b$. By Proposition 4.4.4 this is equivalent to the statement that

$$B(f(a), \epsilon) \in \mathcal{F}.$$

So for every $\epsilon > 0$ there is a ball of radius ϵ in $\mathcal{F}$. $\mathcal{F}$ is therefore a Cauchy filter and f is a Cauchy net. ☐

As a corollary of the previous proposition we have the following.

Proposition 4.5.4. *Suppose (X, d) is a metric space and $\alpha \colon \mathbf{N} \to X$ is a Cauchy sequence. Then $\alpha_*(\mathbf{N})$ is bounded.*

Proof. Choose an $\epsilon > 0$ and then an $m \in \mathbf{N}$ such $d(\alpha_j, \alpha_k) < \epsilon$ for all $k, m \in \mathbf{N}$ such that $j, k \geq m$. In particular, $d(\alpha_j, \alpha_m) < \epsilon$ for all $k \geq m$. Let

$$q = \max_{0 \leq j < m} d(\alpha_j, \alpha_m).$$

Then $d(\alpha_j, \alpha_m) \leq q$ for $j < m$ and $d(\alpha_j, \alpha_m) < \epsilon$ for $j \geq m$. So $d(\alpha_j, \alpha_m) \leq r$ for all $j \in \mathbf{N}$, where

$$r = \max(q, \epsilon).$$

In other words $\alpha_j \in \bar{B}(\alpha_m, r)$ for all $j \in \mathbf{N}$ or, equivalently, $x \in \bar{B}(\alpha_m)$ for all $x \in \alpha_*(\mathbf{N})$. So $\alpha_*(\mathbf{N}) \subseteq \bar{B}(\alpha_m, r)$ and $\alpha_*(\mathbf{N})$ is therefore bounded. ☐

Although Proposition 4.5.2 shows that every convergent filter, prefilter, net or sequence is Cauchy, not every Cauchy filter, net or sequence is convergent. To see this, consider the interval $X = (0, +\infty)$ with the usual metric and the sequence $\alpha \colon \mathbf{N} \to X$ defined by $\alpha_n = 1/2^n$. To see that this is a Cauchy sequence note that for any $\epsilon > 0$ there is an n such that $1/2^n < \epsilon$, and therefore $|\alpha_j - \alpha_k| < \epsilon$ for all $j, k \geq n$. To see that it's not convergent suppose $\lim_{n \to \infty} \alpha_n = x$ for some $x \in X$ and then set $\epsilon = x/2$ and choose $m \in \mathbf{N}$ such that $|\alpha_n - x| < \epsilon$ for all $n \geq m$. Choose $k \in \mathbf{N}$ such that $1/2^n < x$ for all $n \geq k$. If $n > k$ then

$$\alpha_n = 1/2^n < x/2.$$

Let

$$n = \max(m, k + 1).$$

Then

$$|x| \leq |\alpha_n - x| + |\alpha_n| < x,$$

which is impossible, so there is no $x \in X$ such that $\lim_{n \to \infty} \alpha_n = x$. This gives an example of a Cauchy sequence which is not a convergent sequence. It also gives a Cauchy net which is not a convergent net, since nets are sequences. To get a Cauchy filter which is not a convergent filter we take its tail filter. This example might seem artificial in that there is a larger metric space, $\mathbf{R}$, in which this sequence is convergent. We'll see later, when we discuss completions, that this is not an accident.

Definition 4.5.5. A metric space (X, d) is called *complete* if every Cauchy filter is a convergent filter.

It's an immediate consequence of the definitions that if (X, d) is complete and $f \colon D \to X$ is a Cauchy net then f is a convergent net and therefore that Cauchy sequences are convergent.

Proposition 4.5.6. *$\mathbf{R}^n$, with the usual metric, is complete.*

Proof. Every closed ball in $\mathbf{R}^n$ is compact by the Heine-Borel Theorem, Theorem 3.12.14, so the proposition follows from Proposition 4.5.7 below. ☐

Proposition 4.5.7. *Suppose that (X, d) is a metric space such that every closed ball in X is compact. Then X is complete.*

Proof. Suppose $\mathcal{F}$ is a Cauchy filter. Let

$$Q = \{(x, r) \in X \times \mathbf{R} \colon r > 0, B(x, r) \in \mathcal{F}\}$$

and choose some $(y, s) \in Q$. We know there is one because $\mathcal{F}$ is a Cauchy filter. If $(x, r) \in Q$ then $\bar{B}(x, r) \in \mathcal{F}$ because

$$B(x, r) \subseteq \bar{B}(x, r)$$

and $\mathcal{F}$ is upward closed. If $(x_1, r_1), \ldots, (x_m, r_m) \in Q$ then

$$\bigcap_{j=1}^{m} \bar{B}(x_j, r_j) \neq \varnothing.$$

by Lemma 4.4.7. Let $\mathcal{C}$ be the set of sets of the form $\bar{B}(x,r) \cap \bar{B}(y,s)$ for $(x,r) \in Q$. By Lemma 4.4.7 again the intersection of finitely many elements of $\mathcal{C}$ is always non-empty. The elements of $\mathcal{C}$ are closed subsets of the compact set $\bar{B}(y,s)$. By Proposition 3.12.4 the intersection of all elements of $\mathcal{C}$ is non-empty. In other words,

$$\bigcap_{(x,r)\in Q} \left(\bar{B}(x,r) \cap \bar{B}(y,s)\right) \neq \varnothing.$$

But

$$\begin{aligned}
&\bigcap_{(x,r)\in Q} \left(\bar{B}(x,r) \cap \bar{B}(y,s)\right) \\
&\quad = \left(\bigcap_{(x,r)\in Q} \bar{B}(x,r)\right) \cap \bar{B}(y,s) \\
&\quad = \bigcap_{(x,r)\in Q} \bar{B}(x,r)
\end{aligned}$$

since $(y,s) \in Q$ and hence

$$\bigcap_{(x,r)\in Q} \bar{B}(x,r) \subseteq \bar{B}(y,s).$$

So there is a

$$z \in \bigcap_{(x,r)\in Q} \bar{B}(x,r).$$

For any given $r > 0$ choose an x such that $B(x,r) \in \mathcal{F}$. There must be one because $\mathcal{F}$ is a Cauchy filter. Then $(x,r) \in Q$ so $z \in \bar{B}(x,r)$. Suppose $y \in B(x,r)$. Then

$$d(x,y) < r$$

and

$$d(x,z) \leq r$$

so

$$d(y,z) < 2r$$

In other words, $y \in B(z,2r)$. This holds for all $y \in B(x,r)$ so

$$B(x,r) \subseteq B(z,2r).$$

But $B(x,r) \in \mathcal{F}$ and $\mathcal{F}$ is upward closed, so $B(z,2r) \in \mathcal{F}$. So for every $r > 0$ we have $B(z,2r) \in \mathcal{F}$. Therefore $\mathcal{F}$ converges to z by Lemma 4.4.8. □

Proposition 4.5.8. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are filters on a metric space (X,d), $\mathcal{F} \subseteq \mathcal{G}$, $\mathcal{F}$ is Cauchy and $\mathcal{G}$ is convergent. Then $\mathcal{F}$ is convergent.*

Proof. $\mathcal{G}$ is convergent, so there is some $z \in X$ such that $\mathcal{N}(z) \subseteq \mathcal{G}$. $\mathcal{F}$ is Cauchy, so for every $r > 0$ there is an $x \in X$. such that $B(x,r) \in \mathcal{F}$, and hence also $B(x,r) \in \mathcal{F} \cap \mathcal{G}$, since $\mathcal{F} \subseteq \mathcal{G}$. Also $B(z,r) \in \mathcal{N}(z)$. By Lemma 4.4.9 we then have $B(z,3r) \in \mathcal{F}$. By Lemma 4.4.8 then $\mathcal{F}$ converges to $\mathcal{G}$. □

Proposition 4.5.9. *Every compact metric space is complete.*

Proof. Every closed ball is closed by Proposition 1.11.3. Every closed subset of a compact space is compact by Proposition 3.12.6. Therefore every closed ball in X is compact. The result then follows immediately from Proposition 4.5.7. □

Proposition 4.5.10. *Suppose (X,d) is a totally bounded metric space and $\mathcal{F}$ is a filter on X. Then there is a Cauchy filter $\mathcal{G}$ such that $\mathcal{F} \subseteq \mathcal{G}$.*

Proof. We define $\mathcal{H}_j$ inductively by $\mathcal{H}_0 = \mathcal{F}$ and $\mathcal{H}_{k+1}$ is a filter which contains $\mathcal{H}_k$ and a ball of radius $1/2^n$. There must be such a $\mathcal{H}_{k+1}$ by Proposition 4.4.12, since there is a finite set of balls of radius $1/2^n$ which cover X. Then

$$\mathcal{G} = \bigcup_{j=0}^{\infty} \mathcal{H}_j$$

is a filter by Proposition 4.4.2. It contains balls of radius $1/2^n$ for all $n \geq 1$. Since it's upward closed it contains balls of radius r for any $r > 1/2^n$. But every $r > 0$ is greater than $1/2^n$ for some n. So it contains a ball of radius r for each $r > 0$ and therefore is a Cauchy filter. It contains $\mathcal{F}$ since $\mathcal{H}_0 = \mathcal{F}$. □

Proposition 4.5.11. *A metric space (X,d) is compact if and only if it is totally bounded and complete.*

Proof. Suppose (X,d) is compact. Then it's totally bounded by Proposition 4.2.8 and is complete by Proposition 4.5.9.

Suppose (X,d) is totally bounded and complete. If $\mathcal{F}$ is a filter on X then there is a Cauchy filter

$\mathcal{G}$ such that $\mathcal{F} \subseteq \mathcal{G}$ by Proposition 4.5.10. This $\mathcal{G}$ is then convergent by the definition of complete. It then follows from Proposition 4.4.13 that (X, d) is compact. ☐

4.6 Completion

We begin with four simple but useful lemmas.

Lemma 4.6.1. *Suppose $\mathcal{F}$ is a filter on a metric space (X, d). Then there is at most one $z \in X$ such that $\mathcal{F}$ converges to z.*

Proof. Metric spaces are Hausdorff by Proposition 1.10.3 so by Theorem 1.15.8 there is at most one z such that

$$\mathcal{N}(z) \subseteq \mathcal{F}.$$

☐

Lemma 4.6.2. *(a) If $(X, \mathcal{T})$ is a topological space, $\mathcal{F}$ is a convergent filter on X and $\mathcal{G}$ is a filter on X such that $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G}$ is a convergent filter.*

(b) If (X, d) is a metric space, $\mathcal{F}$ is a Cauchy filter on X and $\mathcal{G}$ is a filter on X such that $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G}$ is a Cauchy filter.

Proof. If $\mathcal{F}$ is convergent then there is a $z \in X$ such that

$$\mathcal{N}(z) \subseteq \mathcal{F}.$$

But then

$$\mathcal{N}(z) \subseteq \mathcal{G}$$

because $\mathcal{F} \subseteq \mathcal{G}$.

If $\mathcal{F}$ is Cauchy so for each $r > 0$ there is an $x \in X$ such that $B(x, r) \in \mathcal{F}$. $\mathcal{F} \subseteq \mathcal{G}$ so $B(x, r) \in \mathcal{G}$. So for each $r > 0$ there is an $x \in X$ such that $B(x, r) \in \mathcal{G}$. Therefore $\mathcal{G}$ is a Cauchy filter on X. ☐

Lemma 4.6.3. *Suppose $\mathbf{S}$ is a non-empty set of filters on a set X. Then $\bigcap_{\mathcal{G} \in \mathbf{S}} \mathcal{G}$ is also a filter on X.*

Proof. As usual, we check conditions 1.15.1a through 1.15.1d. Let $\mathcal{F} = \bigcap_{\mathcal{G} \in \mathbf{S}} G$. $X \in \mathcal{G}$ for all $\mathcal{G} \in \mathbf{S}$ so $X \in \mathcal{F}$ and therefore $\mathcal{F} \neq \varnothing$. This establishes 1.15.1a.

$\varnothing \notin \mathcal{G}$ for each $\mathcal{G} \in \mathbf{S}$ so $\varnothing \notin \mathcal{F}$. This establishes 1.15.1b.

If $A, B \in \mathcal{F}$ then $A, B \in \mathcal{G}$ for each $\mathcal{G} \in \mathbf{S}$. Therefore $A \cap B \in \mathcal{G}$ for each $\mathcal{G} \in \mathbf{S}$ and hence $A \cap B \in \mathcal{G}$. This establishes 1.15.1c.

If $A \in \mathcal{F}$ and $A \subseteq B$ then $\mathcal{A} \in \mathcal{G}$ for each $\mathcal{G} \in \mathbf{S}$ and hence $B \in \mathcal{G}$ for each $\mathcal{G} \in \mathbf{S}$. But then $B \in \mathcal{F}$. This establishes 1.15.1d. ☐

Lemma 4.6.4. *Suppose $(X, \mathcal{T})$ is a topological space, $z \in X$, and $A \in \wp(X)$.*

(a) If $\mathcal{G}$ is a filter on X, $A \in \mathcal{G}$ and $\mathcal{G}$ converges to z then $z \in \overline{A}$.

(b) If $z \in \overline{A}$ then there is a filter $\mathcal{G}$ on X such that $A \in \mathcal{G}$ and $\mathcal{G}$ converges to z.

Proof. (a) $\mathcal{G}$ converges to z so $\mathcal{N}(z) \subseteq \mathcal{G}$. In other words, if $V \in \mathcal{N}(z)$ then $V \in \mathcal{G}$. By Proposition 4.4.7 it follows that

$$A \cap V \neq \varnothing$$

for all $V \in \mathcal{N}(z)$. Therefore $z \in \overline{A}$ by Proposition 3.2.2l.

(b) If $z \in \overline{A}$ then $A \cap V \neq \varnothing$ for all $V \in \mathcal{N}(z)$, again by Proposition 3.2.2l. Apply Lemma 4.4.11 with $U = A$ and $\mathcal{F} = \mathcal{N}(z)$. The set $\mathcal{G}$ of $W \in \wp(X)$ such that there is a $V \in \mathcal{N}(z)$ with $A \cap V \subseteq W$ is a filter since $A \cap V \neq \varnothing$ for all $V \in \mathcal{N}(z)$. If $V \in \mathcal{N}(z)$ then $V \in \mathcal{G}$ since $A \cap V \subseteq V$. Therefore $\mathcal{N}(z) \subseteq \mathcal{G}$. In other words, $\mathcal{G}$ converges z. Also $X \in \mathcal{F}$ and $A \cap X \subseteq A$ so $\in \mathcal{G}$.

☐

Definition 4.6.5. A *minimal Cauchy filter* on a metric space (X, d) is a Cauchy filter $\mathcal{G}$ such that if $\mathcal{F}$ is a Cauchy filter and $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{F} = \mathcal{G}$.

Proposition 4.6.6. *Suppose (X, d) is a metric space and $z \in X$. Then the neighbourhood filter $\mathcal{N}(z)$ is a minimal Cauchy filter.*

Proof. $\mathcal{N}(z)$ is a Cauchy filter by Proposition 4.5.2. $\mathcal{N}(z)$ converges to z because $\mathcal{N}(z) \subseteq \mathcal{N}(z)$. If $\mathcal{F}$ is a Cauchy filter and $\mathcal{F} \subseteq \mathcal{N}(z)$ then $\mathcal{F}$ converges to z by Proposition 4.5.8. In other words, $\mathcal{N}(z) \subseteq \mathcal{F}$. Combined with $\mathcal{F} \subseteq \mathcal{N}(z)$ this gives $\mathcal{F} = \mathcal{N}(z)$. So $\mathcal{N}(z)$ is a Cauchy filter and if $\mathcal{F}$ is a Cauchy filter with $\mathcal{F} \subseteq \mathcal{N}(z)$ then $\mathcal{F} = \mathcal{N}(z)$. In other words, $\mathcal{N}(z)$ is a minimal Cauchy filter. ☐

Proposition 4.6.7. *If $\mathcal{H}$ is a Cauchy filter on a metric space (X,d) then there is a unique minimal Cauchy filter $\mathcal{F}$ such that $\mathcal{F} \subseteq \mathcal{H}$.*

Proof. Let
$$\mathcal{F} = \bigcap_{\mathcal{G} \subseteq \mathcal{H}} \mathcal{G}.$$
Then $\mathcal{F}$ is a filter by Lemma 4.6.3. Also, $\mathcal{G} \subseteq \mathcal{H}$ and if $\mathcal{G}$ is a Cauchy filter with $\mathcal{G} \subseteq \mathcal{H}$ then $\mathcal{F} \subseteq \mathcal{G}$. It remains only to prove that $\mathcal{F}$ is a Cauchy filter.

Suppose $\mathcal{G} \subseteq \mathcal{H}$. Both $\mathcal{G}$ and $\mathcal{H}$ are Cauchy filters so there are $x, y \in X$ such that $B(x,r) \in \mathcal{H}$ and $B(y,r) \in \mathcal{G}$. Then $B(y,r) \in \mathcal{G} \cap \mathcal{H}$ since $\mathcal{G} \cap \mathcal{H} = \mathcal{G}$. It follows from Lemma 4.4.9 that $B(x,3r) \in \mathcal{G}$. This holds for all $\mathcal{G}$ such that $\mathcal{G} \subseteq H$ so $B(x,3r) \in \mathcal{F}$. Suppose $s > 0$. Then $r = s/3 > 0$ so by the argument above there is an $x \in X$ such that $B(x,3r) = B(x,s) \in \mathcal{F}$. So $\mathcal{F}$ is a Cauchy filter. □

The following definition and pair of propositions give an alternate characterisation of minimal Cauchy filters.

Definition 4.6.8. Suppose (X,d) is a metric space and $\mathcal{F}$ is a filter on X. $\mathcal{F}$ is said to be *round* if for all $A \in \mathcal{F}$ there is an $r > 0$ such that if $B(x,r) \in \mathcal{F}$ then $B(x,r) \subseteq A$.

Proposition 4.6.9. *If $\mathcal{G}$ is a round Cauchy filter on a metric space (X,d) then it is a minimal Cauchy filter.*

Proof. Suppose $\mathcal{F}$ is a Cauchy filter and $\mathcal{F} \subseteq \mathcal{G}$. $\mathcal{G}$ is round so for every $A \in \mathcal{G}$ there is an $r > 0$ such that if $B(x,r) \in \mathcal{G}$ then $B(x,r) \subseteq A$. $\mathcal{F}$ is Cauchy so there is a $B(x,r) \in \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{G}$ so $B(x,r) \in \mathcal{G}$ and therefore $B(x,r) \subseteq A$. $B(x,r) \in \mathcal{F}$ and $\mathcal{F}$ is upward closed so $A \in \mathcal{F}$. We've just shown that if $A \in \mathcal{G}$ then $A \in \mathcal{F}$, i.e. that $\mathcal{G} \subseteq \mathcal{F}$. But we already had $\mathcal{F} \subseteq \mathcal{G}$ so $\mathcal{F} = \mathcal{G}$. So if $\mathcal{F}$ is a Cauchy filter and $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{F} = \mathcal{G}$. Therefore $\mathcal{G}$ is a minimal Cauchy filter. □

Proposition 4.6.10. *Suppose $\mathcal{G}$ is a filter on a metric space (X,d). Let $b\colon \mathcal{G} \times (0,+\infty) \to \wp(X)$ be defined by*
$$b(A,r) = \bigcup_{x \in A} B(x,r).$$
Let $\mathcal{E} = b_*(\mathcal{G} \times (0,+\infty))$ and let $\mathcal{F}$ be the upward closure of $\mathcal{E}$.

(a) $\mathcal{F}$ is a filter.

(b) $\mathcal{F}$ is round.

(c) $\mathcal{F} \subseteq \mathcal{G}$.

(d) If $\mathcal{G}$ is a Cauchy filter then $\mathcal{F}$ is a Cauchy filter.

(e) If $\mathcal{G}$ is a minimal Cauchy filter then $\mathcal{F} = \mathcal{G}$ and $\mathcal{G}$ is round.

Proof. $b(X,r) = X$ so $X \in \mathcal{E}$ and therefore $\mathcal{E} \neq \varnothing$. If $A \in \mathcal{G}$ and $r > 0$ then $A \neq \varnothing$ and $A \subseteq b(A,r)$ so $b(A,r) \neq \varnothing$. Therefore $\varnothing \notin \mathcal{E}$. Suppose $V_1, V_2 \in \mathcal{E}$, i.e. that there are $A_1, A_2 \in \mathcal{G}$ and $r_1, r_2 > 0$ such that $V_1 = b(A_1, r_1)$ and $V_2 = b(A_2, r_2)$. Then $A_1 \cap A_2 \in \mathcal{G}$ since $\mathcal{G}$ is filter. Also
$$\begin{aligned} b(A_1 \cap A_2, \min(r_1,r_2)) &\subseteq b(A_1,r_1) \cap b(A_2,r_2)) \\ &= V_1 \cap V_2 \end{aligned}$$
and
$$b(A_1 \cap A_2, \min(r_1, r_2)) \in \mathcal{E}.$$
So $\mathcal{E}$ is prefilter and its upward closure $\mathcal{F}$ is therefore a filter. This establishes 4.6.10a.

Suppose $V \in \mathcal{F}$, i.e. that there are $A \in \mathcal{G}$ and $s > 0$ such that $b(A,s) \subseteq V$. Let $r = s/2$. If $B(x,r) \in \mathcal{F}$ then there are $C \in \mathcal{G}$ and $t > 0$ such that $b(C,t) \subseteq B(x,r)$. But $C \subseteq b(C,t)$ so $C \subseteq B(x,r)$ and therefore $B(x,r) \in \mathcal{G}$, since $\mathcal{G}$ is upward closed. Therefore $A \cap B(x,r) \neq \varnothing$. Choose $y \in A \cap B(x,r)$. If $z \in B(x,r)$ then
$$d(y,z) \le d(y,x) + d(x,z) < r + r = s$$
so $z \in B(x,s)$. But $x \in A$ so $z \in b(A,s)$ and therefore $z \in V$. This holds for all $z \in B(x,r)$ so $B(x,r) \subseteq V$. So we've shown that for every $V \in \mathbf{F}$ there is an $r > 0$ such that if $B(x,r) \in \mathcal{F}$ then $B(x,r) \subseteq V$. In other words, $\mathcal{F}$ is round. This establishes 4.6.10b.

Suppose $V \in \mathcal{F}$, i.e. that there are $A \in \mathcal{G}$ and $r > 0$ such that $b(A,r) \subseteq V$. $A \subseteq b(A,r)$ so $A \subseteq V$. $\mathcal{G}$ is upward closed so $V \in \mathcal{G}$. This holds for all $V \in \mathcal{F}$ so $\mathcal{F} \subseteq \mathcal{G}$. This establishes 4.6.10c.

Suppose $r > 0$. Let $s = r/2$. Then $s > 0$. $\mathcal{G}$ is Cauchy so there is an $x \in X$ such that

$B(x,s) \in \mathcal{G}$. If $y \in B(x,s)$ and $z \in B(y,s)$ then $z \in B(x,2s) = B(x,r)$ so $b(B(x,s),s) \subseteq B(x,r)$. Therefore $B(x,r) \in \mathcal{F}$. So for each $r > 0$ there is an $x \in X$ such that $B(x,r) \in \mathcal{F}$. Therefore $\mathcal{F}$ is a Cauchy filter. This establishes 4.6.10d.

If $\mathcal{G}$ is a minimal Cauchy filter then $\mathcal{F} \subseteq \mathcal{G}$ by 4.6.10c and $\mathcal{F}$ is Cauchy by 4.6.10d so $\mathcal{F} = \mathcal{G}$ by the definition of a minimal Cauchy filter. $\mathcal{F}$ is round by 4.6.10b so $\mathcal{G}$ is round. ☐

Together Propositions 4.6.9 and 4.6.10e show that a filter on a metric space is a minimal Cauchy filter if and only if it is a round Cauchy filter.

Proposition 4.6.11. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are filters on a set X and $U \cap V \neq \varnothing$ for all $U \in \mathcal{F}$ and $V \in \mathcal{G}$.*

(a) There is a filter $\mathcal{H}$ on X such that $\mathcal{F} \subseteq \mathcal{H}$, $\mathcal{G} \subseteq \mathcal{H}$ and if $\mathcal{F} \subseteq \mathcal{I}$ and $\mathcal{G} \subseteq \mathcal{I}$ then $\mathcal{H} \subseteq \mathcal{I}$.

(b) If d is a metric on X and $\mathcal{F}$ and $\mathcal{G}$ are Cauchy filters for (X,d) then so is $\mathcal{H}$.

(c) If d is a metric on X and $\mathcal{F}$ and $\mathcal{G}$ are minimal Cauchy filters for (X,d) then $\mathcal{F} = \mathcal{G}$.

Proof. Let $\mathcal{H}$ be the set of $W \in \wp(X)$ such that there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ with $U \cap V \subseteq W$. We prove that $\mathcal{H}$ is a filter by checking the conditions 1.15.1a through 1.15.1d.

$X \in \mathcal{F}$, $X \in \mathcal{G}$, and $X \cap X \subseteq X$ so $W \in \mathcal{H}$ and hence $\mathcal{H} \neq \varnothing$. This establishes 1.15.1a.

If $W \in \mathcal{H}$ then there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ with $U \cap V \subseteq W$. Then $U \cap V \neq \varnothing$ by hypothesis, so $W \neq \varnothing$. This establishes 1.15.1b.

If $W_1, W_2 \in \mathcal{H}$ then there are $U_1, U_2 \in \mathcal{F}$ and $V_1 \cap V_2 \in \mathcal{G}$ such that $U_1 \cap V_1 \subseteq W_1$ and $U_2 \cap V_2 \subseteq W_2$. But then

$$\begin{aligned}(U_1 \cap U_2) \cap (V_1 \cap V_2) &= (U_1 \cap V_1) \cap (U_2 \cap V_2)\\ &\subseteq W_1 \cap W_2.\end{aligned}$$

$U_1 \cap U_2 \in \mathcal{F}$ and $V_1 \cap V_2 \in \mathcal{G}$ so $W_1 \cap W_2 \in \mathcal{H}$. This establishes 1.15.1c.

If $W \in \mathcal{H}$ and $W \subseteq Z$ then there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ such that $U \cap V \subseteq W$. But then $U \cap V \subseteq Z$ so $Z \in \mathcal{H}$. This establishes 1.15.1d. So $\mathcal{H}$ is indeed a filter.

Suppose $U \in \mathcal{F}$. $U \cap X \subseteq U$ and $X \in \mathcal{G}$ so $U \in \mathcal{H}$. This holds for all $U \in \mathcal{F}$ so $\mathcal{F} \subseteq \mathcal{H}$. Suppose $V \in \mathcal{G}$. $X \cap V \subseteq V$ and $X \in \mathcal{F}$ so $V \in \mathcal{H}$. This holds for all $V \in \mathcal{G}$ so $\mathcal{G} \subseteq \mathcal{H}$.

Suppose I is a filter on X such that $\mathcal{F} \subseteq \mathcal{I}$ and $\mathcal{F} \subseteq \mathcal{I}$. Suppose $W \in \mathcal{H}$. Then there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ such that $U \cap V \subseteq W$ Then $U \in \mathcal{I}$ because $\mathcal{F} \subseteq \mathcal{I}$ and $V \in \mathcal{I}$ because $\mathcal{G} \subseteq \mathcal{I}$. $\mathcal{I}$ is a filter so $U \cap V \in \mathcal{I}$ and then $W \in \mathcal{I}$. For all $W \in \mathcal{H}$ we therefore have $W \in \mathcal{I}$. It follows that $\mathcal{H} \subseteq \mathcal{I}$.

We've now finished Part (a). The other two parts are simple consequences of it. $\mathcal{F} \subseteq \mathcal{H}$ and $\mathcal{F}$ is a Cauchy filter so $\mathcal{H}$ is a Cauchy filter by Lemma 4.6.2. If $\mathcal{F}$ is a minimal Cauchy filter then $\mathcal{F} \subseteq \mathcal{G}$ since $\mathcal{F} \subseteq \mathcal{H}$ and $\mathcal{G} \subseteq \mathcal{H}$. Similarly, if $\mathcal{G}$ is a minimal Cauchy filter then $\mathcal{G} \subseteq \mathcal{F}$ since $\mathcal{G} \subseteq \mathcal{H}$ and $\mathcal{F} \subseteq \mathcal{H}$. So if both are minimal Cauchy filters then $\mathcal{F} = \mathcal{G}$. ☐

Proposition 4.6.12. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are minimal Cauchy filters on a metric space (X,d) and $\mathcal{F} \neq \mathcal{G}$. Then there are $x, y \in X$ and $r > 0$ such that $B(x,r) \in \mathcal{F}$, $B(y,r) \in \mathcal{G}$ and $d(x,y) \geq 3r$.*

Proof. $\mathcal{F}$ and $\mathcal{G}$ are minimal Cauchy filters and $\mathcal{F} \neq \mathcal{G}$ then by 4.6.11c there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$. $\mathcal{F}$ is also a round filter by Proposition 4.6.10e so there is an $s > 0$ such that if $B(x,s) \in \mathcal{F}$ then $B(x,s) \subseteq U$. $\mathcal{F}$ is a Cauchy filter and $s/3 > 0$ so there is an $x \in X$ such that $B(x,s/3) \in \mathcal{F}$. $\mathcal{F}$ is upward closed an $B(x,s/3) \subseteq B(x,s)$ so $B(x,s) \in \mathcal{F}$ and hence $B(x,s) \subseteq U$. Similarly there's a $y \in X$ such that $B(y,t/3) \in \mathcal{G}$ and $B(y,t) \subseteq V$. Let $r = \min(s/3, t/3)$. If $d(x,y) < 3r$ then $x \in B(y,3r)$, but

$$B(y,r) \subseteq B(x,t) \subseteq V$$

so $x \in V$. But $x \in U$ so $x \in U \cap V$, which is impossible because $U \cap V = \varnothing$. Therefore $d(x,y) \geq 3r$. From $B(x,s/3) \in \mathcal{F}$ and $B(y,t/3) \in \mathcal{G}$ it follows that $B(x,r) \in \mathcal{F}$ and $B(y,r) \in \mathcal{G}$, since $\mathcal{F}$ and $\mathcal{G}$ are upward closed. ☐

Proposition 4.6.13. *Suppose $\mathcal{F}$ is a filter on a set X and $\mathcal{G}$ is a filter on a set Y. Let $\mathcal{H}$ be the set of sets $W \in \wp(X,Y)$ such that there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ such $U \times V \subseteq W$. Then $\mathcal{H}$ is a filter on $X \times Y$.*

Proof. As usual, we check the conditions 1.15.1a through 1.15.1d.

$X \in \mathcal{F}$ and $Y \in \mathcal{G}$ so $X \times Y \in \mathcal{H}$ and therefore $\mathcal{H} \neq \varnothing$. This proves 1.15.1a.

If $W \in \mathcal{H}$ then there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ such $U \times V \subseteq W$. Then $U \neq \varnothing$ and $V \neq \varnothing$ so $U \times V \neq \varnothing$ and hence $W \neq \varnothing$. So $\varnothing \notin \mathcal{H}$. This proves 1.15.1b.

Suppose $W_1, W_2 \in \mathcal{H}$. Then there are $U_1, U_2 \in \mathcal{F}$ and $V_1, V_2 \in \mathcal{G}$ such $U_1 \times V_1 \subseteq W_1$ and $U_2 \times V_2 \subseteq W_2$. $\mathcal{F}$ is a filter so $U_1 \cap U_2 \in \mathcal{F}$. Similarly, $\mathcal{G}$ is a filter so $V_1 \cap V_2 \in \mathcal{G}$. But

$$\begin{aligned}(U_1 \cap U_2) \times (V_1 \cap V_2) &= (U_1 \times V_1) \cap (U_2 \times V_2) \\ &\subseteq W_1 \cap W_2,\end{aligned}$$

so $W_1 \cap W_2 \in \mathcal{H}$. This proves 1.15.1c.

If $W \in \mathcal{H}$ and $W \subseteq Z$ then there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ such that $U \times V \subseteq W$. But then $U \times V \subseteq Z$ so $Z \in \mathcal{H}$. This proves 1.15.1d. □

The filter $\mathcal{H}$ is called the *product* of the filters $\mathcal{F}$ and $\mathcal{G}$.

You may have noticed a similarity between the proofs of Propositions 4.6.11b and 4.6.13. In fact it's possible to prove Proposition 4.6.13 by applying Proposition 4.6.11 to the filters $\pi_1^{**}\mathcal{F}$ and $\pi_2^{**}\mathcal{G}$ on $X \times Y$, where π_1 and π_2 are the projections onto X and Y respectively, but it's just as easy to prove it directly.

Lemma 4.6.14. *If $\mathcal{F}$ is a filter on a set X, $x, y \in X$, $p, q > 0$, and $B(x,p), B(y,q) \in \mathcal{F}$ then $d(x,y) < p+q$.*

Proof. $B(x,p) \cap B(y,q) \neq \varnothing$ by Lemma 4.4.7. So there is a $z \in B(x,p) \cap B(y,q)$. Then

$$d(x,y) \leq d(x,z) + d(z,y) < p + q.$$

□

Theorem 4.6.15. *Suppose (X, d_X) is a metric space. Let $\mathbf{X}$ be the set of minimal Cauchy filters on X. Define $d_{\mathbf{X}} \colon \mathbf{X} \times \mathbf{X} \to \mathbf{R}$ as follows. If $\mathcal{F}, \mathcal{G} \in \mathbf{X}$ then let $\mathcal{H}$ be the product of $\mathcal{F}$ and $\mathcal{G}$ in the sense of Proposition 4.6.13. Then $\mathcal{I} = d_X^{**}(\mathcal{H})$ is a Cauchy filter on $\mathbf{R}$. $\mathbf{R}$ is a complete metric space so there is a unique $z \in \mathbf{R}$ such that $\mathcal{I}$ converges to z. We define $d_{\mathbf{X}}(\mathcal{F}, \mathcal{G}) = z$. Then $d_{\mathbf{X}}$ is a metric on $\mathbf{X}$. The function $i \colon X \to \mathbf{X}$ defined by $i(x) = \mathcal{N}(x)$ satisfies*

$$d_{\mathbf{X}}(i(x), i(y)) = d_X(x, y)$$

for all $x, y \in X$. Also, $(\mathbf{X}, d_{\mathbf{X}})$ is complete.

The metric space $(\mathbf{X}, d_{\mathbf{X}})$ is called the *completion* of (X, d_X).

Proof. We begin by proving a number of statements which will be needed repeatedly in the proof:

(a) $A \in \mathcal{I}$ if and only if there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ such that for all $s \in U$ and $t \in V$ we have $d_X(s,t) \in A$.

(b) If $B_X(x,r) \in \mathcal{F}$ and $B_X(y,r) \in \mathcal{G}$ then
$$B_{\mathbf{R}}(d_X(x,y), 2r) \in \mathcal{I}.$$

(c) If $B_X(x,r) \in \mathcal{F}$ and $B_X(y,r) \in \mathcal{G}$ then
$$|d_{\mathbf{X}}(\mathcal{F}, \mathcal{G}) - d_X(x,y)| < 3r.$$

(a) is simply a matter of unwrapping the various definitions. Each statement in the following sequence is equivalent to the one which precedes it:

- $A \in \mathcal{I}$.
- $A \in d_X^{**}(\mathcal{H})$.
- $d_X^*(A) \in \mathcal{H}$.
- There are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ such that $U \times V \subseteq d_X^*(A)$.
- There are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ such that for all $s \in U$ and $t \in V$ we have $(s,t) \in d_X^*(A)$.
- There are $U \in \mathcal{F}$ and $V \in \mathcal{G}$ such that for all $s \in U$ and $t \in V$ we have $d_X(s,t) \in A$.

This proves (a).

Suppose $B_X(x,r) \in \mathcal{F}$, $B_X(y,r) \in \mathcal{G}$ and $r > 0$. If $s \in B_X(x,r)$ and $t \in B_X(y,r)$ then

$$\begin{aligned}d_X(s,t) &\leq d_X(s,x) + d_X(x,y) + d_X(y,s) \\ &< d_X(x,y) + 2r\end{aligned}$$

and

$$\begin{aligned} d_X(x,y) &\le d_X(x,s) + d_X(s,t) + d_X(t,y) \\ &< d_X(s,t) + 2r \end{aligned}$$

so

$$|d_X(x,y) - d_X(s,t)| < 2r$$

In other words,

$$d_X(s,t) \in B_{\mathbf{R}}(d_X(x,y), 2r).$$

So there are $U \in \mathcal{F}$ and $V \in \mathcal{G}$, namely $U = B_X(x,r)$ and $V = B_X(y,r)$, such that for all $s \in U$ and $t \in V$ we have

$$d_X(s,t) \in B_{\mathbf{R}}(d_X(x,y), 2r).$$

It follows from (a) that

$$B_{\mathbf{R}}(d_X(x,y), 2r) \in \mathcal{I}.$$

This is (b).

Since $\mathcal{I}$ converges to $d_{\mathbf{X}}(\mathcal{F},\mathcal{G})$ we have

$$\mathcal{N}\left(d_{\mathbf{X}}(\mathcal{F},\mathcal{G})\right) \subseteq \mathcal{I}.$$

From

$$B_{\mathbf{R}}\left(d_{\mathbf{X}}(\mathcal{F},\mathcal{G}), r\right) \in \mathcal{N}\left(d_{\mathbf{X}}(\mathcal{F},\mathcal{G})\right)$$

it follows that

$$B_{\mathbf{R}}\left(d_{\mathbf{X}}(\mathcal{F},\mathcal{G}), r\right) \in \mathcal{I}.$$

By Lemma 4.6.14 applied to $\mathcal{I}$ we then have

$$d_{\mathbf{R}}\left(d_{\mathbf{X}}(\mathcal{F},\mathcal{G}), d_X(x,y)\right) < r + 2r = 3r.$$

This is (c).

Various things need to be checked in order to be sure that $d_{\mathbf{X}}$ is well defined. $\mathcal{H}$ is a filter on $X \times X$ by Proposition 4.6.13 and d_X is a function from $X \times X$ to $\mathbf{R}$. So $d_X^{**}(\mathcal{H})$ is a filter on $\mathbf{R}$ by Proposition 4.4.1. We've defined $\mathcal{I}$ to be $d_X^{**}(\mathcal{H})$. We need to check that it is Cauchy. $\mathcal{F}$ and $\mathcal{G}$ are Cauchy so there are $x, y \in X$ such that $B_X(x,r) \in \mathcal{F}$ and $B_X(y,r) \in \mathcal{G}$. By (b) we have $B_{\mathbf{R}}(d_X(x,y), 2r) \in \mathcal{I}$. So for all $r > 0$ there is a $w \in \mathbf{R}$, namely $w = d_X(x,y)$, such that $B_X(w, 2r) \in \mathcal{I}$. It follows that $\mathcal{I}$ is a Cauchy filter. $\mathbf{R}$ is complete by Proposition 4.5.6 so $\mathcal{I}$ is convergent, i.e. there is a $z \in \mathcal{F}$ such that $\mathcal{N}(z) \subseteq \mathcal{I}$. By Lemma 4.6.1 there is at most one such z. It therefore makes sense to define $d_{\mathbf{X}}(\mathcal{F},\mathcal{G})$ to be this z. If $x \in X$ then $\mathcal{N}(x)$ is a minimal Cauchy filter on X, i.e. and element of $\mathbf{X}$, by Proposition 4.6.6 so i is well defined.

To prove that $d_{\mathbf{X}}$ is a metric we check the conditions 1.6.1a through 1.6.1c.

$\mathcal{F}$ and $\mathcal{G}$ are Cauchy so for each $r > 0$ there are $x, y \in X$ such that $B_X(x,r) \in \mathcal{F}$ and $B_X(y,r) \in \mathcal{G}$. From (c) it follows that

$$d_{\mathbf{X}}(\mathcal{F},\mathcal{G}) > d_X(x,y) - 3r \ge -3r$$

since $d_X(x,y) \ge 0$. This holds for all $r > 0$ so

$$d_{\mathbf{X}}(\mathcal{F},\mathcal{G}) \ge 0.$$

Also,

$$d_{\mathbf{X}}(\mathcal{F},\mathcal{F}) < d_X(x,x) + 3r = -3r$$

for all $r > 0$ so

$$d_{\mathbf{X}}(\mathcal{F},\mathcal{F}) \le 0$$

and hence

$$d_{\mathbf{X}}(\mathcal{F},\mathcal{F}) = 0.$$

$\mathcal{F}$ and $\mathcal{G}$ are minimal Cauchy filters so if $\mathcal{F} \neq \mathcal{G}$ then there are, by Proposition 4.6.12, $x, y \in X$ and $r > 0$ such that $B_X(x,r) \in \mathcal{F}$, $B_X(y,r) \in \mathcal{G}$ and $d_X(x,y) \ge 3r$. If $s \in B_X(x,r)$ and $t \in B_X(y,r)$ then

$$\begin{aligned} d_X(x,y) &\le d_X(x,s) + d_X(s,t) + d_X(t,y) \\ &< r + d_X(s,t) + r \end{aligned}$$

so

$$d_X(s,t) > d_X(x,y) - 2r \ge r$$

This holds for all $s \in B_X(x,r)$ and $t \in B_X(y,r)$ so by (a) it follows that $(r, +\infty) \subseteq \mathcal{I}$.

$$B_{\mathbf{R}}(0,r) \cap (r, +\infty) = \varnothing$$

so $B_{\mathbf{R}}(0,r) \notin \mathcal{I}$. $B_{\mathbf{R}}(0,r) \in \mathcal{N}(0)$ so $\mathcal{N}(0)$ is not a subset of $\mathcal{I}$. Therefore $\mathcal{I}$ does not converge to 0 so $d_{\mathbf{X}}(\mathcal{F},\mathcal{G}) \neq 0$ if $\mathcal{F} \neq \mathcal{G}$. We have now proved that $d_{\mathbf{X}}$ satisfies 1.6.1b.

From (c) we also get

$$\begin{aligned} d_{\mathbf{X}}(\mathcal{F},\mathcal{G}) &< d_X(x,y)+3r \\ &= d_X(y,x)+3r \\ &< d_{\mathbf{X}}(\mathcal{G},\mathcal{F})+6r \end{aligned}$$

for all $r>0$ and hence

$$d_{\mathbf{X}}(\mathcal{F},\mathcal{G}) \le d_{\mathbf{X}}(\mathcal{G},\mathcal{F}).$$

Similarly,

$$\begin{aligned} d_{\mathbf{X}}(\mathcal{G},\mathcal{F}) &< d_X(y,x)+3r \\ &= d_X(x,y)+3r \\ &< d_{\mathbf{X}}(\mathcal{F},\mathcal{G})+6r \end{aligned}$$

for all $r>0$ and hence

$$d_{\mathbf{X}}(\mathcal{G},\mathcal{F}) \le d_{\mathbf{X}}(\mathcal{F},\mathcal{G}).$$

Combining these gives

$$d_{\mathbf{X}}(\mathcal{G},\mathcal{F}) = d_{\mathbf{X}}(\mathcal{F},\mathcal{G}),$$

which is 1.6.1b.

If also $\mathcal{J}$ is a Cauchy filter on X then for all $r>0$ there is a $z\in\mathcal{J}$ such that $B_X(z,r)\in\mathcal{J}$. Then

$$\begin{aligned} d_{\mathbf{X}}(\mathcal{F},\mathcal{J}) &< d_X(x,z)+3r \\ &\le d_X(x,y)+d_X(y,z)+3r \\ &< d_{\mathbf{X}}(\mathcal{F},\mathcal{G})+d_{\mathbf{X}}(\mathcal{G},\mathcal{J})+9r \end{aligned}$$

for all $r>0$ and hence

$$d_{\mathbf{X}}(\mathcal{F},\mathcal{J}) \le d_{\mathbf{X}}(\mathcal{F},\mathcal{G})+d_{\mathbf{X}}(\mathcal{G},\mathcal{J}).$$

which is 1.6.1c, so $d_{\mathbf{X}}$ is a metric on $\mathbf{X}$.

Suppose $x,y\in X$ and $r>0$. Then $B_X(x,r)\in i(x)$ and $B_X(y,r)\in i(y)$ so

$$|d_{\mathbf{X}}(i(x),i(y))-d_X(x,y)|<3r$$

by (c). This holds for all $r>0$ so

$$|d_{\mathbf{X}}(i(x),i(y))-d_X(x,y)|\le 0$$

and hence

$$d_{\mathbf{X}}(i(x),i(y))=d_X(x,y)$$

for all $x,y\in X$.

Suppose $\mathfrak{F}$ is a Cauchy filter on $\mathbf{X}$. Then $\mathfrak{F}$ contains balls of every positive radius so there is a function $\mathcal{G}\colon(0,+\infty)\to\mathbf{X}$ such that for each $r>0$ we have $B_{\mathbf{X}}(\mathcal{G}(r),r/4)\in\mathfrak{F}$. Each $\mathcal{G}(r)$ is also Cauchy so we can find an $x\colon(0,+\infty)$ such that $B_X(x(r),r/4)\in\mathcal{G}(r)$ for each $r>0$. Suppose $3r\ge q,r$. Then $B_{\mathbf{X}}(\mathcal{G}(q),q/4)$ and $B_{\mathbf{X}}(\mathcal{G}(r),r/4)$ belong to $\mathfrak{F}$ so so

$$d_{\mathbf{X}}(\mathcal{G}(q),G(r))<q/4+r/4\le r/2$$

by Lemma 4.6.14. If $s\in B_X(x(q),q/4)$ and $t\in B_X(x(r),r/4)$ then

$$\begin{aligned} d_X(x(q),x(r)) &\le d_X(x(q),s)+d_X(s,t)+d_X(t,x(r)) \\ &< d_X(s,t)+q/4+r/4 \\ &\le d_X(s,t)+r/2 \end{aligned}$$

so

$$d_X(s,t)\in(d_X(x(q),x(r))-r/2,+\infty).$$

This holds for all $s\in B_X(x(q),q/4)$ and $t\in B_X(x(r),r/4)$ and $B_X(x(q),q/4)\in\mathcal{G}(q)$ and $B_X(x(r),r/4)\in\mathcal{G}(r)$ so

$$(d_X(x(q),x(r))-r/2,+\infty)\in\mathcal{I}$$

by (a) and hence

$$d_X(x(q),x(r))-r/2<d_{\mathbf{X}}(\mathcal{G}(q),\mathcal{G}(r))$$

by(b) and so

$$d_{\mathbf{X}}(x(q),x(r))<r$$

If $q\le r$ then $d_{\mathbf{X}}(x(q),x(r))<r$ so considering x as a net with domain the directed set $((0,+\infty),\ge)$ we have that so $B_X(x(r),r)$ belongs to the tail filter of x. In particular, for each $r>0$ there is a ball of radius r in the tail filter, which is therefore Cauchy. Let $\mathcal{F}$ be the unique minimal Cauchy filter contained in the tail filter. $\mathcal{F}$ is Cauchy so there is a $y\in X$ with $B_X(y,r)\in\mathcal{F}$. Then $B_X(x(r),3r)\in\mathcal{F}$ by Lemma 4.4.9.

$$B_X(x(r),r/4)\in\mathcal{G}(r)$$

and

$$B_X(x(r),r/4)\subseteq B_X(x(r),3r)$$

so

$$B_X(x(r), 3r) \in \mathcal{G}(r).$$

From this and $B_X(x(r), 3r) \in \mathcal{F}$ it follows from (c) that

$$d_{\mathbf{X}}(\mathcal{F}, \mathcal{G}(r)) < 9r.$$

Therefore

$$B_{\mathbf{X}}(\mathcal{G}(r), r) \subseteq B_{\mathbf{X}}(\mathcal{F}, 10r).$$

$B_{\mathbf{X}}(\mathcal{G}(r), r/4) \subseteq B_{\mathbf{X}}(\mathcal{G}(r), r)$ so

$$B_{\mathbf{X}}(\mathcal{G}(r), r/4)) \subseteq B_{\mathbf{X}}(\mathcal{F}, 10r).$$

$B_{\mathbf{X}}(\mathcal{G}(r), r/4)) \in \mathfrak{F}$ and $\mathfrak{F}$ is upward closed so

$$B(\mathcal{F}, 10r) \in \mathfrak{F}.$$

Therefore $\mathfrak{F}$ converges to $\mathcal{F}$ by Lemma 4.4.8. We've now shown that every Cauchy filter in $\mathbf{X}$ converges, i.e. that $\mathbf{X}$ is complete. □

4.7 The Banach Fixed Point Theorem

The following theorem is very useful, despite the fact that its proof is not particularly difficult.

Theorem 4.7.1. *Suppose (X, d_X) is a non-empty complete metric space, $c < 1$ and $\varphi\colon X \to X$ satisfies the inequality*

$$d_X(\varphi(x), \varphi(y)) \le c d_X(x, y)$$

for all $x, y \in X$. Then there is a unique $z \in X$ such that $\varphi(z) = z$.

Proof. For the uniqueness, suppose $\varphi(w) = w$ and $\varphi(z) = z$. Then

$$d_X(w, z) = d_X(\varphi(w), \varphi(z)) \le c d_X(w, z)$$

so

$$(1 - c) d_X(w, z) \le 0.$$

$1 - c > 0$ so we can divide both sides by it to obtain

$$d_X(w, z) \le 0.$$

But $d_X(w, z) \ge 0$ so $d_X(w, z) = 0$ and hence $w = z$.

To prove the existence of z we choose an $a \in X$ and define a sequence $\alpha\colon \mathbf{N} \to X$ by

$$\alpha_0 = a, \qquad \alpha_{j+1} = \varphi(\alpha_j).$$

We show by induction on n that for all $m \le n$ we have

$$d_X(\alpha_m, \alpha_n) \le \frac{c^m - c^n}{1 - c} d_X(\alpha_0, \alpha_1).$$

For $n = 0$ the only $m \le n$ is $m = 0$ and the statement is trivially true. For $n = 1$ the only $m \le n$ are $m = 0$, where the statement reduces to

$$d_X(\alpha_0, \alpha_1) \le d_X(\alpha_0, \alpha_1),$$

which is clearly true. Suppose that $n \ge 2$ and the statement is known for all smaller values. If $m = n$ then

$$d_X(\alpha_m, \alpha_n) \le \frac{c^m - c^n}{1 - c} d_X(\alpha_0, \alpha_1)$$

holds trivially. If $m < n$ then

$$d_X(\alpha_m, \alpha_{n-1}) \le \frac{c^m - c^{n-1}}{1 - c} d_X(\alpha_0, \alpha_1).$$

Also,

$$d_X(\alpha_{n-2}, \alpha_{n-1}) \le \frac{c^{n-2} - c^{n-1}}{1 - c} d_X(\alpha_0, \alpha_1).$$

Using the assumption on φ we find

$$\begin{aligned} d_X(\alpha_{n-1}, \alpha_n) &= d_X(\varphi(\alpha_{n-2}), \varphi(\alpha_{n-1})) \\ &\le c d_X(\alpha_{n-2}, \alpha_{n-1}) \\ &\le c \frac{c^{n-2} - c^{n-1}}{1 - c} d_X(\alpha_0, \alpha_1) \\ &= \frac{c^{n-1} - c^n}{1 - c} d_X(\alpha_0, \alpha_1) \end{aligned}$$

Then

$$\begin{aligned} d_X(\alpha_m, \alpha_n) &\le d_X(\alpha_m, \alpha_{n-1}) + d_X(\alpha_{n-1}, \alpha_n) \\ &\le \frac{c^m - c^{n-1}}{1 - c} d_X(\alpha_0, \alpha_1) \\ &\quad + \frac{c^{n-1} - c^n}{1 - c} d_X(\alpha_0, \alpha_1) \\ &= \frac{c^m - c^n}{1 - c} d_X(\alpha_0, \alpha_1). \end{aligned}$$

This completes the induction. Similarly, if $n \leq m$ then

$$d_X(\alpha_m, \alpha_n) \leq \frac{c^n - c^m}{1-c} d_X(\alpha_0, \alpha_1).$$

In general,

$$\begin{aligned} d_X(\alpha_m, \alpha_n) &\leq \frac{c^{\min(m,n)} - c^{\max(m,n)}}{1-c} d_X(\alpha_0, \alpha_1) \\ &< \frac{c^{\min(m,n)}}{1-c} d_X(\alpha_0, \alpha_1) \end{aligned}$$

If $\epsilon > 0$ then there is a k such that

$$\frac{c^k}{1-c} d_X(\alpha_0, \alpha_1) < \epsilon$$

and hence

$$d_X(\alpha_m, \alpha_n) < \epsilon$$

for all $m, n \geq k$. The sequence α is therefore a Cauchy sequence by Proposition 4.5.3. (X, d_X) is complete so α converges to some $z \in X$ by the remarks after Definition 4.5.5. In other words, there is some $z \in X$ such that

$$\lim_{n\to\infty} \alpha_n = z.$$

By Proposition 4.1.7 we have

$$\lim_{n\to\infty} \alpha_{n+1} = \lim_{n\to\infty} \varphi(\alpha_n) = \varphi(\lim_{n\to\infty} \alpha_n) = \varphi(z).$$

But $\lim_{n\to\infty} \alpha_{n+1} = \lim_{n\to\infty} \alpha_n$ so $\varphi(z) = z$. □

4.8 Function spaces

We begin with two propositions and a definition which could have appeared in earlier sections, but which we will need for our discussion of function spaces.

Proposition 4.8.1. *If (X, d_X) and (Y, d_Y) are metric spaces, $f\colon X \to Y$ is a uniformly continuous function and $\mathcal{F}$ is a Cauchy filter on X then $f^{**}(\mathcal{F})$ is a Cauchy filter on Y.*

Proof. $f^{**}(\mathcal{F})$ is a filter by Proposition 4.4.1. Suppose $\epsilon > 0$. f is uniformly continuous so there is a $\delta > 0$ such that

$$B_X(x, \delta) \subseteq f^*(B(f(x), \epsilon))$$

for all $x \in X$. $\mathcal{F}$ is a Cauchy filter so there is an $x \in X$ such that $B(x, \delta) \in \mathcal{F}$. $\mathcal{F}$ is upward closed so

$$f^*(B(f(x), \epsilon)) \in \mathcal{F}.$$

In other words,

$$B(f(x), \epsilon) \in f^{**}(\mathcal{F}).$$

So for every $\epsilon > 0$ there is a $y \in Y$ such that $B(y, \epsilon) \in f^{**}(\mathcal{F})$. Therefore $f^{**}(\mathcal{F})$ is a Cauchy filter. □

Proposition 4.8.2. *Suppose that (X, d_X) is a complete metric space and A is a closed subset of X. Then (A, d_A) is a complete metric space, where d_A is the restriction of d_X.*

Proof. Suppose $\mathcal{F}$ is a Cauchy filter on A. Let $i\colon A \to X$ be the inclusion function $i(x) = x$. i is uniformly continuous so $i^{**}(\mathcal{F})$ is a Cauchy filter on X by Proposition 4.8.1. (X, d_X) is complete so there is a $z \in X$ such that $i^{**}(\mathcal{F})$ converges to z, i.e. such that

$$\mathcal{N}_X(z) \subseteq i^{**}(\mathcal{F}).$$

The subscript X indicates that this is the neighbourhood filter of x with respect to the topology on X coming from the metric d_X. We'll use $\mathcal{N}_A$ to denote the neighbourhood filter of a point in A with respect to the topology coming from the metric d_A. Similarly $\mathcal{O}$ with subscripts will be used for the sets of open neighbourhoods with respect to the two different topologies.

If $z \notin A$ then $X \setminus A \in \mathcal{N}(z)$ and so $X \setminus A \in i^{**}(\mathcal{F})$ Then $i^*(X \setminus A) \in \mathcal{F}$. But $i^*(X \setminus A) = \varnothing$ and $\mathcal{F}$ is a filter so $\varnothing \notin \mathcal{F}$. Therefore $z \in A$.

Suppose $V \in \mathcal{N}_A(z)$. Then there is a $U \in \mathcal{O}_A(z)$ such that $U \subseteq V$. By Proposition 3.8.2 there is a $W \in \mathcal{O}_X(z)$ such that $U = A \cap W$, or, equivalently, such that $U = i^*(W)$. $W \in \mathcal{N}_X(z)$ and $\mathcal{N}_X(z) \subseteq i^{**}(\mathcal{F})$. $i^{**}(\mathcal{F})$ is a filter and so is upward closed, so

$$W \in i^{**}(\mathcal{F})$$

and hence

$$U = i^*(W) \in \mathcal{F}.$$

$\mathcal{F}$ is upward closed and $U \subseteq V$ so

$$V \in \mathcal{F}.$$

So if $V \in \mathcal{N}_A(z)$ then $V \in \mathcal{F}$. In other words,

$$N_A(z) \subseteq \mathcal{F}$$

and $\mathcal{F}$ converges to z. We've shown that for every Cauchy filter $\mathcal{F}$ on A there is a $z \in A$ such that $\mathcal{F}$ converges to z. So (A, d_A) is complete. □

Definition 4.8.3. Suppose X is a non-empty set and (Y, d_Y) is a metric space. Then $f\colon X \to Y$ is called *bounded* if $f_*(X)$ is a bounded subset of Y.

Proposition 4.8.4. *Suppose X is a non-empty set and (Y, d_Y) is a metric space. Let Z be the set of bounded functions from X to Y. Then $d_Z\colon Z \times Z \to \mathbf{R}$, defined by*

$$d_Z(f, g) = \sup_{x \in X} d_Y(f(x), g(x)),$$

is a metric on Z.

Proof. First note that since $f_*(X)$ and $g_*(X)$ are non-empty and bounded there are $y \in f_*(X)$, $z \in g_*(X)$ and $r, s > 0$ such that $f_*(X) \subseteq B_Y(y, r)$ and $g_*(X) \subseteq B_Y(z, s)$. If $x \in X$ then

$$\begin{aligned} d_Y(f(x), g(x)) &\leq d_Y(f(x), y) + d_Y(y, z) \\ &\quad + d_Y(z, g(x)) \\ &< r + d_Y(y, z) + r. \end{aligned}$$

Thus $d_Y(y, z) + 2r$ is an upper bound for $d_Y(f(x), g(x))$ so the supremum exists. To prove that it is a metric we need to check 1.6.1a through 1.6.1c.

The supremum of a set of non-negative numbers is non-negative so $d_Z(f, g) \geq 0$ for all $f, g \in Z$. If $f = g$ then

$$d_Y(f(x), g(x)) = 0$$

for all $x \in X$ so $d_Z(f, g) = 0$. Conversely, if $f \neq g$ then there is an $x \in X$ such that $f(x) \neq g(x)$ and therefore

$$d_Y(f(x), g(x)) > 0.$$

Therefore $d_Z(f, g) > 0$. So $d_Z(f, g) = 0$ if and only if $f = g$. This establishes 1.6.1a.

From

$$d_Y(f(x), g(x)) = d_Y(g(x), f(x))$$

it follows that

$$\begin{aligned} d_Z(f, g) &= \sup_{x \in X} d_Y(f(x), g(x)) \\ &= \sup_{x \in X} d_Y(g(x), f(x)) = d_Z(g, f). \end{aligned}$$

This establishes 1.6.1b.

If $f, g, h \in Z$ and $x \in X$ then

$$\begin{aligned} d_Y(f(x), h(x)) &\leq d_Y(f(x), g(x)) + d_Y(g(x), h(x)) \\ &\leq d_Z(f, g) + d_Z(g, h). \end{aligned}$$

This holds for all $x \in X$ so

$$d_Z(f, g) = \sup_{x \in X} d_Y(f(x), h(x)) \leq d_Z(f, g) + d_Z(g, h).$$

This establishes 1.6.1b. □

Proposition 4.8.5. *Suppose $(X, \mathcal{T})$ is a non-empty topological space and (Y, d_Y) is a metric space. Let Z be the set of bounded functions from X to Y with the metric*

$$d_Z(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Let W be the set of bounded continuous functions from X to Y. Then W is a closed subset of Z.

Proof. Suppose $f \in Z \setminus W$. By Proposition 4.1.6b then there are $x \in X$ and $\epsilon > 0$ such that there is a $U \in \mathcal{O}(x)$ but

$$U \nsubseteq f^*(B_Y(f(x), \epsilon)).$$

The non-inclusion means there is a y such that $y \in U$ and

$$y \notin f^*(B_Y(f(x), \epsilon)).$$

In other words,

$$f(y) \notin B_Y(f(x), \epsilon)$$

or

$$d_Y(f(x), f(y)) \geq \epsilon.$$

Suppose $g \in B_Z(f, \epsilon/3)$. From $g \in B_Z(f, \epsilon/3)$ it follows that

$$d_Y(f(x), g(x)) \leq d_Y(f, g) < \epsilon/3$$

and

$$d_Y(g(y), f(y)) \le d_Y(g, f) < \epsilon/3.$$

From the last three inequalities we deduce that

$$\begin{aligned}\epsilon &\le d_Y(f(x), f(y)) \\ &\le d_Y(f(x), g(x)) + d_Y(g(x), g(y)) + d_Y(g(y), f(y)) \\ &< \epsilon/3 + d_Y(g(x), g(y)) + \epsilon/3\end{aligned}$$

and hence

$$d_Y(g(x), g(y) \ge \epsilon/3.$$

Other ways of expressing this are

$$g(y) \notin B_Y(g(x), \epsilon/3)$$

or

$$y \notin g^*(B_Y(g(x), \epsilon/3)).$$

Since $y \in U$ we conclude that

$$U \nsubseteq g^*(B_Y(g(x), \epsilon/3)).$$

Another application of Proposition 4.1.6b shows that g is not continuous, i.e. that $g \in Z \setminus W$. We've therefore shown that if $f \in W \setminus Z$ then there is an $r > 0$, namely $r = \epsilon/3$ with ϵ as above, such that $B_Z(f, r) \subseteq Z \setminus W$. Therefore $Z \setminus W$ is open and W is closed. □

Proposition 4.8.6. *Suppose that $(X, \mathcal{T}_X)$ is a non-empty topological space and (Y, d_Y) is a complete metric space. Let W be the space of bounded continuous functions from X to Y with the metric*

$$d_W(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Then (W, d_W) is a complete metric space.

Proof. Suppose $\mathcal{F}$ is a Cauchy filter on W. For each $x \in X$ let $e_x : W \to Y$ be defined by

$$e_x(g) = g(x).$$

Then

$$d_Y(e_x(g), e_x(h)) = d_Y(g(x), h(x)) \le d_W(g, h)$$

so e_x is uniformly continuous. By Proposition 4.8.1 then $e_x^{**}(\mathcal{F})$ is a Cauchy filter on Y. (Y, d_Y) is complete so this Cauchy filter is convergent. Define $f : X \to Y$ by saying that $f(x)$ is the element of X to which $e_x^{**}(\mathcal{F})$ converges. We now show that $\mathcal{F}$ converges to f.

Suppose $\epsilon > 0$. $\mathcal{F}$ is a Cauchy filter so there is an $h \in W$ such that

$$B_W(h, \epsilon) \in \mathcal{F}.$$

$e_x^{**}(\mathcal{F})$ converges to $f(x)$. In other words,

$$\mathcal{N}_Y(f(x)) \subseteq e_x^{**}(\mathcal{F}).$$

$B_Y(f(x), \epsilon) \in \mathcal{N}_Y(f(x))$ for all $\epsilon > 0$ so

$$B_Y(f(x), \epsilon) \in e_x^{**}(\mathcal{F}).$$

Therefore

$$e_x^*(B_Y(f(x), \epsilon)) \in \mathcal{F}.$$

Now

$$g \in e_x^*(B_Y(f(x), \epsilon))$$

if and only if

$$e_x(g) \in B_Y(f(x), \epsilon).$$

But $e_x(g) = g(x)$, so

$$g \in e_x^*(B_Y(f(x), \epsilon))$$

if and only if

$$g(x) \in B_Y(f(x), \epsilon),$$

i.e. if and only if

$$d_Y(f(x), g(x)) < \epsilon.$$

In other words,

$$e_x^*(B_Y(f(x), \epsilon)) = \{g \in W : d_Y(f(x), g(x) < \epsilon\}.$$

The set on the right is therefore an element of $\mathcal{F}$.

We've already seen that $B_W(h, \epsilon) \in \mathcal{F}$ so

$$\{g \in W : d_Y(f(x), g(x) < \epsilon\} \cap B_W(h, \epsilon) \neq \varnothing.$$

There is therefore a

$$g \in \{g \in W : d_Y(f(x), g(x) < \epsilon\} \cap B_W(h, \epsilon) \neq \varnothing.$$

For such a g we have

$$\begin{aligned} d_Y(f(x),h(x)) &\leq d_Y(f(x),g(x)) + d_Y(g(x),h(x)) \\ &< \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

This holds for all x so

$$d_W(f,h) \leq 2\epsilon.$$

It follows that

$$B_W(h,\epsilon) \subseteq B_W(f,3\epsilon).$$

$B_W(h,\epsilon) \in \mathcal{F}$ and $\mathcal{F}$ is upward closed so $B_W(f,3\epsilon) \in \mathcal{F}$. This holds for all $\epsilon > 0$ so $\mathcal{F}$ converges to f by Lemma 4.4.8. ☐

Corollary 4.8.7. *Suppose that $(X,\mathcal{T}_X)$ is a non-empty compact topological space and (Y,d_Y) is a complete metric space. Let W be the space of continuous functions from X to Y with the metric*

$$d_W(f,g) = \sup_{x\in X} d_Y(f(x),g(x)).$$

Then (W,d_W) is a complete metric space.

Proof. For every continuous function f from X to Y we $f_*(X)$ is compact by Proposition 3.12.8. It's bounded by Proposition 4.2.8. Therefore f is bounded. So we can replace the words "bounded continuous" in the statement of Proposition 4.8.6 by "continuous". ☐

Definition 4.8.8. Suppose $(X,\mathcal{T}_X)$ is a topological space and (Y,d_Y) is a metric space. A set $\mathcal{A}$ of functions from from X to Y is called *equicontinuous* if for every $x \in A$ and $\epsilon > 0$ there is a $V \in \mathcal{N}(x)$ such that for all $f \in \mathcal{A}$ we have

$$V \subseteq f^*(B(f(x),\epsilon)).$$

Definition 4.8.9. Suppose (X,d_X) and (Y,d_Y) are metric spaces. A set $\mathcal{A}$ of functions from from X to Y is called *uniformly equicontinuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that for all $f \in \mathcal{A}$ and $x \in X$ we have

$$B(x,\delta) \subseteq f^*(B(f(x),\epsilon)).$$

Proposition 4.8.10. *Suppose $(X,\mathcal{T}_X)$ is a non-empty compact topological space, (Y,d_Y) is a metric space and $\mathcal{A}$ is an equicontinuous set of functions from X to Y. Suppose further that for each $x \in X$ the set*

$$I_x = \bigcup_{f\in\mathcal{A}} \{f(x)\}$$

is totally bounded. Then $\mathcal{A}$ is totally bounded.

Proof. Suppose $r > 0$. Let $\epsilon = r/4$. Let

$$\mathcal{R} = \{(x,U) \in X \times \mathcal{T}_X : x \in U, U \subseteq f^*(B(f,\epsilon))\}$$

If $x \in X$ then there is a $V \in \mathcal{N}(x)$ such that

$$V \subseteq f^*(B(f(x),\epsilon)).$$

There is then a $U \in \mathcal{O}(x)$ such that $U \subseteq V$ and hence

$$U \subseteq f^*(B(f(x),\epsilon)).$$

In other words, $(x,U) \in \mathcal{R}$. Therefore

$$X = \bigcup_{(x,U)} U.$$

X is compact so there is a finite subcover. In other words, there are (x_1,U_1), …, (X_m,U_m) in $\mathbf{R}$ such that

$$X = \bigcup_{j=1}^{m} U_m.$$

For each j the set I_{x_j} is totally bounded so the union $\bigcup_{j=1}^m I_j$ is totally bounded. There are therefore y_1, …, y_n in Y such that

$$\bigcup_{j=1}^{m} I_{x_j} \subseteq \bigcup_{k=1}^{n} B_Y(y_k,\epsilon).$$

Let T be the set of functions from $\{1,\ldots,m\}$ to $\{1,\ldots,n\}$. T has n^m elements and so is finite. For each $s \in T$ we define $\mathcal{A}_s$ to be the set of $f \in \mathcal{A}$ such that

$$f(x_j) \in B_Y(y_{s(j)},\epsilon)$$

for each $j \in \{1,\ldots,m\}$. Let S be the set of $s \in T$ such that $\mathcal{A}_s \neq \varnothing$. S is a subset of the finite set T and so is finite. For each $f \in \mathcal{A}$ and each $j \in \{1, \dots, m\}$ we have

$$f(x_j) \in I_{x_j} \subseteq \bigcup_{k=1}^{n} B_Y(y_k, \epsilon).$$

In other words, there is a $k \in \{1, \dots, n\}$ such that

$$f(x_j) \in B_Y(y_k, \epsilon).$$

Therefore every $f \in \mathcal{A}$ belongs to $\mathcal{A}_s$ for some $s \in T$.

Suppose $f, g \in \mathcal{A}_s$. If $x \in X$ then $x \in U_j$ for some $j \in \{1, \dots, m\}$. Then

$$\begin{aligned} d_Y(f(x), g(x)) \le d_Y(f(x), f(x_j)) + d_Y(f(x_j), y_s(j)) \\ + d_Y(y_{s(j)}, g(x_j)) + d_Y(g(x_j), g(x)) \\ < \epsilon + \epsilon + \epsilon + \epsilon = r. \end{aligned}$$

This holds for all $x \in X$ so

$$d_{\mathcal{A}}(f, g) \le r.$$

By definition the set $\mathcal{A}_s$ is non-empty for each $s \in S$. There is therefore an $f_s \in \mathcal{A}_s$ for each $s \in S$. If $g \in \mathcal{A}_s$ then $d_{\mathcal{A}}(f_s, g) \le r$ and hence $g \in \bar{B}_{\mathcal{A}}(f_s, r)$. Every $g \in \mathcal{A}$ is in $\mathcal{A}_s$ for some $s \in S$ so

$$\mathcal{A} = \bigcup_{s \in S} \bar{B}_{\mathcal{A}}(f_s, r).$$

So for every $r > 0$ there is a finite set of elements of $\mathcal{A}$ such that the closed balls of radius r centred at them cover $\mathcal{A}$. Therefore $\mathcal{A}$ is totally bounded. □

The following theorem is known as the *Arzelà-Ascoli Theorem.*

Theorem 4.8.11. *Suppose $(X, \mathcal{T}_X)$ is a compact topological space, (Y, d_Y) is a complete metric space and $\mathcal{A}$ is an equicontinuous set of functions from X to Y. If $\mathcal{A}$ is closed and*

$$\bigcup_{f \in \mathcal{A}} \{f(x)\}$$

is totally bounded for each $x \in X$ then $\mathcal{A}$ is compact.

Proof. $\mathcal{A}$ is totally bounded by Proposition 4.8.10. It is complete by Propositions 4.8.2 and 4.8.6. It is therefore compact by Proposition 4.5.11 □

Corollary 4.8.12. *Suppose $(X, \mathcal{T}_X)$ is a compact topological space, (Y, d_Y) is a compact metric space and $\mathcal{A}$ is an equicontinuous set of functions from X to Y. If $\mathcal{A}$ is closed then $\mathcal{A}$ is compact.*

Proof. (Y, d_Y) is complete and totally bounded by Proposition 4.5.11. The subsets $\bigcup_{f \in \mathcal{A}} \{f(x)\}$ are subsets of Y and hence also totally bounded. Theorem 4.8.11 therefore implies that $\mathcal{A}$ is compact. □

5 Normed vector spaces

5.1 Basic definitions

As stated in Definition 1.4.1, if V is a vector space then we say that $p \colon V \to \mathbf{R}$ is a norm on V if it has the following three properties:

(a) For all $\mathbf{v} \in V$, $p(\mathbf{x}) \ge 0$ and $p(\mathbf{v}) > 0$ unless $\mathbf{v} = \mathbf{0}$.

(b) For all $\alpha \in \mathbf{R}$ and $\mathbf{v} \in V$, $p(\alpha \mathbf{v}) = |\alpha| p(\mathbf{v})$.

(c) For all $\mathbf{v}, \mathbf{w} \in V$, $p(\mathbf{v} + \mathbf{w}) \le p(\mathbf{v}) + p(\mathbf{w})$.

Also, a pair (V, p) consisting of a vector space V and a norm p on V is called a normed vector space. The elementary properties of norms were given in Proposition 1.4.2.

As shown in Theorem 1.6.2, if p is a norm on V then $d \colon V \times V \to \mathbf{R}$, defined by

$$d(\mathbf{x}, \mathbf{y}) = p(\mathbf{x} - \mathbf{y}),$$

is a metric on V. Unless otherwise specified we always assume that a normed vector space is equipped with this metric and the topology consisting of its open sets. This is true in particular in th following definition, since the definition of completeness refers to the metric space structure.

Definition 5.1.1. A complete normed vector space is called a *Banach space*

Definition 5.1.2. Two norms p and q on a vector space V are called *equivalent* if there are $c, C > 0$ such that for all $\mathbf{v} \in V$

$$c p(\mathbf{v}) \le q(\mathbf{v}) \le C p(\mathbf{v}).$$

The sense in which these norms deserve to be called equivalent is explained by the following proposition.

Proposition 5.1.3. *If p and q are equivalent norms on on a vector space V and $\mathcal{T}_p$ and $\mathcal{T}_q$ are the associated topologies then $\mathcal{T}_p = \mathcal{T}_q$.*

Proof. Let c, C be as in the definition of equivalent norms. Suppose $U \in \mathcal{T}_p$. This means that for each $\mathbf{x} \in V$ there is an $r > 0$ such that the ball of radius r about $\mathbf{x}$, with respect to the metric from the norm p, is contained in U. In other words, if $p(\mathbf{x} - \mathbf{y}) < r$ then $\mathbf{y} \in U$. Let $\delta = cr$. If $q(\mathbf{y} - \mathbf{x}) < \delta$ then

$$p(\mathbf{y} - \mathbf{x}) \leq \frac{1}{c} q(\mathbf{y} - \mathbf{x}) < \frac{\delta}{c} = r$$

so $\mathbf{y} \in U$. So $q(\mathbf{y} - \mathbf{x}) < \delta$ implies $\mathbf{y} \in U$. But $q(\mathbf{y} - \mathbf{x}) < \delta$ if and only if $\mathbf{y}$ is in the ball of radius δ about $\mathbf{x}$, with respect to the metric from the norm q, so this ball is contained in U. So for all $\mathbf{x} \in U$ there is a $\delta > 0$ such that the ball of radius δ about $\mathbf{x}$, with respect to the norm q, is contained in U. Therefore $U \in \mathcal{T}_q$. So $\mathcal{T}_p \subseteq \mathcal{T}_q$.

The same argument, with p and q reversed and c replaced by $1/C$, gives the inclusion $\mathcal{T}_q \subseteq \mathcal{T}_p$. Combined with the inclusion found above, this gives

$$\mathcal{T}_p = \mathcal{T}_q.$$

□

The term "equivalent" certainly suggests that equivalence should be an equivalence relation in the sense defined earlier. This is indeed true.

Proposition 5.1.4. *Suppose p, q and r are norms on a vector space V.*

(a) p is equivalent to p.

(b) If p is equivalent to q then q is equivalent to p.

(c) If p is equivalent to q and q is equivalent to r then p is equivalent to r.

Proof.

$$cp(\mathbf{v}) \leq p(\mathbf{v}) \leq Cp(\mathbf{v})$$

with $c = C = 1$. So p is equivalent to p.

If p is equivalent to q then

$$cp(\mathbf{v}) \leq q(\mathbf{v}) \leq Cp(\mathbf{v}).$$

So

$$\frac{1}{C} q(\mathbf{v}) \leq p(\mathbf{v}) \leq \frac{1}{c} q(\mathbf{v}).$$

Therefore q is equivalent to p.

If p is equivalent to q and q is equivalent to r then there are $c, C > 0$ and $k, K > 0$ such that

$$cp(\mathbf{v}) \leq q(\mathbf{v}) \leq Cp(\mathbf{v}).$$

and

$$kq(\mathbf{v}) \leq r(\mathbf{v}) \leq Kq(\mathbf{v}).$$

Then

$$ckp(\mathbf{v}) \leq r(\mathbf{v}) \leq CKp(\mathbf{v}).$$

So p is equivalent to r. □

Although equivalent norms give rise to the same topology they give rise to different metrics and completeness is defined in terms of Cauchy filters, which in turn are defined in terms of a metric. The following proposition is therefore not obvious.

Proposition 5.1.5. *If p and q are equivalent norms on on a vector space V and $\mathcal{F}$ is a Cauchy filter for the norm p then it is also a Cauchy filter for the norm q, and vice versa. If (V, p) is a Banach space then (V, q) is a Banach space, and vice versa.*

Proof. Let c, C be as in the definition of equivalent norms. Suppose $r > 0$. $\mathcal{F}$ is a Cauchy filter for the norm p and $cr > 0$ so there is an $\mathbf{x} \in V$ such that

$$B_p(\mathbf{x}, cr) \in \mathcal{F}.$$

Here B_p is the ball with respect to the metric from the norm p. In other words,

$$B_p(\mathbf{x}, s) = \{\mathbf{y} \in V : p(\mathbf{x} - \mathbf{y}) < s\}.$$

We define balls B_q with respect to the metric from the norm q similarly. If $\mathbf{y} \in B_p(\mathbf{x}, cr)$ then $p(\mathbf{x} - \mathbf{y}) < cr$ so $q(\mathbf{x} - \mathbf{y}) < r$ and hence $\mathbf{y} \in B_q(\mathbf{x}, r)$. In other words,

$$B_p(\mathbf{x}, cr) \subseteq B_q(\mathbf{x}, r).$$

Now $B_p(\mathbf{x}, cr) \in \mathcal{F}$ and $\mathcal{F}$ is upward closed so $B_q(\mathbf{x}, r) \in \mathcal{F}$. There is therefore, for each $r > 0$, an $xinV$ such that $B_q(\mathbf{x}, r) \in \mathcal{F}$. In other words, $\mathcal{F}$ is a Cauchy filter for the norm q.

Conversely, if $\mathcal{F}$ is a Cauchy filter for the norm q then it is a Cauchy filter for the norm p. The proof is the same, but with p and q switched and c replaced by $1/C$.

Suppose (V, p) is a Banach space. If $\mathcal{F}$ is a Cauchy filter for q then it is, by what we've just proved, also a Cauchy filter for p. (V, p) is a Banach space so $\mathcal{F}$ is a convergent filter for p. Convergence is defined purely in terms of the topology and p and q induce the same topology by Proposition 5.1.3 so $\mathcal{F}$ is convergent for q as well. So every Cauchy filter for q is convergent for q. In other words, (V, q) is a Banach space.

Conversely, if (V, q) is a Banach space then (V, p) is a Banach space. The proof is identical, except with p and q switched. □

5.2 Bounded linear transformations

In this section we follow some of the standard notational and terminological conventions of linear algebra. In particular linear functions from a vector space to a vector space will sometimes be called linear transformations. They will generally be written without parentheses, unless the parentheses are required to specify the order of operations. So $A\mathbf{v}$ is the linear transformation A evaluated at the vector $\mathbf{v}$. Also, composition of linear functions will generally be written without the $\circ$ sign. So $(AB)\mathbf{v}$ is the composition of A and B, evaluated at $\mathbf{v}$. By the definition of composition of functions this is the same as A evaluated at the vector $B\mathbf{v}$, i.e. $A(B\mathbf{v})$. We therefore see the the parentheses are unnecessary, since $(AB)\mathbf{v}$ and $A(B\mathbf{v})$ are the same vector. One way in which I will not follow the standard notation of linear algebra is that a superscript $*$ continues to refer to the preimage rather than conjugate transpose, as it often does in linear algebra.

Definition 5.2.1. Suppose (V, p) and (W, q) are normed vector spaces and $A\colon V \to W$ is a linear transformation. $K \geq 0$ is said to be a *bound* for a A if, for all $\mathbf{v} \in V$, we have

$$q(A\mathbf{v}) \leq Kp(\mathbf{v}).$$

A linear transformation is called *bounded* if has a bound.

There is an unfortunate conflict of terminology here, since we already defined boundedness for functions from a set to a metric space. Normed vector spaces are sets and metric spaces, so one might hope that the previous definition would agree with the new one, but it does not. In fact the previous definition, that the image of the function should be a bounded set, is not satisfied by any non-zero linear transformation.

Proposition 5.2.2. *Suppose (V, p) and (W, q) are normed vector spaces and $A\colon V \to W$ is a linear transformation. The following statements are equivalent.*

(a) A is bounded.

(b) A is Lipschitz continuous.

(c) A is uniformly continuous.

(d) A is continuous

(e) A is continuous at $\mathbf{0}$.

(f) There is a $\mathbf{x} \in V$ such that A is continuous at $\mathbf{x}$.

Proof. If K is a bound for A and $\mathbf{x}, \mathbf{y} \in V$ then

$$\begin{aligned} d_W(A\mathbf{x}, A\mathbf{y}) &= q(A\mathbf{x} - A\mathbf{y}) = q(A(\mathbf{x} - \mathbf{y})) \\ &\leq Kq(\mathbf{x} - \mathbf{y}) = Kd_V(\mathbf{x}, \mathbf{y}). \end{aligned}$$

Here d_V and d_W are the expected metrics on V and W, namely the ones coming from p and q respectively. So if A is bounded then it's Lipschitz continuous. Lipschitz continuity implies uniform continuity by Proposition 4.3.3 and uniform continuity implies continuity by the same proposition. Continuity implies continuity at every point by Proposition refcontloc, so in particular it implies continuity at $\mathbf{0}$. If A is continuous at $\mathbf{0}$ then there is certainly an $\mathbf{x} \in V$ such that A is continuous at $\mathbf{x}$. The non-trivial part of the proposition is that the last statement implies the first.

Suppose that A is continuous at $\mathbf{z}$. By Proposition 4.1.6 there is, for every $\epsilon > 0$, a $\delta > 0$ such that

$$B_V(\mathbf{x}, \delta) \subseteq A^*(B_W(A\mathbf{x}, \epsilon)).$$

Choose such an ϵ and δ. If $\mathbf{v} \neq \mathbf{0}$ then set

$$\mathbf{y} = \mathbf{x} + \frac{\delta}{2p(\mathbf{v})}\mathbf{v}.$$

Then

$$\begin{aligned} d_V(\mathbf{y} - \mathbf{x}) &= p(\mathbf{y} - \mathbf{x}) = p\left(\frac{\delta}{2p(\mathbf{v})}\mathbf{v}\right) \\ &= \left|\frac{\delta}{2p(\mathbf{v})}\right| p(\mathbf{v}) = \frac{\delta}{2} < \delta \end{aligned}$$

so

$$\mathbf{y} \in B_V(\mathbf{x}, \delta)$$

and hence

$$\mathbf{y} \in A^*(B_W(A\mathbf{x}, \epsilon)).$$

In other words

$$A\mathbf{y} \in B_W(A\mathbf{x}, \epsilon)$$

or

$$q(A(\mathbf{y} - \mathbf{x})) = q(A\mathbf{y} - A\mathbf{x}) = d_W(A\mathbf{y}, A\mathbf{x}) < \epsilon.$$

Now

$$A\mathbf{v} = A\left(\frac{2p(\mathbf{v})}{\delta}(\mathbf{y} - \mathbf{x})\right) = \frac{2p(\mathbf{v})}{\delta}A(\mathbf{y} - \mathbf{x})$$

so

$$q(A\mathbf{v}) = \left|\frac{2p(\mathbf{v})}{\delta}\right| q(A(\mathbf{y} - \mathbf{x})) \leq \frac{2p(\mathbf{v})}{\delta}\epsilon.$$

Let $K = 2\epsilon/\delta$. Then $K \geq 0$ and

$$q(A\mathbf{v}) \leq Kp(\mathbf{v})$$

for all $\mathbf{v} \neq \mathbf{0}$. This inequality holds trivially though for $\mathbf{v} = \mathbf{0}$ so

$$q(A\mathbf{v}) \leq Kp(\mathbf{v})$$

for all $\mathbf{v} \in V$. K is therefore a bound for A and A is bounded. □

Proposition 5.2.3. *Suppose (V, p) and (W, q) are normed vector spaces and $A\colon V \to W$ is a bounded linear transformation. Then there is a least bound for A.*

Proof. Let S be the set of bounds for A. Then S is non-empty and $K \geq 0$ for all $K \in S$ so

$$L = \inf S$$

exists. If $\epsilon > 0$ then $L + \epsilon > L$ so $L + \epsilon$ is greater than than the greatest lower bound for S and so is not a lower bound. In other words, there is a $K \in S$ such that $K < L + \epsilon$. This K is a bound for A so for all $\mathbf{v} \in V$ we have

$$q(A\mathbf{v}) \leq Kp(\mathbf{v}) \leq (L + \epsilon)p(\mathbf{v}).$$

For any $\mathbf{v} \in V$ we have

$$q(A\mathbf{v}) \leq (L + \epsilon)p(\mathbf{v})$$

for all $\epsilon > 0$ so

$$q(A\mathbf{v}) \leq Lp(\mathbf{v})$$

L is therefore a bound for A, i.e. an element of S. The infimum of S belongs to S and is therefore a minimum of S. □

Proposition 5.2.4. *Suppose (V, p) and (W, q) are normed vector spaces. The set of bounded linear transformations from V to W is a vector space. For each a bounded linear transformation $A\colon V \to W$ let $r(A)$ be the minimum bound for A, the existence of which was proved in the previous proposition. Then r is a norm on the space of bounded linear transformations.*

Proof. The set of bounded linear transformations is a subset of the set of all linear transformations, which is a vector space. To show that the bounded linear transformations are a vector space it therefore suffices to show that any scalar multiple of any bounded linear transformation is bounded and that the sum of any two bounded linear transformations is bounded. This will be done in the course of checking that r satisfies the requirements to be a norm.

We defined bounds to be non-negative so $r(A) \geq 0$ for all A. If A is the zero linear transformation then

$$q(A\mathbf{v}) = q(\mathbf{0}) \leq 0p(\mathbf{v})$$

for all $\mathbf{v} \in V$ so 0 is a bound for A. If must be the least bound so $r(A) = 0$. Conversely, if A is a non-zero linear transformation then there is a $\mathbf{v} \in V$ such that $A\mathbf{v} \neq \mathbf{0}$ and hence $q(A\mathbf{v}) > 0$. This is incompatible with

$$q(A\mathbf{v}) \leq 0p(\mathbf{v})$$

so 0 is not a bound and so the minimum bound $r(A)$ must be positive. This establishes 1.4.1a.

If $\alpha \in \mathbf{R}$, A is a bounded linear transformation from V to W and $\mathbf{v} \in V$ then

$$q((\alpha A)\mathbf{v}) = q(\alpha(A\mathbf{v}) = |\alpha| q(A\mathbf{v}) \leq |\alpha| r(A) p(\mathbf{v}).$$

So $|\alpha| r(A)$ is a bound for αA. It follows that αA is bounded and

$$r(\alpha A) \leq |\alpha| r(A),$$

since the minimum bound is less than or equal to any other bound. If $\alpha \neq 0$ then we also have

$$\begin{aligned} q(A\mathbf{v}) &= q(\alpha^{-1}(\alpha A\mathbf{v})) = \left|\alpha^{-1}\right| q(\alpha A\mathbf{v}) \\ &\leq |\alpha|^{-1} r(\alpha A) p(\mathbf{v}) \end{aligned}$$

so $|\alpha|^{-1} r(\alpha A)$ is a bound for A. Therefore

$$r(A) \leq |\alpha|^{-1} r(\alpha A)$$

since the minimum bound is less than or equal to any other bound. Equivalently,

$$r(\alpha A) \geq |\alpha| r(A).$$

This was proved for $\alpha \neq 0$ but clearly holds for $\alpha = 0$ as well. Combined with the inequality we already obtained, this gives

$$r(\alpha A) \geq |\alpha| r(A).$$

This establishes 1.4.1b.

Next, observe that

$$\begin{aligned} q((A+B)\mathbf{v}) &= q(A\mathbf{v} + B\mathbf{v}) \leq q(A\mathbf{v}) + q(B\mathbf{v}) \\ &\leq r(A)p(\mathbf{v}) + r(B)p(\mathbf{v}) \\ &= (r(A) + r(B))p(\mathbf{v}) \end{aligned}$$

since $r(A)$ and $r(B)$ are bounds for A and B. It follows from

$$q((A+B)\mathbf{v}) \leq (r(A) + r(B))p(\mathbf{v})$$

that $r(A) + r(B)$ is a bound for $A + B$. Therefore $A + B$ is bounded and, since the minimum bound is less than or equal to any other bound,

$$r(A+B) \leq r(A) + r(B).$$

This establishes 1.4.1c. □

We are therefore justified in referring to the minimum bound of A as the *operator norm* of A. The operator norm is submultiplicative in the following sense.

Proposition 5.2.5. *Suppose (U, p_U), (V, p_V) and (W, p_W) are normed vector spaces. Let $p_{V,U}$, $p_{W,V}$ and $p_{W,U}$ be the operator norms on the spaces of bounded linear transformations from U to V, from V to W and from U to W, respectively. If A is a bounded linear transformation from V to W and B is a bounded linear transformation from U to V then*

$$p_{W,U}(AB) \leq p_{W,V}(A) p_{V,U}(B).$$

Proof.

$$\begin{aligned} p_W((AB)\mathbf{v}) &= p_W(A(B\mathbf{v})) \leq p_{W,V}(A) p_V(B\mathbf{v}) \\ &\leq p_{W,V}(A) p_{V,U}(B) p_U(\mathbf{v}) \end{aligned}$$

so $p_{W,V}(A) p_{V,U}(B)$ is a bound for AB and therefore

$$p_{W,U}(AB) \leq p_{W,V}(A) p_{V,U}(B).$$

□

5.3 Equivalence of norms on finite dimensional normed spaces

Proposition 5.3.1. *Suppose that p and q are norms on a finite dimensional vector space V. Then p and q are equivalent.*

Proof. The assumption that V is finite dimensional means that there is a finite basis $\mathbf{u}_1, \ldots \mathbf{u}_n$ for V.

$$p(\mathbf{w}) \leq p(\mathbf{w} - \mathbf{z}) + p(\mathbf{z})$$

and

$$p(\mathbf{z}) \leq p(\mathbf{z}-\mathbf{w}) + p(\mathbf{w})$$

so

$$-p(\mathbf{z}-\mathbf{w}) \leq p(\mathbf{w}) - p(\mathbf{z}) \leq p(\mathbf{w}-\mathbf{z})$$

In other words,

$$|p(\mathbf{w}) - p(\mathbf{z})| \leq p(\mathbf{w}-\mathbf{z}).$$

Define $\mathbf{f}\colon \mathbf{R}^n \to V$ by

$$\mathbf{f}(\mathbf{x}) = \sum_{j=1}^{n} x_j \mathbf{u}_j, \qquad g(\mathbf{x}) = p(\mathbf{f}(\mathbf{x})).$$

Then

$$\begin{aligned}|g(\mathbf{x}) - g(\mathbf{y})| &= |p(\mathbf{f}(\mathbf{x})) - p(\mathbf{f}(\mathbf{y}))| \leq p(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})) \\ &= p(\mathbf{f}(\mathbf{x}-\mathbf{y})) = p\left(\sum_{j=1}^{n} (x_j - y_j)\mathbf{u}_j\right) \\ &\leq \sum_{j=1}^{n} |x_j - y_j| p\left(\mathbf{u}_j\right) \\ &\leq \sqrt{\sum_{j=1}^{n} |x_j - y_j|^2} \sqrt{\sum_{j=1}^{n} p\left(\mathbf{u}_j\right)^2} \\ &= K\|\mathbf{x}-\mathbf{y}\|\end{aligned}$$

where $K = \sqrt{\sum_{j=1}^{n} p\left(\mathbf{u}_j\right)^2}$. The norm in $\|\mathbf{x}-\mathbf{y}\|$ is the usual Euclidean norm on $\mathbf{R}^n$. So g is Lipschitz, hence continuous.

Let $S = \{\mathbf{x} \in \mathbf{R}^n \colon \|\mathbf{x}\| = 1\}$. If $\mathbf{x} \in S$ then $\mathbf{x} \neq \mathbf{0}$ and hence $\mathbf{f}(\mathbf{x}) = \sum_{j=1}^{n} x_j \mathbf{u}_j$ is a non-trivial linear combination of basis vectors of V. So $\mathbf{f}(\mathbf{x}) \neq \mathbf{0}$ and therefore $g(\mathbf{x}) = p(\mathbf{f}(\mathbf{x}) > 0$.

Similarly if q is a norm on V and $h(\mathbf{x}) = q(\mathbf{f}(\mathbf{x}))$ then h is continuous and is positive on S. q/p is therefore a continuous positive function on S.

S is closed and bounded, hence compact, so h/g has a minimum and maximum on S, both of which must both be positive. There are therefore $c, C > 0$ such that

$$c \leq \frac{h(\mathbf{x})}{g(\mathbf{x})} \leq C$$

for all $\mathbf{x} \in S$. If $\mathbf{y} \neq \mathbf{0}$ then

$$\mathbf{x} = \frac{1}{\|\mathbf{y}\|}\mathbf{y}$$

is an element of S.

$$c \leq \frac{h(\mathbf{x})}{g(\mathbf{x})} \leq C$$

and

$$\begin{aligned}\frac{q(\mathbf{f}(\mathbf{y}))}{p(\mathbf{f}(\mathbf{y}))} &= \frac{q(\mathbf{f}(\|\mathbf{y}\|\mathbf{x}))}{p(\mathbf{f}(\|\mathbf{y}\|\mathbf{x}))} = \frac{q(\|\mathbf{y}\|\mathbf{f}(\mathbf{x}))}{p(\|\mathbf{y}\|\mathbf{f}(\mathbf{x}))} \\ &= \frac{\|\mathbf{y}\|q(\mathbf{f}(\mathbf{x}))}{\|\mathbf{y}\|p(\mathbf{f}(\mathbf{x}))} = \frac{h(\mathbf{x})}{g(\mathbf{x})}\end{aligned}$$

so

$$cp(\mathbf{f}(\mathbf{y})) \leq q(\mathbf{f}(\mathbf{y})) \leq Cp(\mathbf{f}(\mathbf{y})).$$

This was proved for $\mathbf{y} \neq \mathbf{0}$ but clearly also holds for $\mathbf{y} = \mathbf{0}$.

$\mathbf{u}_1, \ldots, \mathbf{u}_n$ is a basis for V so if $\mathbf{v} \in V$ then $\mathbf{v} = \sum_{j=1}^{n} y_j \mathbf{u}_j$ for some $y_1, \ldots, y_n$. In other words, $\mathbf{v} = \mathbf{f}(\mathbf{y})$ for some $\mathbf{y}$. Therefore

$$cp(\mathbf{v}) \leq q(\mathbf{v}) \leq Cp(\mathbf{v}).$$

So p and q are equivalent. □

5.4 Useful inequalities

Proposition 5.4.1. *Suppose $\mathbf{w}, \mathbf{x} \in \mathbf{R}^n$ and $w_j \geq 0$ for all j. Suppose that $\varphi\colon \mathbf{R} \to \mathbf{R}$ is strictly convex. Then*

$$\varphi\left(\frac{\sum_{j=1}^{n} w_j x_j}{\sum_{j=1}^{n} w_j}\right) \leq \frac{\sum_{j=1}^{n} w_j \varphi\left(x_j\right)}{\sum_{j=1}^{n} w_j}$$

with equality if and only if all x's are equal.

Proof. This is clear if $n = 1$. For $n > 1$ it is proved by induction. It's convenient to introduce the quantities

$$\alpha_m = \sum_{j=1}^{m} w_j x_j, \quad \beta_m = \sum_{j=1}^{m} w_j,$$

$$\gamma_m = \sum_{j=1}^{m} w_j \varphi\left(x_j\right).$$

Our induction hypothesis is then

$$\varphi((\alpha_k/\beta_k) \leq \gamma_k/\beta_k$$

and we wish to prove the same with k replaced by $k+1$. Let

$$s = \frac{\beta_k}{\beta_{k+1}}, \quad t = \frac{w_{k+1}}{\beta_{k+1}}, \quad \mu = \frac{\alpha_k}{\beta_k}, \quad \nu = x_{k+1}.$$

Then $s, t \geq 0$ and $s + t = 1$ so, by the definition of strict convexity,

$$\varphi(s\mu + t\nu) \leq s\varphi(\mu) + t\varphi(\nu).$$

with equality if and only if $\mu = \nu$. Now

$$s\mu = \frac{\alpha_k}{\beta_{k+1}}, \quad t\nu = \frac{w_{k+1}x_{k+1}}{\beta_{k+1}}, \quad s\mu + t\nu = \frac{\alpha_{k+1}}{\beta_{k+1}},$$

and hence

$$\varphi\left(\frac{\alpha_{k+1}}{\beta_{k+1}}\right) \leq \frac{\beta_k\varphi\left(\alpha_k/\beta_k\right) + w_k\varphi\left(x_{k+1}\right)}{\beta_{k+1}}$$

with equality if and only if

$$\frac{\alpha_k}{\beta_k} = x_{k+1}.$$

Combining this with the induction hypothesis,

$$\varphi\left(\frac{\alpha_{k+1}}{\beta_{k+1}}\right) \leq \frac{\gamma_k + w_k\varphi\left(x_{k+1}\right)}{\beta_{k+1}} = \frac{\gamma_{k+1}}{\beta_{k+1}}$$

with equality if and only if both

$$\frac{\alpha_k}{\beta_k} = x_{k+1}.$$

and $x_1 = x_2 = \cdots = x_k$. This happens if and only if $x_1 = x_2 = \cdots = x_{k+1}$, so the inductive proof is complete. ☐

We are mostly interested in the special case

$$\sum_{j=1}^{n} w_j = 1,$$

in which case the inequality simplifies to

$$\varphi\left(\sum_{j=1}^{n} w_j x_j\right) \leq \sum_{j=i}^{n} w_j \varphi\left(x_j\right).$$

This is known as *Jensen's Inequality*. Clearly we can allow some of the w's to be zero, but nothing is really gained by doing so.

Corollary 5.4.2. *Suppose* $\mathbf{w}, \mathbf{x} \in \mathbf{R}^n$ *and* $a_j \geq 0$ *and* $w_j \geq 0$ *for all* j.

$$\prod_{j=1}^{n} a_j^{w_j} \leq \sum_{j=1}^{n} w_j a_j$$

with equality if and only if all a's are equal.

Proof. If any of the a's are zero then then the product on the left is zero while the sum on the right is non-negative, so the inequality holds. If all a's are positive then we take $x_j = \log a_j$ and $\varphi = \exp$ in Jensen's Inequality. ☐

The special case $w_j = 1/n$,

$$\left(\prod_{j=1}^{n} a_j\right)^{1/n} \leq \frac{\sum_{j=1}^{n} a_j}{n},$$

is known as the *Arithmetic-Geometric Mean Inequality*.

Proposition 5.4.3. *Suppose* B *is an* $m \times n$ *matrix,* $w \in \mathbf{R}^n$, *and* $b_{j,k} \geq 0$, $w_j > 0$ *for all* k *and* j. *Suppose also that* $\sum_{k=1}^{n} w_k = 1$ *Then*

$$\sum_{j=1}^{m} \prod_{k=1}^{n} b_{j,k} \leq \prod_{k=1}^{n} \left(\sum_{j=1}^{m} b_{j,k}^{1/w_k}\right)^{w_k}.$$

Proof. We begin by noting that if there is a k such that $b_{j,k} = 0$ for all j then both the left and right hand sides of the inequality are both zero and so the proposition holds. We therefore only need to consider the case where for each k there is a j with $b_{j,k} > 0$. Let

$$c_{j,k} = b_{j,k}^{w_k}.$$

Then $c_{j,k} \geq 0$ and $c_{j,k} > 0$ if and only if $b_{j,k} > 0$ so we only need to consider the case where for each k there is a j with $c_{j,k} > 0$. We may therefore apply the inequality

$$\prod_{k=1}^{n} a_k^{w_k} \leq \sum_{k=1}^{n} w_k a_k$$

to

$$a_k = \frac{c_{j,k}}{\sum_{i=1}^{m} c_{i,k}}$$

since the denominator is non-zero. We find

$$\prod_{k=1}^{n}\left(\frac{c_{j,k}}{\sum_{i=1}^{m} c_{i,k}}\right)^{w_k} \le \sum_{k=1}^{n} w_k \frac{c_{j,k}}{\sum_{i=1}^{m} c_{i,k}}.$$

Summing over $1 \le j \le m$ gives

$$\sum_{j=1}^{m}\prod_{k=1}^{n}\left(\frac{c_{j,k}}{\sum_{i=1}^{m} c_{i,k}}\right)^{w_k} \le 1$$

or

$$\sum_{j=1}^{m}\prod_{k=1}^{n} c_{j,k}^{w_k} \le \prod_{k=1}^{n}\left(\sum_{i=1}^{m} c_{i,k}\right)^{w_k}.$$

This is just

$$\sum_{j=1}^{m}\prod_{k=1}^{n} b_{j,k} \le \prod_{k=1}^{n}\left(\sum_{i=1}^{m} b_{i,k}^{1/w_k}\right)^{w_k}.$$

This, except for the name of the index of summation on the right hand side, is exactly the inequality from the proposition. □

The proposition below is known as *Hölder's Inequality*.

Proposition 5.4.4. *Suppose that* $\mathbf{x}, \mathbf{y} \in \mathbf{R}^m$, *and that* $p, q \in (1, +\infty)$ *are such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$\left|\sum_{j=1}^{m} x_j y_j\right| \le \left(\sum_{j=1}^{m} |x_j|^p\right)^{1/p} \left(\sum_{j=1}^{m} |y_j|^q\right)^{1/q}.$$

Proof. We apply the preceding inequality with $n = 2$, $w_1 = 1/p$, $w_2 = 1/q$, $b_{j,1} = |x_j|$ and $b_{j,2} = |y_j|$. This produces

$$\sum_{j=1}^{m} |x_j y_j| \le \left(\sum_{j=1}^{m} |x_j|^p\right)^{1/p} \left(\sum_{j=1}^{m} |y_j|^q\right)^{1/q}.$$

The result now follows from the elementary inequality

$$\left|\sum_{j=1}^{m} x_j y_j\right| \le \sum_{j=1}^{m} |x_j y_j|.$$

□

The special case $p = q = 2$ is known as the *Cauchy-Schwarz Inequality*.

$$\left|\sum_{j=1}^{m} x_j y_j\right| \le \sqrt{\sum_{j=1}^{m} |x_j|^2}\sqrt{\sum_{j=1}^{m} |y_j|^2}.$$

The following is known as *Minkowski's Inequality*.

Proposition 5.4.5. *Suppose* $r \ge 1$ *and* A *is an* $m \times n$ *matrix. Let* $\mathbf{z} \in \mathbf{R}^m$ *be defined by*

$$z_j = \sum_{k=1}^{n} a_{j,k}.$$

Then

$$\left(\sum_{j=1}^{m} |z_j|^r\right)^{1/r} \le \sum_{k=1}^{n}\left(\sum_{j=1}^{m} |a_{j,k}|^r\right)^{1/r}.$$

Proof. We begin by noting that

$$|z_j| = \left|\sum_{k=1}^{n} a_{j,k}\right| \le \sum_{k=1}^{n} |a_{j,k}|.$$

Summing this over j from 1 to m gives the case $r = 1$ of the proposition, so it only remains to consider the case $r > 1$. Then

$$\begin{aligned}|z_j|^r = |z_j|^{r-1}|z_j| &\le |z_j|^{r-1}\sum_{k=1}^{n} |a_{j,k}| \\ &= \sum_{k=1}^{n} |z_j|^{r-1}|a_{j,k}|.\end{aligned}$$

Summing over j,

$$\sum_{j=1}^{m} |z_j|^r \le \sum_{j=1}^{m}\sum_{k=1}^{n} |z_j|^{r-1}|a_{j,k}| = \sum_{k=1}^{n}\sum_{j=1}^{m} |z_j|^{r-1}|a_{j,k}|$$

We apply Hölder's Inequality with $p = r/(r-1)$, $q = r$, $x_j = |z_j|^{r-1}$ and $y_j = |a_{j,k}|$. This gives

$$\sum_{j=1}^{m} |z_j|^{r-1}|a_{j,k}| \le \left(\sum_{j=1}^{m} |z_j|^r\right)^{\frac{r-1}{r}} \left(\sum_{j=1}^{m} |a_{j,k}|^r\right)^{1/r}.$$

Therefore

$$\sum_{j=1}^{m}|z_j|^r \le \sum_{k=1}^{n}\left(\sum_{j=1}^{m}|z_j|^r\right)^{\frac{r-1}{r}}\left(\sum_{j=1}^{m}|a_{j,k}|^r\right)^{1/r}$$
$$= \left(\sum_{j=1}^{m}|z_j|^r\right)^{\frac{r-1}{r}}\sum_{k=1}^{n}\left(\sum_{j=1}^{m}|a_{j,k}|^r\right)^{1/r}.$$

Dividing both sides by $\left(\sum_{j=1}^{m}|z_j|^r\right)^{\frac{r-1}{r}}$ gives

$$\left(\sum_{j=1}^{m}|z_j|^r\right)^{1/r} \le \sum_{k=1}^{n}\left(\sum_{j=1}^{m}|a_{j,k}|^r\right)^{1/r},$$

which is what we were looking for. □

5.5 Inequalities for infinite sums

All of the inequalities above have analogues for infinite sums. Of course infinite sums needn't converge so we need to be careful to note which sums we need to assume converge in the hypotheses of the propositions and which sums converge as part of the conclusion of the propositions.

Proposition 5.5.1. *Suppose that $p, q \in (1, +\infty)$ are such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Suppose that S is a set and f, g are functions from S to $\mathbf{R}$ such that

$$\sum_{s\in S}|f(s)|^p$$

and

$$\sum_{s\in S}|g(s)|^q$$

are convergent. Then

$$\sum_{s\in S} f(s)g(s)$$

is also convergent and

$$\left|\sum_{s\in S} f(s)g(s)\right| \le \left(\sum_{s\in S}|f(s)|^p\right)^{1/p}\left(\sum_{s\in S}|g(s)|^q\right)^{1/q}.$$

Proof. This will require some properties of infinite sums which we won't prove until a later section, but the proofs there won't depend on anything in this section so the argument isn't circular.

Let

$$u = \sum_{s\in S}|f(s)|^p$$

and

$$v = \sum_{s\in S}|g(s)|^q.$$

These exist, by the hypotheses of the proposition. The sums are of non-negative terms so in fact

$$u = \sup \sum_{s\in H}|f(s)|^p$$

and

$$v = \sup \sum_{s\in H}|g(s)|^q,$$

where the supremum is over finite subsets H of S. For each such H we have

$$\sum_{s\in H}|f(s)g(s)| \le \left(\sum_{s\in H}|f(s)|^p\right)^{1/p}\left(\sum_{s\in H}|g(s)|^q\right)^{1/q}$$
$$\le u^{1/p}v^{1/q}$$

by the finite version of Hölder's Inequality, which we proved in the last section. This holds for all F so

$$\sup \sum_{s\in H}|f(s)g(s)| \le u^{1/p}v^{1/q},$$

where the sum is again over all finite $H \subseteq S$. Using again the fact that this is a sum of non-negative terms we get

$$\sum_{s\in S}|f(s)g(s)| = \sup \sum_{s\in H}|f(s)g(s)|.$$

As with series, absolute convergence implies convergence so

$$\sum_{s\in S} f(s)g(s)$$

exists. The same theorem shows that

$$\left|\sum_{s\in S} f(s)g(s)\right| \le \sum_{s\in S}|f(s)g(s)| \le u^{1/p}v^{1/q}.$$

□

In a similar way we get the infinite version of Minkowski's Inequality.

$$\left(\sum_{s\in S}\left|\sum_{k=1}^{n} a_k(s)\right|^r\right)^{1/r} \le \sum_{k=1}^{n}\left(\sum_{s\in S}|a_k(s)|^r\right)^{1/r}.$$

The sum over $s \in S$ on the left hand side converges provided that the sums on the right do.

5.6 The spaces $\ell^p(\mathbf{N})$

Definition 5.6.1. Suppose $p \in [1, +\infty)$. Then $\ell^p(\mathbf{N})$ is the set of functions $\alpha\colon \mathbf{N} \to \mathbf{R}$, i.e. sequences in $\mathbf{R}$, such that

$$\sum_{j=0}^{\infty} |\alpha_j|^p$$

is convergent.

Proposition 5.6.2. *Then*

$$\|\alpha\|_p = \left(\sum_{j=0}^{\infty} |\alpha_j|^p\right)^{1/p}$$

is a norm on $\ell^p(\mathbf{N})$.

Proof. The first two properties of norms are straightforward. The third is *Minkowski's inequality.* □

$\ell^p(\mathbf{N})$ also complete, as will be proved in a later section, and so $\ell^p(\mathbf{N})$ is a Banach space.

Proposition 5.6.3. *Suppose* $1 \le p \le q < +\infty$ *Then*

$$\ell^p(\mathbf{N}) \subseteq \ell^q(\mathbf{N}),$$

and

$$\|\alpha\|_q \le \|\alpha\|_p.$$

Also,

$$\ell^p(\mathbf{N}) \subset \ell^q(\mathbf{N})$$

if $p < q$.

Proof. Suppose $\alpha \in \ell^p(\mathbf{N})$. If $\alpha \ne 0$ then set $\beta_j = \alpha_j/\|\alpha\|_p$. Then

$$\sum_{j=0}^{\infty} |\beta_j|^p = \sum_{j=0}^{p} |\alpha_j|^p/\|\alpha\|_p = \|\alpha\|_p/\|\alpha\|_p = 1.$$

Each summand is non-negative so $|\beta_j|^p \le 1$. It follows that

$$|\beta_j|^q = (|\beta_j|^p)^{q/p} \le |\beta_j|^p$$

Multiplying by $\|\alpha\|_p^q$,

$$|\alpha_j|^q \le |\alpha_j|^p \|\alpha\|_p^{p-q}$$

By the comparison test $\sum_{j=0}^{\infty} |\alpha_j|^q$ is convergent and $\|\alpha\|_q^q \le \|\alpha\|_p^p \|\alpha\|_p^{p-q}$ This also holds if $\alpha = 0$. In other words,

$$\ell^p(\mathbf{N}) \subseteq \ell^q(\mathbf{N}), \qquad , \|\alpha\|_q \le \|\alpha\|_p,$$

as promised.

Define $\gamma\colon \mathbf{N} \to \mathbf{R}$ by $\gamma_j = 2^{-n/r}$ if $2^n \le j < 2^{n+1}$. Then

$$\sum_{j=0}^{\infty} |\gamma_j|^q = \sum_{n=0}^{\infty} 2^n 2^{-nq/r} = \sum_{n=0}^{\infty} 2^{-n(q-r)/r}.$$

If $q > r$ then this geometric series converges. If $q \le r$ then it doesn't. So $\gamma \in \ell^q(\mathbf{N})$ exactly for $q > r$ If $p < q$ then we therefore have a strict inclusion

$$\ell^p(\mathbf{N}) \subset \ell^q(\mathbf{N})$$

□

Another way to state the proposition above is that the inclusion function $i\colon \ell^p(\mathbf{N}) \to \ell^q(\mathbf{N})$ is a continuous injection whose image is a proper (linear) subspace. It's continuous because $\|\alpha\|_q \le \|\alpha\|_p$, so $K = 1$ is a bound.

Proposition 5.6.4. *Let F be the subset of $\ell^q(\mathbf{N})$ consisting of sequences with only finitely many non-zero elements. Then F is dense in $\ell^q(\mathbf{N})$ for all $q \ge 1$. Also, if $1 \le p < q$ then $\ell^p(\mathbf{N})$ is also a dense proper (linear) subspace of $\ell^q(\mathbf{N})$.*

Proof. Suppose $\alpha \in \ell^q(\mathbf{N})$. Define $\alpha^{[k]} \in \ell^q(\mathbf{N})$ by

$$\alpha_j^{[k]} = \begin{cases} \alpha_j & \text{if } j < k, \\ 0 & \text{if } j \geq k. \end{cases}$$

$$\alpha_j - \alpha_j^{[k]} = \begin{cases} \alpha_j & \text{if } j \geq k, \\ 0 & \text{if } j < k. \end{cases}$$

So

$$\|\alpha - \alpha^{[k]}\|_q^q = \sum_{j=k}^{\infty} |\alpha_j|^q.$$

This tends to zero as k tends to infinity, because $\sum_{j=0}^{\infty} |\alpha_j|^q$ is convergent, so

$$\lim_{k\to\infty} \|\alpha - \alpha^{[k]}\|_q^q = 0$$

from which it follows that $\|\alpha - \alpha^{[k]}\|_q \to 0$ and $\alpha^{[k]} \to \alpha$. $\alpha^{[k]} \in F$ for all k and $\alpha^{[k]} \to \alpha$ so $\alpha \in \overline{F}$. $\alpha \in \ell^q(\mathbf{N})$ was arbitrary, so $\overline{F} = \ell^q(\mathbf{N})$. In other words, F is dense in $\ell^q(\mathbf{N})$ for all $q \geq 1$.

If $1 \leq p < q$ then

$$F \subseteq \ell^p(\mathbf{N}) \subseteq \ell^q(\mathbf{N})$$

so $\ell^p(\mathbf{N})$ is also a dense (linear) subspace of $\ell^q(\mathbf{N})$. We've already seen that it's a proper subset. □

If your intuition is based on finite dimensional normed spaces then it can be hard to imagine a dense proper subspace!

Proposition 5.6.5. *Closed balls in $\ell^p(\mathbf{N})$ are not compact for any p.*

Proof. Suppose $\alpha \in \ell^p(\mathbf{N})$ and $r > 0$. For each $k \in \mathbf{R}$ define $\beta^{[k]} \in \ell^p(\mathbf{N})$ by

$$\beta_j^{[k]} = \begin{cases} \alpha_j + r & \text{if } j = k, \\ \alpha_j & \text{if } j \neq k. \end{cases}$$

Then

$$\|\beta^{[k]} - \alpha\| = r$$

and

$$\|\alpha^{[k]} - \alpha^{[l]}\| = 2^{1/p} r$$

for all $k, l \in \mathbf{N}$. It follows that $\beta^{[k]} \in \bar{B}(\alpha, r)$ for each k. Let C be the set of all $\beta^{[k]}$ as k ranges over $\mathbf{N}$. Then C is a closed subset of $\bar{B}(\alpha, r)$. For any $k, l \in \mathbf{N}$ and any $\xi \in \bar{B}(\alpha, r)$ we have

$$\|\alpha^{[k]} - \xi\|_p \|\xi - \alpha^{[l]}\|_p \geq \|\alpha^{[k]} - \alpha^{[l]}\|_p = 2^{1/p} r$$

so at least one of $\|\alpha^{[k]} - \xi\|_p$ or $\|\xi - \alpha^{[l]}\|_p$ is greater than or equal to $2^{1/p-1} r$. Equivalently, at most one of the $\alpha^{[m]}$ belongs to $B(\xi, 2^{1/p-1} r)$, so there is no finite collection of balls of that radius which covers $\bar{B}(\alpha, r)$. Therefore $\bar{B}(\alpha, r)$ is not totally bounded and so is not compact. □

Proposition 5.6.6. *$\| \ \|_p$ and $\| \ \|_q$ are inequivalent norms on $\ell^p(\mathbf{N})$ if $1 \leq p < q$.*

Proof. Choose $r \in (p, q)$ and define $\gamma_j = 2^{-n/r}$ if $2^n \leq j < 2^{n+1}$ as before and then

$$\gamma_j^{[k]} = \begin{cases} \gamma_j & \text{if } j < k, \\ 0 & \text{if } j \geq k. \end{cases}$$

Then $\lim_{k\to\infty} \gamma^{[k]} = \gamma$ in $\ell^q(\mathbf{N})$ but the sequence is unbounded, and hence does not converge, in $\ell^p(\mathbf{N})$. Convergence is defined in terms of the topology so the two topologies are different, but equivalent norms give rise to the same topology. □

In fact all the (uncountably many) norms $\| \ \|_r$ for $p \leq r \leq q$ are inequivalent.

Proposition 5.6.7. *The inclusion $i \colon \ell^p(\mathbf{N}) \to \ell^q(\mathbf{N})$ is an injection for $1 \leq p < q$, but has no bounded left inverse.*

6 Infinite sums

Infinite sums of functions defined on arbitrary sets were defined in an earlier section as limits of nets, with the directed subset being the set of finite subsets and the net being the function with assigns to each finite subset the sum over that subset. We already derived those properties of sums which are immediate consequences of the properties of limits, namely uniqueness, linearity and monotonicity. There are a number of other properties of series which we would like to extend to infinite sums, such as the comparison test, which are not immediate consequences of any familiar property of limits. We will derive those properties in this chapter. Many properties of sums hold for sums with values in a normed vector space while others require a Banach space or even a finite dimensional space. For simplicity we'll consider only real valued sums, although it's convenient to consider also sums with values in the extended reals, which we now define.

6.1 The extended reals

Sums of real numbers can fail to converge in either of two ways. $\sum_{j=1}^{\infty} j$ fails to converge because the partial sums grow without bound while $\sum_{j=1}^{\infty}(-1)^j$ has partial sums which are bounded, but oscillatory. We'd like to distinguish between these two situations, both for sums and later for integrals. One way to do this is to work with the extended real numbers and allow sums to be infinite.

We write $[-\infty, +\infty]$ for the set consisting of the $\mathbf{R}$ and two additional points, labeled $+\infty$ and $-\infty$. The order structure and topology on the reals are extended to the set $[-\infty, +\infty]$ and the arithmetic operations are partially extended.

We extend the order relation by saying that

$$-\infty \leq x \leq +\infty$$

for all $x \in [-\infty, +\infty]$. In addition to the empty set we have four types of intervals:

- $(a, b) = \{x \in [-\infty, +\infty] \colon a < x < b\}$,
- $[a, b) = \{x \in [-\infty, +\infty] \colon a \leq x < b\}$,
- $(a, b] = \{x \in [-\infty, +\infty] \colon a < x \leq b\}$,
- $[a, b] = \{x \in [-\infty, +\infty] \colon a \leq x \leq b\}$.

Here $a, b \in [-\infty, +\infty]$. This notation is consistent with the notation for intervals in $\mathbf{R}$. It is also self-consistent, since $-\infty \leq x \leq +\infty$ for all $x \in [-\infty, +\infty]$. The open intervals in $[-\infty, +\infty]$ are those of the form (a, b), $(a, +\infty]$, $[-\infty, b)$, $[-\infty, +\infty]$ or $\varnothing$, where $a, b \in [+\infty, -\infty]$. A subset of $[-\infty, +\infty]$ is said to be open if it is a union of open intervals. As in $\mathbf{R} = (-\infty, +\infty)$ we define upper and lower bounds in terms of the order structure and we define infima and suprema to be greatest lower bounds and least upper bounds, respectively.

The following propositions illustrate some ways in which $[-\infty, +\infty]$ is better behaved than $\mathbf{R} = (-\infty, +\infty)$.

Proposition 6.1.1. *Every subset of $[-\infty, +\infty]$ has an infimum and a supremum in $[-\infty, +\infty]$.*

Proof. Suppose $A \in \wp([-\infty, +\infty])$. One of the following three statements is true:

1. $+\infty$ is a supremum for A.
2. $x \leq -\infty$ for all $x \in A$.
3. $+\infty$ is not a supremum for A and there is an $x \in A$ such that $x > -\infty$.

In each case we can show that A has a supremum. In the first case that supremum is $+\infty$.

In the second case $-\infty$ is an upper bound. There are no elements of $[-\infty, +\infty]$ less than $-\infty$ and hence no upper bounds of A less than $-\infty$. $-\infty$ is therefore an infimum of A.

The interesting case is the third one. $+\infty$ is an upper bound for A because there are no larger elements in $[-\infty, +\infty]$. If $+\infty$ were an element of A then nothing less than $+\infty$ could be an upper bound and so $+\infty$ would be a supremum and we would not be in the third case. Therefore $+\infty \notin A$. There is an $x \in A$ such that $x > -\infty$ and we've just seen that $+\infty \notin A$ so $x \neq +\infty$. Therefore $-\infty < x < +\infty$, i.e. $x \in \mathbf{R}$. Thus $\mathbf{R} \cap A$ is non-empty. $+\infty$ is an upper bound for A but not a least upper bound so there is an upper bound, which we can call z. Thus $w \leq z < +\infty$ for all $w \in A$. In particular, $x \leq z < +\infty$ so $-\infty < z < +\infty$, i.e. $z \in \mathbf{R}$. Now $w \leq z$ for all $w \in A$ and hence for all $w \in \mathbf{R} \cap A$. So z is an upper bound in $\mathbf{R}$ for $\mathbf{R} \cap A$. $\mathbf{R} \cap A$ is a non-empty subset of $\mathbf{R}$ which is bounded by $z \in \mathbf{R}$ and hence has a least upper bound in $\mathbf{R}$. Call this bound y. y is an upper bound for $\mathbf{R} \cap A$. If $w \in A$ then $w \neq +\infty$, so $w \in \mathbf{R} \cap A$ or $w = -\infty$. In either case $w \leq y$, so y is an upper bound for A as well. Any lesser upper bound for A would also be an upper bound for $\mathbf{R} \cap A$, which is impossible because y is a least upper bound for $\mathbf{R} \cap A$. So y is a least upper bound for A, i.e. a supremum.

The proof that A has an infimum is identical, except that the roles of $+\infty$ and $-\infty$ are reversed and all inequalities run in the opposite direction. $\square$

Every subset of $\mathbf{R}$ is a subset of $[-\infty, +\infty]$ so it follows from the proposition that every subset of $\mathbf{R}$ has an infimum and a supremum in $[-\infty, +\infty]$, although it may not have an infimum or supremum in $\mathbf{R}$.

Proposition 6.1.2. *Suppose $(D, \preccurlyeq)$ is a non-empty directed set and $\varphi\colon D \to [-\infty, +\infty]$ is a monotone function. Then φ converges to $\sup \varphi_*(D)$.*

Proof. Let $z = \sup \varphi_*(D)$. Suppose U is an open neighbourhood of z. U is a union of open intervals so there is an open interval I such that $z \in I$ and $I \subseteq U$.

If $z = -\infty$ then $\varphi(a) \leq -\infty$ for all $a \in D$ and hence $\varphi(a) = -\infty$ for all $a \in D$. Constant nets are always convergent, so in this case the conclusion of the proposition follows trivially. We can therefore restrict our attention to the case $z > -\infty$.

$z \in I$ so $I \neq \varnothing$. It must of one of the four types (α, β), $[\alpha, \beta)$, $(\alpha, \beta]$ or $[\alpha, \beta]$. If $\alpha = -\infty$ and $z = +\infty$ then $0 \in I$ and $0 < z$. If $\alpha = -\infty$ and $z < +\infty$ then $z - 1 \in I$ and $z - 1 < z$. If $\alpha > -\infty$ then $(\alpha + z)/2 \in I$ and $(\alpha + z)/2 < z$. For the latter inequality we need to use the fact that I is an open interval rather than just an interval, so cannot be of the from $[\alpha, \beta)$ or $[\alpha, \beta]$ if $\alpha > -\infty$. So in every case there is a $w \in I$ such that $w < z$. z is a least upper bound for $\varphi_*(D)$ so w is not an upper bound for $\varphi_*(D)$. In other words, there is an $x \in \varphi_*(D)$ such that $x > w$. $x \in \varphi_*(D)$ so there is an $a \in D$ such that $\varphi(a) = x$, and hence $\varphi(a) > w$. φ is monotone so

$$\varphi(b) \geq \varphi(b) \geq \varphi(a) > w$$

for all $b \in D$ such that $a \preccurlyeq b$. Now z is an upper bound for $\varphi_*(D)$ so $\varphi(b) \leq z$. $w, z \in I$ and $w \leq \varphi(b) \leq z$ so $\varphi(b) \in I$ and hence $\varphi(b) \in U$. So for every open neighbourhood U of z there is an $a \in D$ such that $\varphi(b) \in U$ whenever $a \preccurlyeq b$. Therefore φ converges to z. $\square$

Corollary 6.1.3. *Every monotone sequence in $[-\infty, +\infty]$ converges.*

Proof. Sequences are nets. $\square$

Proposition 6.1.4. *Suppose S is a set and $u\colon S \to [0, +\infty]$ is a function. Then $\sum_{s \in S} u(s)$ converges to*

$$\sup \sum_{s \in F} u(s)$$

where the supremum is over all finite $F \subseteq S$.

Proof. Let D be the set of finite subsets of S, ordered by $\subseteq$. Define $\varphi\colon D \to [0, +\infty]$ by

$$\varphi(F) = \sum_{s \in F} u(s).$$

If $F, G \in \mathcal{D}$ and $F \subseteq G$ then

$$\varphi(F) = \sum_{s \in F} u(s) \leq \sum_{s \in G} u(s) = \varphi(G)$$

so φ is monotone. This is the point at which we use the fact that $\varphi(s) \geq 0$ for all s, and so in particular for $s \in G \setminus F$. The limit of φ is $\sum_{s \in S} u(s)$, by the definition of sums. We can now apply the previous proposition to conclude that $\sum_{s \in S} u(s)$ converges to

$$\sup \sum_{s \in F} u(s).$$

$\square$

The arithmetic operations are extended to $[-\infty, +\infty]$ in more or less the way one might expect. Addition is defined by

$$w + +\infty = +\infty + w = +\infty,$$
$$x + -\infty = -\infty + x = -\infty$$

for $w \in (-\infty, +\infty]$ and $x \in [-\infty, +\infty)$. $+\infty + -\infty$ and $-\infty + +\infty$ are deliberately left undefined. The additive inverse is defined by

$$- + \infty = -\infty, \qquad - - \infty = +\infty$$

and subtraction is defined in the usual way in terms of these operations:

$$u - v = u + (-v).$$

Multiplication is defined by

$$\begin{aligned} y \cdot +\infty &= +\infty \cdot y = +\infty, \\ z \cdot +\infty &= z \cdot +\infty = -\infty \\ y \cdot -\infty &= -\infty \cdot y = -\infty, \\ z \cdot -\infty &= z \cdot -\infty = +\infty \end{aligned}$$

for $y \in (0, +\infty]$ and $z \in [-\infty, 0)$ and

$$0 \cdot +\infty = +\infty \cdot 0 = 0 = 0 \cdot -\infty = -\infty \cdot 0.$$

We define the multiplicative inverse by

$$(+\infty)^{-1} = 0 = (-\infty)^{-1}$$

and division by

$$u/v = uv^{-1}.$$

Division by zero remains undefined.

These definitions preserve most of the algebraic properties of the real numbers, including the commutative, associative and distributive laws. They don't fully obey the usual cancellation laws though. From $x + z = y + z$ it doesn't follow, for example that $x = y$, since $0 + +\infty = 1 + +\infty$ but $0 \neq 1$. The arithmetic operations are continuous everywhere they're defined, with the exception of multiplication at the points $(0, +\infty)$, $(0, -\infty)$, $(+\infty, 0)$ and $(-\infty, 0)$.

One consequence of the continuity of addition is the limit of a finite sum is the sum of the limits in $[-\infty, +\infty]$, just as it is for limits in $\mathbf{R}$, provided the sum of the limits is defined, i.e. doesn't involve both $+\infty$ and $-\infty$ as summands.

6.2 Comparison

Theorem 6.2.1. *Suppose that u and v are functions from a set S to $\mathbf{R}$. If*

$$|u(s)| \leq |v(s)|$$

for all $s \in S$ and

$$\sum_{s \in S} |v(s)| < +\infty$$

then

$$\sum_{s \in S} u(s)$$

converges.

Proof. $\sum_{s \in S} |v(s)|$ is a sum of non-negative terms so it must converge to some element of $[0, +\infty]$. The assumption

$$\sum_{s \in S} |v(s)| < +\infty$$

means that it converges to an element of $[0, +\infty)$. By the definition of sums this means that the net of partial sums is convergent. It must therefore be Cauchy. In other words, for each $\epsilon > 0$ there is a finite subset F of S such that if $F \subseteq G$ and $F \subseteq H$ then

$$\left| \sum_{s \in G} |v(s)| - \sum_{s \in H} |v(s)| \right| < \epsilon.$$

This holds in particular for $H = F$, so

$$\sum_{s \in G} |v(s)| - \sum_{s \in F} |v(s)| < \epsilon$$

if $F \subseteq G$. Removing the outer layer of absolute value signs is permissible because

$$\sum_{s \in G} |v(s)| - \sum_{s \in F} |v(s)| = \sum_{s \in G \setminus F} |v(s)|$$

is a sum of non-negative terms and hence non-negative. Now $-|v(s)| \leq u(s) \leq |v(s)|$ for all $s \in S$ and hence

$$-\epsilon < -\sum_{s \in G \setminus F} |v(s)| \leq \sum_{s \in G \setminus F} u(s) \leq \sum_{s \in G \setminus F} |v(s)| < \epsilon.$$

We can rewrite this as

$$-\epsilon < \sum_{s \in G} u(s) - \sum_{s \in F} u(s) < \epsilon$$

Similarly,

$$-\epsilon < \sum_{s \in H} u(s) - \sum_{s \in F} u(s) < \epsilon$$

if $F \subseteq H$. Writing

$$\begin{aligned} \sum_{s \in G} u(s) - \sum_{s \in H} u(s) = &\left(\sum_{s \in G} u(s) - \sum_{s \in F} u(s) \right) \\ &- \left(\sum_{s \in H} u(s) - \sum_{s \in F} u(s) \right) \end{aligned}$$

we see that

$$-2\epsilon < \sum_{s\in G} u(s) - \sum_{s\in H} u(s) < 2\epsilon$$

or

$$\left|\sum_{s\in G} u(s) - \sum_{s\in H} u(s)\right| < 2\epsilon.$$

Thus the net of partial sums of u is Cauchy. Since this takes values in $\mathbf{R}$ and $\mathbf{R}$ is complete we conclude that this net of partial sums is convergent. Using the definition of infinite sums again we see that $\sum_{s\in S} u(s)$ converges. $\square$

Two special cases are worth singling out.

Corollary 6.2.2. *Suppose that u and v are functions from a set S to $\mathbf{R}$ and $[0,+\infty)$ respectively. If*

$$|u(s)| \leq v(s)$$

for all $s \in S$ and

$$\sum_{s\in S} v(s) < +\infty$$

then

$$\sum_{s\in S} u(s)$$

converges.

Proof. In this case $|v(s)| = v(s)$. $\square$

Corollary 6.2.3. *Suppose that u is a function from a set S to $\mathbf{R}$. If*

$$\sum_{s\in S} |u(s)| < +\infty$$

then

$$\sum_{s\in S} u(s)$$

converges.

Proof. This is just the special case $u = v$ of the theorem. $\square$

This corollary says that absolutely convergent sums are convergent. We know that series can be convergent without being absolutely convergent so it is perhaps surprising to see that this corollary has a converse.

Proposition 6.2.4. *Suppose that u is a function from a set S to $\mathbf{R}$. If*

$$\sum_{s\in S} u(s)$$

converges then

$$\sum_{s\in S} |u(s)| < +\infty.$$

Proof. Since

$$\sum_{s\in S} u(s)$$

converges it is Cauchy, i.e. for any $\epsilon > 0$ there is a finite $F \subseteq S$ such that if $G, H \subseteq S$ are finite, $F \subseteq G$ and $F \subseteq H$ then

$$\left|\sum_{s\in G} u(s) - \sum_{s\in H} u(s)\right| < \epsilon.$$

We choose any $\epsilon > 0$ and a corresponding F. Having done so, suppose $K \subseteq S$ is finite. Define

$$G = \{s \in F \cup K : s \in F \text{ or } u(s) > 0\}$$

and

$$H = \{s \in F \cup K : s \in F \text{ or } u(s) < 0\}$$

Then G, H are finite, $F \subseteq G$ and $F \subseteq H$ so we have

$$\left|\sum_{s\in G} u(s) - \sum_{s\in H} u(s)\right| < \epsilon.$$

If $s \in F$ then $u(s)$ appears in both sums, with opposite signs, so those terms cancel. If $s \in K \setminus F$ then $u(s)$ appears in at most one of the sums, with a sign which gives us a net contribution of $|u(s)|$. We also get such a contribution, trivially, if $u(s) = 0$, in which case s is an element of neither G nor H. So

$$\sum_{s\in G} u(s) - \sum_{s\in H} u(s) = \sum_{s\in K\setminus F} |u(s)|.$$

Taking the absolute value of the sum on the right would have no effect, since this is a sum of non-negative terms. It follows that

$$\sum_{s\in K\setminus F} |u(s)| < \epsilon$$

and therefore

$$\begin{aligned}\sum_{s\in K} |u(s)| &= \sum_{s\in K\cap F} |u(s)| + \sum_{s\in K\setminus F} |u(s)| \\ &< \sum_{s\in F} |u(s)| + \epsilon.\end{aligned}$$

This bound is independent of K so

$$\sup \sum_{s\in K} |u(s)| \le \sum_{s\in F} |u(s)| + \epsilon.$$

and hence

$$\sum_{s\in S} |u(s)| \le \sum_{s\in F} |u(s)| + \epsilon < +\infty.$$

□

6.3 Convergence theorems for sums

Suppose S is a set and u a sequence of real valued functions on S. Is it true that

$$\lim_{n\to\infty} \sum_{s\in S} u_n(s) = \sum_{s\in S} \lim_{n\to\infty} u_n(s)?$$

Without further assumptions the answer can certainly be no. The limits or sums may fail to converge, but even if they do the left and right hand sides needn't be equal. Consider, for example, what happens when $S = \mathbf{N}$ and $u_n(s) = 1$ if $s = n$ and $u_n(s) = 0$ otherwise. The limit of sums on the left hand side is then equal to 1 while the sum of limits on the right hand side is equal to 0. So some further hypotheses are clearly needed.

Theorem 6.3.1. *Suppose $(D, \preccurlyeq)$ is a non-empty directed set, S is a set, and $f\colon D\times S \to [0,+\infty]$ is a function such that if $a, b\in D$, $s\in S$ and $a \preccurlyeq b$ then $f(a,s) \le f(b,s)$. Then*

$$\sum_{s\in S} \lim_{a\in D} f(a,s) = \lim_{a\in D} \sum_{s\in S} f(a,s)$$

This theorem, or rather the corollary below for sequences, is known as the *Monotone Convergence Theorem* for sums.

Proof. By Proposition 6.1.2 we have

$$\lim_{a\in D}(s) = \sup_{a\in D} f(a,s)$$

so

$$f(a,s) \le \lim_{a\in D} f(a,s).$$

By the monotonicity property of sums we have

$$\sum_{s\in S} f(a,s) \le \sum_{s\in S} \lim_{a\in D} f(a,s).$$

Using monotonicity of limits we then get

$$\lim_{a\in D} \sum_{s\in S} f(a,s) \le \sum_{s\in S} \lim_{a\in D} f(a,s).$$

We now establish the reverse inequality. Suppose that F is a finite subset of S. We've already seen that for finite sums the limit of the sum is the sum of the limits, so

$$\sum_{s\in F} \lim_{a\in D} f(a,s) = \lim_{a\in D} \sum_{s\in F} f(a,s).$$

$F \subseteq S$ and the summands are non-negative so

$$\sum_{s\in F} f(a,s) \le \sum_{s\in S} f(a,s)$$

for each $a\in D$. Using the monotonicity of limits then

$$\lim_{a\in D} \sum_{s\in F} f(a,s) \le \lim_{a\in D} \sum_{s\in S} f(a,s).$$

Combining all of these we find that

$$\sum_{s\in F} \lim_{a\in D} f(a,s) \le \lim_{a\in D} \sum_{s\in S} f(a,s).$$

If we take the limit over all finite subsets F of S then we get

$$\sum_{s\in S} \lim_{a\in D} f(a,s) \le \lim_{a\in D} \sum_{s\in S}(a,s).$$

□

Corollary 6.3.2. *Suppose $f\colon \mathbf{N} \times S \to [0, +\infty]$ is such that if $m \le n$ then $f_m(s) \le f_n(s)$ for all $s \in S$. Then*

$$\lim_{n\to\infty} \sum_{s\in S} f_n(s) = \sum_{s\in S} \lim_{n\to\infty} f_n(s).$$

Proof. This is just the special case $(D, \preccurlyeq) = (\mathbf{N}, \le)$ of the previous theorem. □

Theorem 6.3.3. *Suppose $(D, \preccurlyeq)$ is a non-empty directed set, S is a set, and $f\colon D \times S \to [0, +\infty]$ is a function. Let*

$$T_a = \{b \in D\colon a \preccurlyeq b\}.$$

Then

$$\sum_{s\in S} \sup_{a\in D} \inf_{b\in T_a} f(b, s) \le \sup_{a\in D} \inf_{b\in T_a} \sum_{s\in S} f(b, s).$$

This theorem is known as *Fatou's Lemma* for sums.

Proof. Define $g\colon D \times S \to [0, +\infty]$ by

$$g(a, s) = \inf_{c\in T_a} f(c, s).$$

This exists because all subsets of $[0, +\infty]$ have infima. Also, if $a \preccurlyeq b$ then $T_b \subseteq T_a$ and so

$$\inf_{c\in T_a} f(c, s) \le \inf_{c\in T_b} f(c, s).$$

In other words, if $a \preccurlyeq b$ then

$$g(a, s) \preccurlyeq g(b, s).$$

It follows from the Monotone Convergence Theorem that

$$\sum_{s\in S} \lim_{a\in D} g(a, s) = \lim_{a\in D} \sum_{s\in s} g(a, s).$$

These are monotone nets so the limit is the same as the supremum and therefore

$$\sum_{s\in S} \sup_{a\in D} g(a, s) = \sup_{a\in D} \sum_{s\in s} g(a, s).$$

Now if $a \preccurlyeq b$ then $b \in \{c \in D\colon c \in T_a\}$ and so

$$g(a, s) = \inf_{c\in T_a} f(c, s) \le f(b, s)$$

so

$$\sum_{s\in S} g(a, s) \le \sum_{s\in S} f(b, s).$$

This holds for all $b \in T_a$ so

$$\sum_{s\in S} g(a, s) \le \inf_{b\in T_a} \sum_{s\in S} f(b, s)$$

and

$$\sup_{a\in D} \sum_{s\in S} g(a, s) \le \sup_{a\in D} \inf_{b\in T_a} \sum_{s\in S} f(b, s).$$

Combining this with the equation

$$\sum_{s\in S} \sup_{a\in D} g(a, s) = \sup_{a\in D} \sum_{s\in s} g(a, s).$$

obtained earlier, we find that

$$\sum_{s\in S} \sup_{a\in D} g(a, s) \le \sup_{a\in D} \inf_{b\in T_a} \sum_{s\in S} f(b, s),$$

or, in view of how g was defined,

$$\sum_{s\in S} \sup_{a\in D} \inf_{b\in T_a} f(b, s) \le \sup_{a\in D} \inf_{b\in T_a} \sum_{s\in S} f(b, s).$$

□

Again, there's a corollary for sequences.

Corollary 6.3.4. *Suppose S is a set and $f\colon \mathbf{N} \times S \to [0, +\infty]$ is a function. Then*

$$\sum_{s\in S} \sup_{m\in\mathbf{N}} \inf_{n\ge m} f_n(s) \le \sup_{m\in\mathbf{N}} \inf_{n\ge m} \sum_{s\in S} f_n(s).$$

Proof. This is just the case $(D, \preccurlyeq) = (\mathbf{N}, \le)$ of the theorem. □

The following lemma extends one of the standard properties of sequences with values in $\mathbf{R}$ to nets with values in $[-\infty, +\infty]$. It will be needed in the proof of the theorem which follows it.

Lemma 6.3.5. *Suppose $(D, \preccurlyeq)$ is a non-empty directed set and $\varphi\colon D \to [-\infty, +\infty]$ is a net. Then*

$$\sup_{a\in D} \inf_{b\in T_a} \varphi(b) \le \inf_{a\in D} \sup_{b\in T_a} \varphi(b).$$

If

$$\inf_{a\in D}\sup_{b\in T_a}\varphi(b) \le \sup_{a\in D}\inf_{b\in T_a}\varphi(b)$$

then φ *is convergent and*

$$\inf_{a\in D}\sup_{b\in T_a}\varphi(b) = \lim\varphi = \sup_{a\in D}\inf_{b\in T_a}\varphi(b).$$

Note that all the infima and suprema exist because we are working in $[-\infty,+\infty]$.

Proof. Suppose that $c,d\in D$. D is a directed set so there is an $e\in D$ such that $c\preccurlyeq e$ and $d\preccurlyeq e$, i.e. such that $e\in T_c$ and $e\in T_d$ It follows that

$$\inf_{b\in T_c}\varphi(b) \le \varphi(e)$$

and

$$\varphi(e) \le \sup_{b\in T_d}\varphi(b).$$

Therefore

$$\inf_{b\in T_c}\varphi(b) \le \sup_{b\in T_d}\varphi(b).$$

Taking the supremum over c we find that

$$\sup_{c\in D}\inf_{b\in T_c}\varphi(b) \le \sup_{b\in T_d}\varphi(b).$$

Then taking the infimum over d we get

$$\sup_{c\in D}\inf_{b\in T_c}\varphi(b) \le \inf_{d\in D}\sup_{b\in T_d}\varphi(b).$$

This is the same as

$$\sup_{a\in D}\inf_{b\in T_a}\varphi(b) \le \inf_{a\in D}\sup_{b\in T_a}\varphi(b).$$

If the reverse inequality holds then we must have

$$\sup_{a\in D}\inf_{b\in T_a}\varphi(b) = \inf_{a\in D}\sup_{b\in T_a}\varphi(b).$$

In this case we let y be their common value. We will now show that φ converges to y.

Suppose $x<y$ and $I=(x,+\infty]$. Then

$$x < \sup_{a\in D}\inf_{b\in T_a}\varphi(b)$$

so, by the definition of the supremum, x is not an upper bound for $\inf_{b\in T_a}\varphi$. In other words, there is an $a\in D$ such that

$$\inf_{b\in T_a}\varphi(b) > x.$$

It follows that if $b\in T_a$ then $\varphi(b)>x$. In other words, if $a\preccurlyeq b$ then $\varphi(b)\in I$. Similarly, if $y<z$ and $J=[-\infty,z)$ then

$$\inf_{a\in D}\sup_{b\in T_a}\varphi(b) < z$$

so, by the definition of the infimum, z is not a lower bound for $\sup_{b\in T_a}\varphi$. In other words, there is an $a\in D$ such that

$$\inf_{b\in T_a}\varphi(b) < z.$$

It follows that if $b\in T_a$ then $\varphi(b)<z$. In other words, if $a\preccurlyeq b$ then $\varphi(b)\in J$.

If we have a finite set $\{K_1,\ldots,K_m\}$ of intervals all of the form I or J as above then for each K_j there is an a_j such that if $b\in D$ and $a_j\preccurlyeq b$ then $\varphi(b)\in K_j$. D is a directed set so there is an $a\in D$ such that $a_j\preccurlyeq a$ for each j and therefore if $a\preccurlyeq b$ then

$$\varphi(b)\in\bigcap_{j=1}^{m}K_j.$$

Because the topology of $[-\infty,+\infty]$ is generated by sets of the form $(x,+\infty]$ and $[-\infty,z)$ every neighbourhood of y contains an intersection of the form above, so for every $V\in\mathcal{N}(y)$ there is an $a\in D$ such that if $b\in D$ and $a\preccurlyeq b$ then

$$\varphi(b)\in V.$$

Thus φ converges y. □

Theorem 6.3.6. *Suppose* $(D,\preccurlyeq)$ *is a non-empty directed set,* S *is a set, and* $f\colon D\times S\to\mathbf{R}$ *is a function and* $g\colon S\to[0,+\infty]$ *is a function such that*

$$\lim_{a\in D} f(a,s)$$

exists for all $s\in S$*,*

$$\sum_{s\in S} g(s) < +\infty$$

and

$$|f(a,s)| \leq g(a)$$

for all $a \in D$. Then

$$\lim_{a\in D} \sum_{s\in S} f(a,s) = \sum_{s\in S} \lim_{a\in D} f(a,s).$$

This is known as the *Dominated Convergence Theorem* for sums.

Proof. Define

$$h(a,s) = g(s) + f(a,s).$$

Then $h(a,s) \geq 0$ for all $a \in D$ and $s \in S$. By Fatou's Lemma,

$$\sum_{s\in S} \sup_{a\in D} \inf_{b\in T_a} h(b,s) \leq \sup_{a\in D} \inf_{b\in T_a} \sum_{s\in S} h(b,s).$$

Now

$$\begin{aligned}\sup_{a\in D} \inf_{b\in T_a} h(b,s) &= g(s) + \sup_{a\in D} \inf_{b\in T_a} f(b,s)\\ &= g(s) + \lim_{a\in D} f(a,s).\end{aligned}$$

Also,

$$\sum_{s\in S} h(a,s) = \sum_{s\in S} g(s) + \sum_{s\in S} f(a,s)$$

so

$$\sup_{a\in D} \inf_{b\in T_a} \sum_{s\in S} h(b,s) = \sum_{s\in S} g(s) + \sup_{a\in D} \inf_{b\in T_a} \sum_{s\in S} f(b,s).$$

Therefore

$$\begin{aligned}&\sum_{s\in S} g(s) + \sum_{s\in S} \lim_{a\in D} f(a,s)\\ &\qquad \leq \sum_{s\in S} g(s) + \sup_{a\in D} \inf_{b\in T_a} \sum_{s\in S} f(b,s)\end{aligned}$$

Because

$$\sum_{s\in S} g(s) < +\infty$$

we can conclude that

$$\sum_{s\in S} \lim_{a\in D} f(a,s) \leq \sup_{a\in D} \inf_{b\in T_a} \sum_{s\in S} f(b,s).$$

We can apply the same argument with $-f(a,s)$ in place of $f(a,s)$ to get

$$\sum_{s\in S} \lim_{a\in D} -f(a,s) \leq \sup_{a\in D} \inf_{b\in T_a} \sum_{s\in S} -f(b,s),$$

or, equivalently,

$$\inf_{a\in D} \sup_{b\in T_a} \sum_{s\in S} f(b,s) \leq \sum_{s\in S} \lim_{a\in D} f(a,s).$$

It follows that

$$\sup_{a\in D} \inf_{b\in T_a} \sum_{s\in S} f(b,s) \leq \inf_{a\in D} \sup_{b\in T_a} \sum_{s\in S} f(b,s)$$

and therefore the lemma above shows that

$$\lim_{a\in D} \sum_{s\in S} f(a,s)$$

exists and is equal to their common value. So

$$\lim_{a\in D} \sum_{s\in S} f(a,s) = \sum_{s\in S} \lim_{a\in D} f(a,s).$$

□

Corollary 6.3.7. *Suppose S is a set and $f\colon D\times S \to$ $\mathbf{R}$ is a function and $g\colon S \to [0,+\infty]$ is a function such that*

$$\lim_{n\to\infty} f_n(s)$$

exists for all $s \in S$,

$$\sum_{s\in S} g(s) < +\infty$$

and

$$|f_n(s)| \leq g(a)$$

for all $n \in \mathbf{N}$. Then

$$\lim_{n\to\infty} \sum_{s\in S} f_n(s) = \sum_{s\in S} \lim_{n\to\infty} f_n(s).$$

6.4 Partitioning sums

Proposition 6.4.1. *Suppose $\mathcal{A}$ is a set of disjoint sets. In other words if $P, Q \in \mathcal{A}$ and $P \neq Q$ then $P \cap Q = \varnothing$. Let $S = \bigcup_{P \in \mathcal{A}} P$. Suppose $f\colon S \to [0, +\infty]$ is a function. Then*

$$\sum_{s \in S} f(s) = \sum_{P \in \mathcal{A}} \sum_{s \in P} f(s).$$

Note that the sums all exist because these are sums of elements of $[0, +\infty]$.

Proof. Suppose $F \subseteq S$ is finite. Then

$$F = \bigcup_{\substack{P \in \mathcal{A} \\ P \cap F \neq \varnothing}} P \cap F.$$

This is a finite union of finite disjoint sets so

$$\sum_{s \in F} f(s) = \sum_{\substack{P \in \mathcal{A} \\ P \cap F \neq \varnothing}} \sum_{s \in P \cap F} f(s).$$

But $P \cap F \subseteq P$ so

$$\sum_{s \in P \cap F} f(s) \leq \sum_{s \in P} f(s)$$

and therefore

$$\sum_{\substack{P \in \mathcal{A} \\ P \cap F \neq \varnothing}} \sum_{s \in P \cap F} f(s) \leq \sum_{\substack{P \in \mathcal{A} \\ P \cap F \neq \varnothing}} \sum_{s \in P} f(s).$$

Also,

$$\{P \in \mathcal{A} \colon P \cap F \neq \varnothing\} \subseteq \mathcal{A}$$

and

$$\sum_{s \in P} f(s) \in [0, +\infty]$$

for all $P \in \mathcal{A}$ so

$$\sum_{\substack{P \in \mathcal{A} \\ P \cap F \neq \varnothing}} \sum_{s \in P} f(s) \leq \sum_{P \in \mathcal{A}} \sum_{s \in P} f(s).$$

Combining the previous results,

$$\sum_{s \in F} f(s) \leq \sum_{P \in \mathcal{A}} \sum_{s \in P} f(s).$$

Taking limits with respect to the net of finite subsets F of S gives

$$\sum_{s \in S} f(s) \leq \sum_{P \in \mathcal{A}} \sum_{s \in P} f(s).$$

It remains to prove the reverse inequality.

Suppose that $\mathcal{G} \subseteq \mathcal{A}$ is finite. Then

$$\sum_{P \in \mathcal{G}} \sum_{s \in P} f(s) = \sum_{P \in \mathcal{G}} \sup \sum_{s \in F_P} f(s)$$

where the supremum is over finite subsets F_P of P. This is the same as

$$\sup \sum_{P \in \mathcal{G}} \sum_{s \in F_P} f(s)$$

where the supremum is over all choices of an F_P for each $P \in \mathcal{G}$. Each such choice is uniquely determined by

$$H = \bigcup_{P \in \mathcal{G}} F_P,$$

which is a finite subset of S with the property that $H \cap Q = \varnothing$ if $Q \notin \mathcal{G}$, since P_F is determined from H by

$$F_P = H \cap P.$$

Therefore

$$\sum_{P \in \mathcal{G}} \sum_{s \in P} f(s) = \sup \sum_{P \in \mathcal{G}} \sum_{s \in H \cap P} f(s) = \sup \sum_{s \in H} f(s)$$

where the supremum is over all such finite subsets H. This is less than or equal to the supremum over all finite subsets, which is just $\sum_{s \in S} f(s)$ by definition, so

$$\sum_{P \in \mathcal{G}} \sum_{s \in P} f(s) \leq \sum_{s \in S} f(s).$$

Taking the supremum over all finite subsets $\mathcal{G}$ of $\mathcal{A}$ gives

$$\sum_{P \in \mathcal{A}} \sum_{s \in P} f(s) \leq \sum_{s \in S} f(s).$$

Since we already have the reverse inequality we conclude that

$$\sum_{s \in S} f(s) = \sum_{P \in \mathcal{A}} \sum_{s \in P} f(s).$$

□

There are a number of important corollaries to the proposition above.

Proposition 6.4.2. *Suppose $P \cap Q = \varnothing$ and $f\colon P \cup Q \to [0 + \infty]$ is a function. Then*

$$\sum_{s \in P \cup Q} f(s) = \sum_{s \in P} f(s) + \sum_{s \in Q} f(s).$$

Proof. We just apply the proposition above with $S = P \cup Q$ and $\mathcal{A} = \{P, Q\}$. □

Theorem 6.4.3. *Suppose A and B are sets and $f\colon A \times B \to [0, +\infty]$ is a function. Then*

$$\sum_{a \in A} \sum_{b \in B} f(a, b) = \sum_{(a,b) \in A \times B} f(a, b) = \sum_{b \in B} \sum_{a \in A} f(a, b).$$

This is known as *Tonelli's Theorem* for sums.

Proof. To prove the first equation apply the proposition with $S = A \times B$ and $\mathcal{A}$ the set of subsets of S of the form $\{a\} \times B$ where a ranges over A. To prove the second equation we take $S = A \times B$ again but take $\mathcal{A}$ to be the set of subsets of S of the form $A \times \{b\}$ where b ranges over B. □

Theorem 6.4.4. *Suppose A and B are sets and $g\colon A \times B \to \mathbf{R}$ is a function such that*

$$\sum_{(a,b) \in A \times B} g(a, b)$$

is convergent. Then

$$\sum_{a \in A} \sum_{b \in B} g(a, b) = \sum_{(a,b) \in A \times B} g(a, b) = \sum_{b \in B} \sum_{a \in A} g(a, b).$$

This is *Fubini's Theorem* for sums.

Proof. By Proposition 6.2.4 we have

$$\sum_{(a,b) \in A \times B} |g(a, b)| < +\infty.$$

Let

$$S = A \times B$$

and let D be the set of finite subsets of S, ordered by inclusion. Define $f\colon D \times S \to \mathbf{R}$ by

$$f(H, (a, b)) = \begin{cases} g(a, b) & \text{if } (a, b) \in H, \\ 0 & \text{if } (a, b) \notin H. \end{cases}$$

Then

$$\lim_{H \in D} f(H, (a, b)) = g(a, b)$$

and

$$|f(H, (a, b))| \leq |g(a, b)|$$

for all $H \in D$. It follows from the Dominated Convergence Theorem, Theorem 6.3.6, that

$$\begin{aligned} \lim_{H \in D} \sum_{(a,b) \in S} f(H, (a, b)) &= \sum_{(a,b) \in S} \lim_{H \in D} f(H, (a, b)) \\ &= \sum_{(a,b) \in S} g(a, b). \end{aligned}$$

Also

$$\begin{aligned} \lim_{H \in D} \sum_{a \in A} \sum_{b \in B} f(H, (a, b)) &= \sum_{a \in A} \lim_{H \in D} \sum_{b \in B} f(H, (a, b)) \\ &= \sum_{a \in A} \sum_{b \in B} \lim_{H \in D} f(H, (a, b)) \\ &= \sum_{a \in A} \sum_{b \in B} g(a, b). \end{aligned}$$

For each H we have

$$\sum_{a \in A} \sum_{b \in B} f(H, (a, b)) = \sum_{(a,b) \in A \times B} f(H, (a, b))$$

because there are only finitely many non-zero terms in these sums, due to the finiteness of H, and so the order of summation doesn't matter. Taking limits,

$$\begin{aligned} &\lim_{H \in D} \sum_{a \in A} \sum_{b \in B} f(H, (a, b)) \\ &\quad = \lim_{H \in D} \sum_{(a,b) \in A \times B} f(H, (a, b)). \end{aligned}$$

Combining all of these, we find that

$$\sum_{a \in A} \sum_{b \in B} g(a, b) = \sum_{(a,b) \in A \times B} g(a, b).$$

The equation

$$\sum_{(a,b)\in A\times B} g(a,b) = \sum_{b\in B}\sum_{a\in A} g(a,b)$$

is proved similarly. □

7 Content and measure

7.1 Boolean algebras

Definition 7.1.1. A *Boolean algebra* on a set X is a $\mathcal{B} \in \wp(\wp(X))$ such that

(a) $\varnothing \in \mathcal{B}$.

(b) If $E \in \mathcal{B}$ then $X \setminus E \in \mathcal{B}$.

(c) If $E, F \in \mathcal{B}$ then $E \cup F \in \mathcal{B}$.

Proposition 7.1.2. *Suppose $\mathcal{B}$ is a Boolean algebra on X. Then*

(a) $X \in \mathcal{B}$.

(b) If $E, F \in \mathcal{B}$ then $E \cap F \in \mathcal{B}$.

(c) If $E, F \in \mathcal{B}$ then $E \setminus F \in \mathcal{B}$.

(d) If $E, F \in \mathcal{B}$ then $E \triangle F \in \mathcal{B}$.

Proof. $X = X \setminus \varnothing$ and $\varnothing \in \mathcal{B}$. Also

$$E \cap F = X \setminus ((X \setminus E) \cup (X \setminus F)),$$

$$E \setminus F = X \setminus ((X \setminus E) \cup F),$$

and

$$E \triangle F = (X \setminus ((X \setminus E) \cup F)) \cup (X \setminus (E \cup (X \setminus E)))$$

so each belongs to $\mathcal{B}$ if E and F do. □

Proposition 7.1.3. *The following are examples of Boolean algebras.*

(a) For any set X, $\mathcal{B} = \{\varnothing, X\}$ is an algebra, called the trivial Boolean algebra on X.

(b) For any set X, $\mathcal{B} = \wp(X)$ is an algebra, called the discrete Boolean algebra on X.

(c) The set $\mathcal{B}$ of finite unions of intervals is a Boolean algebra on $\mathbf{R}$.

Proof. In the first two cases it's trivial to verify the conditions, so we'll concentrate on the last one. The empty set is the union of an empty collection of intervals, establishing 7.1.1a.

If $\mathcal{P}$ and $\mathcal{Q}$ are finite sets of intervals then $\mathcal{P} \cup \mathcal{Q}$ is a finite set of intervals and

$$\left(\bigcup_{E\in\mathcal{P}} E\right) \cup \left(\bigcup_{E\in\mathcal{Q}} E\right) = \bigcup_{E\in\mathcal{P}\cup\mathcal{Q}} E$$

so $\bigcup_{E\in\mathcal{P}\cup\mathcal{Q}} E$ is a finite union of intervals. This establishes 7.1.1c.

7.1.1b is geometrically obvious, but surprisingly tricky to prove. In order to prove it we need to define the term "interval", which we have so far managed to avoid doing in these notes. In lecture the ten different types of intervals were listed, namely

- (a, b), where $a < b$
- $[a, b]$, where $a \le b$
- $[a, b)$, where $a < b$
- $(a, b]$, where $a < b$
- $(a, +\infty)$
- $[a, +\infty)$
- $(-\infty, b)$
- $(-\infty, b]$
- $(-\infty, +\infty)$
- $\varnothing$

This list is extremely awkward to use as a definition though since it leads to case by case analysis with a large number of cases. If we wanted to prove the elementary fact that the intersection of intervals is an interval then we'd have to consider 55 different possibilities for the types of the two intervals.[1] A better option is to choose a single defining property.

[1] This assumes we exploit the fact that $E \cap F = F \cap E$. If we don't then there are 100 cases.

Of course we do sometimes need to know that the list above is an exhaustive list of the types of intervals, so we must show that the sets with whatever property we've chosen are all of one of the ten types above and then any set of one of those types is an interval. One option would be to define intervals to be connected subsets of $\mathbf{R}$. This can be made to work, but few of the properties of intervals follow directly from connectedness. It's more convenient to define an interval as a subset $I \in \wp(\mathbf{R})$ such that if $x \leq y \leq z$ and $x, z \in I$ then $y \in \mathbf{R}$, so that's what we'll do. After proving a few elementary properties we'll be able to complete the proof of 7.1.1b, as Corollary 7.1.6. Finally, we'll prove that the intervals are according to this definition are actually of the familiar ten types listed above. □

Definition 7.1.4. $I \in \wp(\mathbf{R})$ is said to be an *interval* if $y \in I$ whenever $x \leq y \leq z$ and $x, z \in I$.

Proposition 7.1.5. *(a) The intersection of any non-empty collection of intervals is an interval.*

(b) If I is an interval then so are

$$I_< = \bigcap_{y \in I} (-\infty, y)$$

and

$$I_> = \bigcap_{y \in I} (y, +\infty)$$

Also,

$$\mathbf{R} \setminus I = I_< \cup I_>.$$

Proof. Suppose $\mathcal{A}$ is a set of intervals and

$$I = \bigcap_{J \in \mathcal{A}} J.$$

If $x \leq y \leq z$ and $x, z \in I$ then $x, z \in J$ for each $J \in \mathcal{A}$. Since each $J \in \mathcal{A}$ is an interval it follows that $y \in J$. Since this holds for all $J \in \mathcal{A}$ we therefore have $y \in I$. So if $x \leq y \leq z$ and $x, z \in I$ then $y \in I$. In other words, I is an interval. This establishes the first part.

From the first part it follows immediately that $I_<$ and $I_>$ are intervals. If $y \in I_<$ and $y \in I$ then $y \in (-\infty, y)$, which is impossible. So if $y \in I_<$ then $y \notin I$, i.e. $y \in \mathbf{R} \setminus I$. In other words,

$$I_< \subseteq \mathbf{R} \setminus I.$$

Similarly,

$$I_> \subseteq \mathbf{R} \setminus I.$$

and hence

$$I_< \cup I_> \subseteq \mathbf{R} \setminus I.$$

Now

$$\begin{aligned} \mathbf{R} \setminus I_< &= \mathbf{R} \setminus \bigcap_{x \in I} (-\infty, x) \\ &= \bigcup_{x \in I} \mathbf{R} \setminus (-\infty, x) \\ &= \bigcup_{x \in I} [x, +\infty). \end{aligned}$$

Similarly,

$$\mathbf{R} \setminus I_> = \bigcup_{z \in I} (-\infty, z].$$

Therefore

$$\begin{aligned} \mathbf{R} \setminus (I_< \cup I_>) &= (\mathbf{R} \setminus I_<) \cap (\mathbf{R} \setminus I_>) \\ &= \bigcup_{x, z \in I} [x, +\infty) \cap (-\infty, z] \\ &= \bigcup_{x, z \in I} [x, z] \end{aligned}$$

and so So if $y \notin I_< \cup I_>$ then there are $x, z \in I$ such that $x \leq y \leq z$. I is an interval so then $y \in I$. So if $y \notin I_< \cup I_>$ then $y \in I$ or, equivalently, if $y \notin I$ then $y \in\notin I_< \cup I_>$, i.e.

$$\mathbf{R} \setminus I \subseteq I_< \cup I_>.$$

We already have the reverse inclusion, so

$$\mathbf{R} \setminus I \subseteq I_< \cup I_>.$$

□

Corollary 7.1.6. *If E is a finite union of intervals then $X \setminus E$ is a finite union of intervals.*

Proof. Suppose $E = I_1 \cup \cdots \cup I_m$ where $I_1, \ldots, I_m$ are intervals. Then

$$\begin{aligned}\mathbf{R} \setminus E &= (\mathbf{R} \setminus I_1) \cap \cdots \cap (\mathbf{R} \setminus I_m) \\ &= (I_{1<} \cup I_{1>}) \cap \cdots \cap (I_{m<} \cup I_{m>})\,.\end{aligned}$$

Writing the intersection of unions as a union of intersections we see that $\mathbf{R} \setminus E$ is a union of 2^m sets, each of which is an intersection of intervals and hence is an interval. So $\mathbf{R} \setminus E$ is a union of intervals. □

The proof shows not only that the complement of the union of m intervals is a finite union of intervals but that we can write it as a union of at most 2^m intervals. A more careful argument shows that we need at most $m+1$ intervals, but we won't need this and so won't prove it.

Proposition 7.1.7. *The intervals are precisely the sets of one of the following ten forms:*

(a) (a, b)*, where* $a < b$

(b) $[a, b]$*, where* $a \leq b$

(c) $[a, b)$*, where* $a < b$

(d) $(a, b]$*, where* $a < b$

(e) $(a, +\infty)$

(f) $[a, +\infty)$

(g) $(-\infty, b)$

(h) $(-\infty, b]$

(i) $(-\infty, +\infty)$

(j) $\varnothing$

Proof. Verifying that each of these sets is an interval is straightforward. Checking that every interval is of one of these forms is more complicated. Suppose I is a non-empty interval. Every non-empty subset of $\mathbf{R}$ which is bounded from above has a supremum, which may or may not belong to the subset. This gives three possibilities: I has a supremum belonging to I, i.e. a maximum, I has a supremum not in I, or I has no upper bound. Similarly, I has an infimum belonging to I, i.e. a minimum, an infimum not belonging to I, or I has no lower bound. These two three-way distinctions give nine types of non-empty interval. We can check that each of them corresponds to one of the first nine classes listed above.

For example, if I has both a minimum and a maximum then we call the former a and the latter b. Then $a, b \in I$ and so if $a \leq x \leq b$ then $x \in I$, since I is an interval. Conversely, if $x \in I$ then $a \leq x \leq b$ since a is a minimum and b a maximum for I. So I is precisely the set of x such that $a \leq x \leq b$, i.e. the set $[a, b]$.

If I has a minimum and an supremum, but not a maximum then we call the minimum a and the supremum b. If $a \leq x < b$ then x is not an upper bound for I so there is a $y \in I$ such that $x < y$. But then $a \leq x \leq y$ and $a, y \in I$ so $x \in I$, since I is an interval. Conversely, if $x \in I$ then $a \leq x \leq b$ since a is a lower bound for I and b is an upper bound. $b \notin I$ though so $a \leq x < b$. Thus $x \in I$ if and only if $a \leq x < b$, i.e. if and only if $x \in [a, b)$.

If I has a minimum but no upper bound then we again call the minimum a. If $a \leq x < +\infty$ then x is again not an upper bound for I so $x \in I$, by the same argument as above. Conversely, if $x \in I$ then $a \leq x < +\infty$ because a is a minimum for I. So $x \in I$ if and only if $a \leq x < +\infty$, i.e. if and only if $x \in [a, +\infty)$.

The six remaining cases are similar. □

Proposition 7.1.8. *Suppose* $\mathbf{A}$ *is a non-empty set of Boolean algebras on a set* X*. Then* $\bigcap_{\mathcal{C} \in \mathbf{A}}$ *is a Boolean algebra.*

Proof. Let $\mathcal{B} = \bigcap_{\mathcal{C} \in \mathbf{A}}$. Then $\varnothing \in \mathcal{B}$ since $\varnothing \in \mathcal{C}$ for each $\mathcal{C} \in \mathbf{A}$.

If $E \in \mathcal{B}$ then $E \in \mathcal{C}$ for each $\mathcal{C} \in \mathbf{A}$ so $X \setminus E \in \mathcal{C}$ for each $\mathcal{C} \in \mathbf{A}$. Therefore $X \setminus E \in \mathcal{B}$.

If $E, F \in \mathcal{B}$ then $E, F \in \mathcal{C}$ for each $\mathcal{C} \in \mathbf{A}$ so $E \cup F \in \mathcal{C}$ for each $\mathcal{C} \in \mathbf{A}$. Therefore $E \cup F \in \mathcal{B}$. □

Proposition 7.1.9. *Suppose* $A \in \wp(\wp(X))$*. Then there is a smallest Boolean algebra which contains* $\mathcal{A}$*.*

In the setting of the proposition above the Boolean algebra $\mathcal{B}$ is said to be *generated* by the set $\mathcal{A}$.

Proof. Let $\mathbf{A}$ be the set of Boolean algebras which contain $\mathcal{A}$. $\mathcal{A}$ is non-empty because $\wp(\wp(X))$ is a Boolean algebra which contains $\mathcal{A}$. $\mathcal{B} = \bigcap_{\mathcal{C}\in\mathbf{A}}$ is a Boolean algebra by the preceding proposition. It contains $\mathcal{A}$. It is also a subset of any Boolean algebra which contains $\mathcal{A}$ and so is the the smallest Boolean algebra which contains $\mathcal{A}$. □

As an example, the set of finite unions of intervals in $\mathbf{R}$, which we've already seen is a Boolean algebra, is generated by the set of intervals, since it is a Boolean algebra and clearly any Boolean algebra which contains the set of intervals must contain it.

We also have the following corollary to the proposition.

Corollary 7.1.10. *Suppose $\mathbf{A}$ is a set of Boolean algebras on a set X. Then there is a smallest Boolean algebra which contains $\bigcup_{\mathcal{B}\in\mathbf{A}} \mathcal{B}$.*

Proof. We just apply the proposition to $A = \bigcup_{\mathcal{B}\in\mathbf{A}} \mathcal{B}$. □

Proposition 7.1.11. *Suppose X, Y are sets and $f\colon X \to Y$ is a function. If $\mathcal{B}$ is a Boolean algebra on X then $f^{**}(\mathcal{B})$ is a Boolean algebra on X.*

Proof. Let
$$\mathcal{A} = f^{**}(\mathcal{B}).$$
Then
$$f^*(\varnothing) = \varnothing \in \mathcal{B}$$
so
$$\varnothing \in f^{**}(\mathcal{B}) = \mathcal{A}.$$

If $E \in \mathcal{A}$ then $f^*(E) \in \mathcal{B}$ so $Y \setminus f^*(E) \in \mathcal{B}$. But
$$Y \setminus f^*(E) = f^*(X) \setminus f^*(E) = f^*(X \setminus E)$$
so $f^*(X \setminus E) \in \mathcal{B}$ and hence
$$X \setminus E \in f^{**}(\mathcal{B}) = \mathcal{A}.$$

If $E, F \in \mathcal{A}$ then $f^*(E), f^*(F) \in \mathcal{B}$ so
$$f^*(E \cup F) = f^*(E) \cup f^*(F) \in \mathcal{B}.$$
Therefore
$$E \cup F \in f^{**}(\mathcal{B}) = \mathcal{A}.$$
Thus $\mathcal{A}$ satisfies all three conditions to be a Boolean algebra. □

7.2 σ-algebras

Definition 7.2.1. A *σ-algebra* on a set X is a $\mathcal{B} \in \wp(\wp(X))$ such that

(a) $\varnothing \in \mathcal{B}$.

(b) If $E \in \mathcal{B}$ then $X \setminus E \in \mathcal{F}$.

(c) If $\mathcal{A}$ is a countable subset of $\mathcal{B}$ then
$$\bigcup_{E\in\mathcal{A}} E \in \mathcal{B}.$$

A pair $(X, \mathcal{B})$ where $\mathcal{B}$ is a σ-algebra on X is called a *measurable space*.

Proposition 7.2.2. *If $\mathcal{B}$ is then it is a σ-algebra on X a Boolean algebra on X.*

Proof. The first two conditions in the definitions are the same. For the third condition, if $\mathcal{A}$ is a countable subset of a σ-algebra $\mathcal{B}$ then
$$\bigcup_{E\in\mathcal{A}} E \in \mathcal{B}.$$
Apply this to $\mathcal{A} = \{E, F\}$ where $E, F \in \mathcal{A}$ to get that
$$E \cup F \in \mathcal{B}.$$
□

Proposition 7.2.3. *If $\mathcal{A}$ is a non-empty countable subset of a σ-algebra $\mathcal{B}$ then*
$$\bigcap_{E\in\mathcal{A}} E \in \mathcal{B}.$$

Proof.
$$\bigcap_{E\in\mathcal{A}} E \in \mathcal{B} = X \setminus (\bigcup_{E\in\mathcal{A}} (X \setminus E)).$$
□

The set $\mathcal{B}$ of finite unions of intervals is not a σ-algebra on $\mathbf{R}$. Perhaps more surprisingly, neither is the set of countable unions of intervals. If it were then the preceding proposition could be used to show that the Cantor set is a countable union of intervals.

Between any two elements of the Cantor set there is a point which is not in the Cantor set, so none of these intervals could contain more than one point. But there are uncountably many elements of the Cantor set.

Proposition 7.2.4. *Suppose X, Y are sets and $f\colon X \to Y$ is a function. If $\mathcal{B}$ is a σ-algebra on X then $f^{**}(\mathcal{B})$ is a σ-algebra on X.*

Proof. Every σ-algebra is a Boolean algebra so $\mathcal{B}$ is a Boolean algebra on X. Then

$$\mathcal{A} = f^{**}(\mathcal{B})$$

is a Boolean algebra on Y. It therefore satisfies the first two conditions to be a σ-algebra and we need only check the last one.

Suppose $\mathcal{C}$ is a countable subset of $\mathcal{A}$. Then

$$f^*\left(\bigcup_{E\in\mathcal{C}} E\right) = \bigcup_{E\in\mathcal{C}} f^*(E).$$

$f^*E \in \mathcal{B}$ so

$$\bigcup_{E\in\mathcal{C}} E \in \mathcal{B}$$

and therefore

$$\bigcup_{E\in\mathcal{C}} E \in f^{**}(\mathcal{B}) = \mathcal{A}.$$

□

Proposition 7.2.5. *Suppose $\mathbf{A}$ is a non-empty set of σ-algebras on a set X. $\bigcap_{\mathcal{C}\in\mathbf{A}}$ is a σ-algebra.*

Proof. Let

$$\mathcal{B} = \bigcap_{\mathcal{C}\in\mathbf{A}} \mathcal{C}.$$

Each $\mathcal{C}$ in $\mathbf{A}$ is a Boolean algebra so their intersection $\mathcal{B}$ is a Boolean algebra by a previous proposition. To show that it is a σ-algebra it thus suffices to check the only condition where the definitions differ, namely that if $\mathcal{A}$ is a countable subset of $\mathcal{B}$ then

$$\bigcup_{E\in\mathcal{A}} E \in \mathcal{B}.$$

If $E \in \mathcal{B}$ then $E \in \mathcal{C}$ for all $\mathcal{C} \in \mathbf{A}$. Since $\mathcal{C}$ is a σ-algebra it follows that

$$\bigcup_{E\in\mathcal{A}} E \in \mathcal{C}.$$

This holds for all $\mathcal{C} \in \mathbf{A}$, so

$$\bigcup_{E\in\mathcal{A}} E \in \bigcap_{\mathcal{C}\in\mathbf{A}} \mathcal{C} = \mathcal{B}.$$

□

Proposition 7.2.6. *Suppose $A \in \wp(\wp(X))$. Then there is a smallest σ-algebra which contains $\mathcal{A}$.*

This smallest σ-algebra which contains $\mathcal{A}$ is said to be *generated* by $\mathcal{A}$.

Proof. We apply the preceding proposition with $\mathbf{A}$ being the set of all σ-algebras on X which contain $\mathcal{A}$. □

Corollary 7.2.7. *Suppose $\mathbf{A}$ is a set of σ-algebras on a set X. There is a smallest σ-algebra which contains $\bigcup_{\mathcal{B}\in\mathbf{A}} \mathcal{B}$.*

Proof. We apply the preceding proposition to $\mathcal{A} = \bigcup_{\mathcal{B}\in\mathbf{A}} \mathcal{B}$. □

Definition 7.2.8. Suppose $(X, \mathcal{T})$ is a topological space. The *Borel σ-algebra* on X is the σ-algebra generated by $\mathcal{T}$. The *Borel sets* are the elements of the Borel σ-algebra.

Proposition 7.2.9. *Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces $f\colon X \to Y$ is a continuous function. If E is a Borel subset of Y then $f^*(E)$ is a Borel subset of X.*

Proof. Let $\mathcal{B}_X$ be the set of Borel subsets of X and let $\mathcal{B}_Y$ be the set of Borel subsets of Y. Let

$$\mathcal{A} = f^{**}(\mathcal{B}_X)$$

$\mathcal{B}_X$ is generated by $\mathcal{T}_X$ so

$$\mathcal{T}_X \subseteq \mathcal{B}_X$$

and therefore

$$f^{**}(\mathcal{T}_X) \subseteq f^{**}(\mathcal{B}_X) = \mathcal{A}.$$

Now

$$\mathcal{T}_Y \subseteq f^{**}(\mathcal{T}_X)$$

by Proposition 3.6.2 so

$$\mathcal{T}_Y \subseteq \mathcal{A}.$$

$\mathcal{A}$ is a σ-algebra by Proposition 7.2.4 and $\mathcal{B}_Y$ is the smallest σ-algebra containing $\mathcal{T}_Y$ by definition so

$$\mathcal{B}_Y \subseteq \mathcal{A},$$

i.e.

$$\mathcal{B}_Y \subseteq f^{**}(\mathcal{B}_X).$$

So if $E \in \mathcal{B}_Y$ then $E \in f^{**}(\mathcal{B}_X)$ and therefore $f^*(E) \in \mathcal{B}_X$. □

Proposition 7.2.10. *Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and that E is a Borel subset of X and F is a Borel subset of Y. Then $E \times F$ is a Borel subset of $X \times Y$, with respect to the product topology.*

Proof.

$$E \times F = \pi_1^*(E) \cap \pi_2^*(F)$$

where π_1 and π_2 are the projections from $X \times Y$ onto its first and second factors, respectively. These projections are continuous so $\pi_1^*(E)$ and $\pi_2^*(F)$ are Borel subsets of $X \times Y$ by the preceding proposition. Their intersection is therefore a Borel subset by Proposition 7.2.3. □

Proposition 7.2.11. *Show that the σ-algebra generated by the set $\mathcal{I}$ of finite unions of intervals is the Borel σ-algebra.*

Proof. Finite unions are countable unions and every interval is Borel set so every finite union of intervals is a Borel set. In other words, $\mathcal{I} \subseteq \mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra.

Suppose $\mathcal{C}$ is a σ-algebra which contains $\mathcal{I}$. Let $\mathcal{T}$ be the usual topology on $\mathbf{R}$. Suppose $U \in \mathcal{T}$, i.e. that U is an open subset of $\mathbf{R}$. Let $\mathcal{A}$ be the set of intervals (a, b) such that $a, b \in \mathbf{Q}$ and $(a, b) \subseteq U$. U is open so if $x \in U$ then there is some $r > 0$ such that $B(x, r) \subseteq U$, i.e. such that $(x - r, x + r) \subseteq U$. There are rational numbers $a \in (x - r, x)$ and $b \in (x, x + r)$. Then $x \in (a, b)$ and $(a, b) \in \mathcal{A}$. so $x \in \bigcup_{E \in \mathcal{A}} E$. This holds for all $x \in U$ so $U \subseteq \bigcup_{E \in \mathcal{A}} E$. On the other hand, $E \subseteq U$ for all $E \in \mathcal{A}$ so $\bigcup_{E \in \mathcal{A}} E \subseteq U$. Therefore $U = \bigcup_{E \in \mathcal{A}} E$. $E \in \mathcal{I}$ for all $E \in \mathcal{A}$ and $\mathcal{I} \subseteq \mathcal{C}$ so $E \in \mathcal{C}$. This holds for all $E \in \mathcal{A}$ so $\mathcal{A} \subseteq \mathcal{C}$. Now $\mathcal{C}$ is a σ-algebra and $\mathcal{A}$ is countable so $U = \bigcup_{E \in \mathcal{A}} E \in \mathcal{C}$. This holds for all $U \in \mathcal{T}$ so $\mathcal{T} \subseteq \mathcal{C}$. So $\mathcal{C}$ is a σ-algebra containing $\mathcal{T}$ and $\mathcal{B}$ was defined to be the smallest σ-algebra containing $\mathcal{T}$ so $\mathcal{B} \subseteq \mathcal{C}$.

The σ-algebra generated by $\mathcal{I}$ is the smallest σ-algebra containing $\mathcal{I}$. We've now seen that $\mathcal{B}$ is such a σ-algebra and that every other such σ-algebra contains $\mathcal{B}$ so $\mathcal{B}$ is indeed the smallest such σ-algebra. □

7.3 Contents

Definition 7.3.1. Suppose $\mathcal{B}$ is a Boolean algebra on set X. A *content* on $(X, \mathcal{B})$ is a function $\mu \colon \mathcal{B} \to [0, +\infty]$ such that

(a)

$$\mu(\varnothing) = 0.$$

(b) If $E, F \in \mathcal{B}$ and $E \cap F = \varnothing$ then

$$\mu(E \cup F) = \mu(E) + \mu(F)$$

A triple $(X, \mathcal{B}, \mu)$ where X is a set, $\mathcal{B}$ is a Boolean algebra on X and μ is a content on $(X, \mathcal{B})$ is called a *content space*.

Another, more common, name for a content is a *finitely additive measure*. That name can be confusing though because we will define measures later and will see that not all contents are measures. The term content space is not used outside of these notes but it's convenient to have a name for such objects and there is no standard name.

Proposition 7.3.2. *Suppose $\mathcal{B}$ is a Boolean algebra on a set X. The following are examples of contents.*

(a) $\mu(E) = 0$ *for all* $E \in \mathcal{B}$.

(b) $\mu(\varnothing) = 0$ *and* $\mu(E) = +\infty$ *for all other* $E \in \mathcal{B}$.

(c)

$$\mu(E) = \begin{cases} 1 & \text{if } y \in E, \\ 0 & \text{if } y \notin E. \end{cases}$$

where $y \in X$.

(d) $\mu(E) = n$ *if* E *is a finite set with* n *elements and* $\mu(E) = +\infty$ *if* E *is infinite.*

(e) $\mu(E) = \sum_{x\in X} w(x)$ *where* $w\colon X \to [0, +\infty]$ *is a function.*

Proof. The first four are all special cases of the last one. The first corresponds to $w(x) = 0$. The second corresponds to $w(x) = +\infty$. The third corresponds to

$$w(x) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

The fourth corresponds to $w(x) = 1$. So we only need to show that the fifth is a content.

$$\mu(\varnothing) = \sum_{x\in\varnothing} w(x) = 0.$$

Suppose $E \cap F = \varnothing$. By Proposition 6.4.2 we have

$$\sum_{x\in E\cap F} w(x) = \sum_{x\in E} w(x) + \sum_{x\in F} w(x),$$

which is just

$$\mu(E \cup F) = \mu(E) + \mu(F).$$

□

Proposition 7.3.3. *Suppose* $\mathcal{B}$ *is a content on a set* X *and* μ *is a content on* $(X, \mathcal{B})$*. Then*

(a) If $E, F \in \mathcal{B}$ *then*

$$\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F).$$

(b) If $E, F \in \mathcal{B}$ *and* $E \subseteq F$ *then* $\mu(E) \leq \mu(F)$*.*

(c) If $E, F \in \mathcal{B}$ *then*

$$\mu(E \cup F) \leq \mu(E) + \mu(F).$$

(d) If $\mathcal{A}$ *is a finite subset of* $\mathcal{B}$ *and* $E \cap F = \varnothing$ *whenever* $E, F \in \mathcal{A}$ *and* $E \cap F = \varnothing$ *then*

$$\mu\left(\bigcup_{E\in\mathcal{A}} E\right) = \sum_{E\in\mathcal{A}} \mu(E).$$

(e) If $\mathcal{A}$ *is a finite subset of* $\mathcal{B}$ *then*

$$\mu\left(\bigcup_{E\in\mathcal{A}} E\right) \leq \sum_{E\in\mathcal{A}} \mu(E).$$

Proof. We have

$$E \cup F = E \cup (F \setminus E)$$

and

$$E \cap (F \setminus E) = \varnothing$$

so

$$\mu(E \cup F) = \mu(E) + \mu(F \setminus E).$$

Also,

$$F = (F \setminus E) \cup (E \cap F)$$

and

$$F \cap (E \setminus F) = \varnothing$$

so

$$\mu(F) = \mu(F \setminus E) + \mu(E \cap F).$$

It follows that

$$\begin{aligned} \mu(E) + \mu(F) &= \mu(E) + \mu(F \setminus E) + \mu(E \cap F). \\ &= \mu(E \cup F) + \mu(E \cap F). \end{aligned}$$

This is 7.3.3a.

If $E \subseteq F$ then $E \cap F = E$ so the equation

$$\mu(F) = \mu(F \setminus E) + \mu(E \cap F).$$

above becomes

$$\mu(F) = \mu(F \setminus E) + \mu(E).$$

Now $\mu(F \setminus E) \geq 0$ so $\mu(E) \leq \mu(F)$. This is 7.3.3b

By 7.3.3a we have

$$\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F).$$

But $\mu(E \cap F) \geq 0$ so

$$\mu(E \cup F) \leq \mu(E) + \mu(F),$$

which is 7.3.3c.

If $\mathcal{A}$ is a finite subset of $\mathcal{B}$ then we can write it as

$$\mathcal{A} = \{E_1, E_2, \ldots, E_m\}$$

for distinct $E_1, E_2, \ldots, E_m \in \mathcal{B}$. We have

$$\mu\left(\bigcup_{j=1}^{0} E_j\right) = \mu(\varnothing) = 0 = \sum_{j=1}^{0} \mu(E_j).$$

Suppose

$$\mu\left(\bigcup_{j=1}^{k} E_j\right) = \sum_{j=1}^{k} \mu(E_j).$$

Now

$$\bigcup_{j=1}^{k+1} E_j = \left(\bigcup_{j=1}^{k} E_j\right) \cup E_{k+1}.$$

If the E's are disjoint then

$$\left(\bigcup_{j=1}^{k} E_j\right) \cap E_{k+1} = \varnothing$$

so

$$\begin{aligned}\mu\left(\bigcup_{j=1}^{k+1} E_j\right) &= \mu\left(\bigcup_{j=1}^{k} E_j\right) + \mu(E_{k+1}) \\ &= \sum_{j=1}^{k} \mu(E_j) + \mu(E_{k+1}) \\ &= \sum_{j=1}^{k+1} \mu(E_j).\end{aligned}$$

This gives us

$$\mu\left(\bigcup_{j=1}^{k} E_j\right) = \sum_{j=1}^{k} \mu(E_j).$$

but with $k+1$ in place of k. Since we've already seen that the equation holds for $k=0$ we conclude that it holds for all k. In particular it holds for $k=m$, so

$$\mu\left(\bigcup_{j=1}^{k} E_j\right) = \sum_{j=1}^{k} \mu(E_j).$$

This is 7.3.3d

The proof of 7.3.3e is similar. We have

$$\mu\left(\bigcup_{j=1}^{0} E_j\right) = \mu(\varnothing) = 0 \le 0 = \sum_{j=1}^{0} \mu(E_j).$$

Suppose

$$\mu\left(\bigcup_{j=1}^{k} E_j\right) \le \sum_{j=1}^{k} \mu(E_j).$$

It's still true that

$$\bigcup_{j=1}^{k+1} E_j = \left(\bigcup_{j=1}^{k} E_j\right) \cup E_{k+1}.$$

From 7.3.3c we get

$$\begin{aligned}\mu\left(\bigcup_{j=1}^{k+1} E_j\right) &\le \mu\left(\bigcup_{j=1}^{k} E_j\right) + \mu(E_{k+1}) \\ &\le \sum_{j=1}^{k} \mu(E_j) + \mu(E_{k+1}) \\ &= \sum_{j=1}^{k+1} \mu(E_j).\end{aligned}$$

This gives us

$$\mu\left(\bigcup_{j=1}^{k} E_j\right) \le \sum_{j=1}^{k} \mu(E_j).$$

but with $k+1$ in place of k. Since we've already seen that the inequality holds for $k=0$ we conclude that it holds for all k, and in particular for $k=m$, so

$$\mu\left(\bigcup_{j=1}^{k} E_j\right) \le \sum_{j=1}^{k} \mu(E_j).$$

□

Theorem 7.3.4. *Suppose $\mathcal{B}$ is a Boolean algebra on a set X and μ is a content on $(X, \mathcal{B})$. Let $\mathcal{B}^\dagger$ be the set of $F \in \wp(X)$ such that for every $\epsilon > 0$ there are $D, H \in \mathcal{B}$ such that*

$$F \triangle H \subseteq D$$

and

$$\mu(D) < \epsilon.$$

Then $\mathcal{B}^\dagger$ is a Boolean algebra on X and $\mathcal{B} \subseteq \mathcal{B}^\dagger$. For $F \in \mathcal{B}^\dagger$ we define

$$\mu^-(F) = \sup_{\substack{E \in \mathcal{B} \\ E \subseteq F}} \mu(E)$$

and

$$\mu^+(F) = \inf_{\substack{G \in \mathcal{B} \\ F \subseteq G}} \mu(G).$$

Then $\mu^-(F) = \mu^+(F)$ for all $F \in \mathcal{B}^\dagger$. Let $\mu^\dagger(F)$ be their common value. Then $\mu^\dagger$ is a content on $(X, \mathcal{B})$ and

$$\mu^\dagger(F) = \mu(F)$$

for all $F \in \mathcal{B}$.

$(X, \mathcal{B}^\dagger, \mu^\dagger)$ is called the *completion* of $(X, \mathcal{B}, \mu)$.

Proof. First we show that $\mathcal{B} \subseteq \mathcal{B}^\dagger$. For any $F \in \mathcal{B}$ and $\epsilon > 0$ we choose $D = \varnothing$ and $H = F$. Then $D, H \in \mathcal{B}$,

$$F \triangle H = \varnothing \subseteq D$$

and

$$\mu(D) = \mu(\varnothing) = 0 < \epsilon$$

so $F \in \mathcal{B}^\dagger$. So if $F \in \mathcal{B}$ then $F \in \mathcal{B}^\dagger$. In other words,

$$\mathcal{B} \subseteq \mathcal{B}^\dagger.$$

Next we show that $\mathcal{B}^\dagger$ is a Boolean algebra on X. $\varnothing \in \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{B}^\dagger$ so $\varnothing \in \mathcal{B}^\dagger$. This is the first of the required properties for a Boolean algebra.

Suppose $F \in \mathcal{B}^\dagger$, i.e. that for all $\epsilon > 0$ there are $D, H \in \mathcal{B}$ such that

$$F \triangle H \subseteq D$$

and

$$\mu(D) < \epsilon.$$

Then

$$(X \setminus F) \triangle (X \setminus H) = F \triangle H \subseteq D,$$

$X \setminus H \in \mathcal{B}$ and

$$\mu(D) < \epsilon.$$

So $X \setminus F \in \mathcal{B}^\dagger$. This is the second of the required properties for a Boolean algebra.

Suppose $F_1, F_2 \in \mathcal{B}^\dagger$, i.e. that for any $\delta > 0$ there are $D_1, H_1, D_2, H_2 \in \mathcal{B}$ such that

$$F_i \triangle H_i \subseteq D_i$$

and

$$\mu(D_i) < \delta.$$

Here and in the rest of this proof statements involving i are to be interpreted as valid for $i = 1$ and for $i = 2$. If $\epsilon > 0$ then $\epsilon/2 > 0$ so there are $D_1, H_1, D_2, H_2 \in \mathcal{B}$ such that

$$F_i \triangle H_i \subseteq D_i$$

and

$$\mu(D_i) < \epsilon/2.$$

Let

$$D = D_1 \cup D_2$$

and

$$H = H_1 \cup H_2.$$

Then $D, H \in \mathcal{B}$,

$$\begin{aligned}(F_1 \cup F_2) \triangle H &= (F_1 \cup F_2) \triangle (H_1 \cup H_2) \\ &\subseteq (F_1 \triangle H_1) \cup (F_2 \triangle H_2) \\ &\subseteq D_1 \cup D_2 = D\end{aligned}$$

and

$$\mu(D) = \mu(D_1 \cup D_2) \leq \mu(D_1) + \mu(D_2) < \epsilon/2 + \epsilon/2 = \epsilon.$$

So for every $\epsilon > 0$ there are $D, H \in \mathcal{B}$ such that

$$(F_1 \cup F_2) \triangle H \subseteq D$$

and

$$\mu(D) < \epsilon.$$

Therefore $F_1 \cup F_2 \in \mathcal{B}^\dagger$. This is the third and last required property for a Boolean algebra so $\mathcal{B}^\dagger$ is a Boolean algebra.

Next we show that if $F \in \mathcal{B}^\dagger$ then $\mu^-(F) = \mu^+(F)$. If $E, B \in \mathcal{B}$ and $E \subseteq F \subseteq G$ then $E \subseteq G$ and hence

$$\mu(E) \subseteq \mu(G).$$

Taking the supremum over E and the infimum over G gives

$$\mu^-(F) = \sup_{\substack{E \in \mathcal{B} \\ E \subseteq F}} \mu(E) \leq \inf_{\substack{G \in \mathcal{B} \\ F \subseteq G}} \mu(G) = \mu^+(F).$$

So

$$\mu^-(F) \le \mu^+(F).$$

Now we show the reverse inequality. By hypothesis $F \in \mathcal{B}^\dagger$ so for any $\epsilon > 0$ there are $D, H \in \mathcal{B}$ such that

$$F \triangle H \subseteq D$$

and

$$\mu(D) < \epsilon.$$

Let

$$E = H \setminus D$$

and

$$G = H \cup D.$$

Then $E, G \in \mathcal{B}$ and

$$E \subseteq F \subseteq G$$

so

$$\mu(E) \le \mu^-(F)$$

and

$$\mu^+(F) \le \mu(G).$$

On the other hand, $G = E \cup D$ so

$$\mu(G) \le \mu(E) + \mu(D) < \mu(E) + \epsilon.$$

Combining these inequalities,

$$\mu^+(F) < \mu^-(F) + \epsilon.$$

This holds for all $\epsilon > 0$ so

$$\mu^+(F) \le \mu^-(F).$$

Together with the reverse inequality, which we've already proved, this gives

$$\mu^+(F) = \mu^-(F).$$

Next we show that $\mu^\dagger$ is a content.

$$\mu^\dagger(\varnothing) = \mu^-(\varnothing) = \sup_{\substack{E \in \mathcal{B} \\ E \subseteq \varnothing}} \mu(E)$$
$$= \sup_{E = \varnothing} \mu(E) = \mu(\varnothing) = 0.$$

That's the first of the properties of a content.

Suppose $F_1, F_2 \in \mathcal{B}^\dagger$ and $F_1 \cap F_2 = \varnothing$.

$$\{E \in \mathcal{B} \colon E \subseteq F_i\} \subseteq \{E \in \mathcal{B} \colon E \subseteq F_1 \cup F_2\}$$

so

$$\mu(F_i) = \sup_{\substack{E \in \mathcal{B} \\ E \subseteq F_i}} \mu(E) \le \sup_{\substack{E \in \mathcal{B} \\ E \subseteq F_1 \cup F_2}} \mu(E) = \mu(F_1 \cup F_2).$$

It follows that if $\mu(F_i) = +\infty$ then $\mu(F_1 \cup F_2) = +\infty$. In that case

$$\mu(F_1 \cup F_2) = \mu(F_1) + \mu(F_2).$$

It remains to prove this equation when both $\mu(F_1)$ and $\mu(F_2)$ are less than $+\infty$.

Now

$$\mu^+(F_i) = \inf_{\substack{G_i \in \mathcal{B} \\ F_i \subseteq G_i}} \mu(G_i).$$

If $\epsilon > 0$ then $\mu^+(F_i) + \epsilon$ is greater than the infimum and so is not a lower bound. In other words, there is a $G_i \in \mathcal{B}$ such that $F_i \subseteq G_i$ and

$$\mu(G_i) < \mu^+(F_i) + \epsilon.$$

Then

$$\mu(G_1 \cup G_2) \le \mu(G_1) + \mu(G_2) < \mu^+(F_1) + \mu^+(F_2) + 2\epsilon.$$

Now

$$F_1 \cup F_2 \subseteq G_1 \cup G_2$$

and $G_1 \cup G_2 \in \mathcal{B}$ so

$$\mu^+(F_1 \cup F_2) = \inf_{\substack{G \in \mathcal{B} \\ F_1 \cup F_2 \subseteq G}} \mu(G) \le \mu(G_1 \cup G_2).$$

Combining these,

$$\mu^+(F_1 \cup F_2) < \mu^+(F_1) + \mu^+(F_2) + 2\epsilon.$$

This holds for all $\epsilon > 0$ so

$$\mu^+(F_1 \cup F_2) \le \mu^+(F_1) + \mu^+(F_2).$$

Similarly,

$$\mu^-(F_i) = \sup_{\substack{E_i \in \mathcal{B} \\ E_i \subseteq F_i}} \mu(E_i).$$

If $\epsilon > 0$ then $\mu^-(F_i) - \epsilon$ is less than the supremum and so is not an upper bound. In other words, there is an $E_i \in \mathcal{B}$ such that $E_i \subseteq F_i$ and

$$\mu(E_i) > \mu^-(F_i) - \epsilon.$$

Then

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2) > \mu^-(F_1) + \mu^-(F_2) - 2\epsilon.$$

Here we've used the fact that $E_1 \cap E_2 = \varnothing$, which follows from $E_1 \subseteq F_1$, $E_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$. Now

$$E_1 \cup E_2 \subseteq F_1 \cup F_2$$

and $E_1 \cup E_2 \in \mathcal{B}$ so

$$\mu^-(F_1 \cup F_2) = \sup_{\substack{E \in \mathcal{B} \\ E \subseteq F_1 \cup F_2}} \mu(E) \geq \mu(E_1 \cup E_2).$$

Combining these,

$$\mu^-(F_1 \cup F_2) > \mu^-(F_1) - \mu^+(F_2) - 2\epsilon.$$

This holds for all $\epsilon > 0$ so

$$\mu^-(F_1 \cup F_2) \geq \mu^-(F_1) + \mu^-(F_2).$$

We now have

$$\mu^-(F_1 \cup F_2) \geq \mu^-(F_1) + \mu^-(F_2)$$

and

$$\mu^+(F_1 \cup F_2) \leq \mu^+(F_1) + \mu^+(F_2),$$

but

$$\mu^-(F_1) = \mu^\dagger(F_1) = \mu^+(F_1),$$
$$\mu^-(F_2) = \mu^\dagger(F_2) = \mu^+(F_2),$$

and

$$\mu^-(F_1 \cup F_2) = \mu^\dagger(F_1 \cup F_2) = \mu^+(F_1 \cup F_2)$$

so

$$\mu^\dagger(F_1 \cup F_2) = \mu^\dagger(F_1) + \mu^\dagger(F_2).$$

This is the second required property for a content.

Now we show that $\mu^\dagger(F) = \mu(F)$ if $F \in \mathcal{B}$. If $E \in \mathcal{B}$ and $E \subseteq F$ then $\mu(E) \leq \mu^-(F)$, by the definition of μ^-. This applies in particular to $E = F$, so $\mu(F) \leq \mu^-(F)$. Similarly, if $G \in \mathcal{B}$ and $F \subseteq G$ then $\mu^+(F) \leq \mu(G)$. Applying this to $G = F$ gives $\mu^+(F) \leq \mu(F)$. But

$$\mu^-(F) = \mu^\dagger(F) = \mu^+(F)$$

so

$$\mu^\dagger(F) = \mu(F).$$

□

7.4 Jordan content on R

As we already saw, the set of finite unions of intervals in $\mathbf{R}$ is a Boolean algebra, generated by the set of all intervals. There is a natural content on it.

Definition 7.4.1. If I is a non-empty interval then its *length* is defined to be

$$\ell(I) = \sup I - \inf I.$$

The length of $\varnothing$ is defined to be 0.

Proposition 7.4.2. *Suppose $\mathcal{I}$ is the Boolean algebra on $\mathbf{R}$ generated by the intervals. Then there is a unique content μ on $(\mathbf{R}, \mathcal{I})$ such that if $I_1, \ldots, I_m$ are disjoint intervals then*

$$\mu\left(\bigcup_{j=1}^m I_j\right) = \sum_{j=1}^m \ell(I_j).$$

Note that the equation above isn't suitable as a definition because there may be more than one way to write a given set as a union of disjoint intervals and it's not immediately obvious that the right hand side is independent of which way is chosen.

Proof. We define

$$\mu(E) = \lim_{n \to +\infty} \frac{p_n(E)}{2^n}$$

where $p_n(E)$ is the number of points $x \in E$ such that $2^n x \in \mathcal{Z}$. This limit exists when

$$E = \bigcup_{j=1}^m I_j$$

for disjoint intervals $I_1, I_2, \ldots, I_m$ because the number of points $x \in I_j$ such that $2^n x \in \mathbf{Z}$ is within 1 of $2^n \ell(I_j)$ so $p_n(E)$ is within m of $2^n \sum_{j=1}^m \ell(I_j)$. It follows that

$$\lim_{n\to+\infty} \frac{p_n(E)}{2^n} = \sum_{j=1}^m \ell(I_j).$$

It's clear that $\mu(\varnothing) = \varnothing$. If $E \cap F = \varnothing$ then

$$p_n(E \cup F) = p_n(E) + p_n(F)$$

for each $n \in \mathbf{N}$ and hence

$$\mu(E \cup F) = \mu(E) + \mu(F).$$

So μ is a content.

The uniqueness of μ is immediate since every element of $\mathcal{I}$ is of the form $E = \bigcup_{j=1}^m I_j$ for some disjoint intervals $I_1, \ldots, I_m$ and the equation

$$\mu\left(\bigcup_{j=1}^m I_j\right) = \sum_{j=1}^m \mu(\ell(I)).$$

determines its value on any such element. ☐

Definition 7.4.3. The *Jordan algebra* $\mathcal{J}$ on $\mathbf{R}$ and the *Jordan content* μ are the Boolean algebra and content obtained by completing $\mathcal{I}$ and $\mu_{\mathcal{I}}$ as in Theorem 7.3.4.

It's worth noting that the formula

$$\mu(E) = \lim_{n\to+\infty} \frac{p_n(E)}{2^n}$$

holds for $E \in \mathcal{I}$, but needn't hold for $E \in \mathcal{J}$.

Proposition 7.4.4. *Let C be the Cantor set. Then $C \in \mathcal{J}$ and $\mu(C) = 0$ but $C \notin \mathcal{I}$.*

Proof. Let C_n be the union of the 2^n intervals of the form $\left[\frac{j}{3^n}, \frac{j+1}{3^n}\right]$ which contain an element of C. Then $C_n \in \mathcal{I}$, $C \subseteq C_n$ and

$$\mu(C^n) = \left(\frac{2}{3}\right)^n.$$

For any $\epsilon > 0$ there is an n such that

$$\left(\frac{2}{3}\right)^n < \epsilon$$

Then

$$C_n \triangle C = C \setminus C_n \subseteq C_n$$

and

$$\mu(C_n) < \epsilon.$$

In other words, for every $\epsilon > 0$ there are $D, H \in \mathcal{I}$ such that $D \triangle C \subseteq H$ and $\mu(H) < \epsilon$, namely $D = H = C_n$. Therefore $C \in \mathcal{I}^\dagger = \mathcal{J}$. Also, $\varnothing \subseteq C \subseteq C_n$ so

$$0 \le \mu(C) \le \mu(C_n) = \left(\frac{2}{3}\right)^n.$$

This holds for all $n \in \mathbf{N}$ so

$$\mu(C) = 0.$$

If C were an element of $\mathcal{I}$ then we could write it as a finite union of intervals, the sum of the lengths of which is zero. The only intervals of length 0 are empty or singletons, so C would be a finite set. But we've already seen that it's uncountable. ☐

Proposition 7.4.5. $\mathbf{Q} \notin \mathcal{J}$.

Proof. Suppose that I is an interval of positive length which is a subset of $\mathbf{Q}$. Then $\inf I \le \sup I$ so there are $x, z \in I$ such that $x < z$. In between any two rational numbers there is an irrational number so there is a $y \in \mathbf{R} \setminus \mathbf{Q}$ such that $x \le y \le z$. I is an interval so $y \in I$. But $I \subseteq \mathbf{Q}$, so we have a contradiction. Therefore there any interval which is a subset of $\mathbf{Q}$ must have length 0 and hence if E is an element of $\mathcal{I}$ then $\mu(E) = 0$. If $\mathbf{Q} \in \mathcal{J}$ then we have

$$\mu(\mathbf{Q}) = \mu^-(\mathbf{Q}) = \sup_{\substack{E \in \mathcal{I} \\ E \subseteq \mathbf{Q}}} \mu(E) = \sup_{\substack{E \in \mathcal{I} \\ E \subseteq \mathbf{Q}}} 0 = 0.$$

The argument above remains valid if we swap the roles of $\mathbf{Q}$ and $\mathbf{R} \setminus \mathbf{Q}$, so if $\mathbf{R} \setminus \mathbf{Q} \in \mathcal{J}$ then

$$\mu(\mathbf{R} \setminus \mathbf{Q}) = 0.$$

$\mathcal{J}$ is a σ-algebra so if $\mathbf{Q} \in \mathcal{J}$ then $\mathbf{R} \setminus \mathbf{Q} \in \mathcal{J}$ and we have both $\mu(\mathbf{Q}) = 0$ and $\mu(\mathbf{R} \setminus \mathbf{Q}) = 0$). But then

$$\mu(\mathbf{R}) = \mu((\mathbf{R} \setminus \mathbf{Q}) \cup \mathbf{Q}) \le \mu(\mathbf{R} \setminus \mathbf{Q}) + \mu(\mathbf{Q}) = 0 + 0 = 0.$$

But of course $\mu(\mathbf{R}) = +\infty$, so $\mathbf{Q} \notin \mathcal{J}$. ☐

7.5 Banach-Tarski

The following theorem is due to Banach and Tarski:

Theorem 7.5.1. *There are sets $E_1, \ldots, E_m$ and $F_1, \ldots, F_m$ in $\mathbf{R}^3$ with the following properties:*

(a) E_i is congruent to F_i for each i.

(b) The E's are disjoint, i.e. $E_j \cap E_j = \varnothing$ if $i \neq j$.

(c) The F's are disjoint, i.e. $F_j \cap F_j = \varnothing$ if $i \neq j$.

(d) $\bigcup_{i=1}^m E_i$ is a ball of radius 1.

(e) $\bigcup_{i=1}^m F_i$ is the union of two disjoint balls of radius 1.

Banach and Tarski's argument doesn't give a particular value of m, it just shows that there is one. It was shown subsequently by Robinson that one can take $m = 5$ or any higher value but not $m = 4$ or any lower value.

We're not going to prove the Banach-Tarski Theorem, but we will prove the following corollary, assuming the validity Banach-Tarski Theorem.

Corollary 7.5.2. *Suppose $\mathcal{B}$ is a Boolean algebra on $\mathbf{R}^3$ and μ is a content on $(\mathbf{R}^3, \mathcal{B})$ with the following properties:*

(a) If $E \in \mathcal{B}$ and F is congruent to E then $F \in \mathcal{B}$ and $\mu(E) = \mu(F)$.

(b) $B(\mathbf{x}, r) \in \mathcal{B}$ and $\mu(B(\mathbf{x}, r)) = \frac{4}{3}\pi r^3$ for all $\mathbf{x} \in \mathbf{R}^3$ and $r > 0$.

Then $\mathcal{B} \neq \wp(\mathbf{R}^3)$.

This corollary shows that any reasonable theory of volumes of subsets of $\mathbf{R}^3$ must avoid assigning volumes, even infinite volumes, to certain sets. The meaning of the word "reasonable" is incorporated in the definitions of Boolean algebras and contents. We would like to assign a volume to any subsets which arise naturally in examples, but we can't hope to assign one to all subsets.

Proof. Suppose there were such a μ with $\mathcal{B} = \wp(\mathbf{R}^3)$. Let $E_1, \ldots, E_m$ and $F_1, \ldots, F_m$ be as in the Banach-Tarski Theorem. Then $E_1, \ldots, E_m \in \mathcal{B}$ so by the first condition above we have $F_1, \ldots, F_m \in \mathcal{B}$ and

$$\mu(E_i) = \mu(F_i).$$

The E's are disjoint so

$$\mu\left(\bigcup_{i=1}^m E_i\right) = \sum_{i=1}^m \mu(E_i).$$

Similarly, the F's are disjoint so

$$\mu\left(\bigcup_{i=1}^m F_i\right) = \sum_{i=1}^m \mu(F_i).$$

It follows that

$$\mu\left(\bigcup_{i=1}^m E_i\right) = \mu\left(\bigcup_{i=1}^m F_i\right).$$

Now $\bigcup_{i=1}^m E_i$ is a ball of radius 1, so

$$\mu\left(\bigcup_{i=1}^m E_i\right) = \frac{4}{3}\pi.$$

$\bigcup_{i=1}^m F_i$ is a union of two disjoint balls of radius 1, i.e there are B_1 and B_2 which balls of radius 1 such that $\bigcup_{i=1}^m F_i = B_1 \cup B_2$ and $B_1 \cup B_2 = \varnothing$. Therefore

$$\mu\left(\bigcup_{i=1}^m F_i\right) = \mu(B_1) + \mu(B_2)$$

and

$$\mu(B_n) = \frac{4}{3}\pi.$$

It follows that

$$\begin{aligned} \frac{4}{3}\pi = \mu\left(\bigcup_{i=1}^m E_i\right) &= \mu\left(\bigcup_{i=1}^m F_i\right) \\ &= \mu(B_1) + \mu(B_2) = \frac{4}{3}\pi + \frac{4}{3}\pi = \frac{8}{3}\pi \end{aligned}$$

But of course

$$\frac{4}{3}\pi \neq \frac{8}{3}\pi,$$

so the assumption that there is such a μ for $\mathcal{B} = \wp(\mathbf{R}^3)$ must be false. □

Both the theorem and the corollary require the Axiom of Choice and are known not to be true in some versions of Set Theory which do not include this axiom or which include certain weaker versions of it.

7.6 Measures

Definition 7.6.1. Suppose $(X, \mathcal{B})$ is a measurable space. A *measure* on $(X, \mathcal{B})$ is a function $\mu\colon \mathcal{B} \to [0, +\infty]$ such that

(a)
$$\mu(\varnothing) = 0.$$

(b) If $\mathcal{A}$ is a countable subset of $\mathcal{B}$ and $E \cap F = \varnothing$ whenever $E, F \in \mathcal{A}$ and $E \neq F$ then
$$\mu\left(\bigcup_{E\in\mathcal{A}} E\right) = \sum_{E\in\mathcal{A}} \mu(E).$$

A triple $(X, \mathcal{B}, \mu)$ where $(X, \mathcal{B})$ is a measurable space and μ is a measure on $(X, \mathcal{B})$ is called a *measure space*.

Definition 7.6.2. If $(X, \mathcal{B}, \mu)$ is a measure space then a subset E of X is called a *null set* if $E \in \mathcal{B}$ and $\mu(E) = 0$.

Proposition 7.6.3. *If $(X, \mathcal{B}, \mu)$ is a measure space then μ is a content on $(X, \mathcal{B})$.*

Note that every σ-algebra on X is a Boolean algebra on X, so this statement is meaningful.

Proof. The first condition for a content, that $\mu(\varnothing) = 0$, is part of the definition of a measure. For the second condition, suppose $E, F \in \mathcal{B}$ and $E \cap F = \varnothing$. Let $\mathcal{A} = \{E, F\}$. Then $\mathcal{A}$ is a countable subset of $\mathcal{B}$ and $G \cap H = \varnothing$ whenever $G, H \in \mathcal{A}$ and $G \neq H$, so
$$\mu\left(\bigcup_{G\in\mathcal{A}} G\right) = \sum_{G\in\mathcal{A}} \mu(E),$$
which just means
$$\mu(E \cup F) = \mu(E) + \mu(F).$$
So μ also satisfies the second condition for a content. □

Proposition 7.6.4. *All of the contents in Proposition 7.3.2 are measures, if $\mathcal{B}$ is a σ-algebra.*

Proof. As noted in the proof of Proposition 7.3.2, the first four examples are all special cases of the fifth and last, so we only need to establish that
$$\mu(E) = \sum_{x\in E} w(x)$$
is a measure. We still have $\mu(\varnothing) = 0$ so we need only check that if $\mathcal{A} \subseteq \mathcal{B}$ and $E \cap F = \varnothing$ whenever $E \neq F$ then
$$\mu\left(\bigcup_{E\in\mathcal{A}} E\right) = \sum_{E\in\mathcal{A}} \mu(E).$$
Let $S = \bigcup_{E\in\mathcal{A}} E$. By Proposition 6.4.1 we have
$$\sum_{x\in S} w(x) = \sum_{E\in\mathcal{A}} \sum_{x\in E} w(x),$$
which is just
$$\mu\left(\bigcup_{E\in\mathcal{A}} E\right) = \sum_{E\in\mathcal{A}} \mu(E).$$
□

Not every content on a σ-algebra is a measure though.

Proposition 7.6.5. *Suppose X is an infinite set and $\mathcal{B} = \wp(X)$. Define $\mu\colon \mathcal{B} \to [0, +\infty]$ by setting $\mu(E) = 0$ if E is finite and $\mu(E) = +\infty$ if E is infinite. Then μ is a content but is not a measure.*

Proof. $\mu(\varnothing) = 0$ because $\varnothing$ is finite. Suppose $E, F \in \mathcal{B}$. If E and F are finite then $E \cup F$ is finite and
$$\mu(E \cup F) = 0 = 0 + 0 = \mu(E) + \mu(F).$$
If E or F is infinite then $E \cup F$ is infinite and
$$\mu(E \cup F) = +\infty = \mu(E) + \mu(F)$$
because at least one of $\mu(E)$ or $\mu(F)$ is equal to $+\infty$ and the sum of $+\infty$ and anything in $[0, +\infty]$ is $+\infty$. So $\mu(E \cup F) = \mu(E) + \mu(F)$ in all cases. Therefore μ is a content.

To show that μ is not a measure we chose a countable subset $S \subseteq X$ and let $\mathcal{A}$ be the set of subsets of X which have a single element, which belongs to S. Then $\mathcal{A} \subseteq \wp(X) = \mathcal{B}$. If $E, F \in \mathcal{A}$ and $E \neq F$ then $E \cup F = \varnothing$. $\mu(E) = 0$ for all $E \in \mathcal{A}$ since a set with only a single element is finite, so

$$\sum_{E \in \mathcal{A}} \mu(E) = \sum_{E \in \mathcal{A}} 0 = 0.$$

On the other hand,

$$S \subseteq \bigcup_{E \in \mathcal{A}} E$$

since if $x \in S$ then $\{x\} \in E$ and hence $x \in \bigcup_{E \in \mathcal{A}} E$. It follows that $\bigcup_{E \in \mathcal{A}} E$ is infinite, so

$$\mu\left(\bigcup_{E \in \mathcal{A}} E\right) = +\infty.$$

Thus

$$\mu\left(\bigcup_{E \in \mathcal{A}} E\right) \neq \sum_{E \in \mathcal{A}} \mu(E),$$

so μ is not a measure. □

Measures have properties analogous to those proved for contents in Proposition 7.3.3.

Proposition 7.6.6. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space. Then*

(a) If $E, F \in \mathcal{B}$ then

$$\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F).$$

(b) If $E, F \in \mathcal{B}$ and $E \subseteq F$ then $\mu(E) \leq \mu(F)$.

(c) If $E, F \in \mathcal{B}$ then

$$\mu(E \cup F) \leq \mu(E) + \mu(F).$$

(d) If $\mathcal{A}$ is a countable subset of $\mathcal{B}$ and $E \cap F = \varnothing$ whenever $E, F \in \mathcal{A}$ and $E \cap F = \varnothing$ then

$$\mu\left(\bigcup_{E \in \mathcal{A}} E\right) = \sum_{E \in \mathcal{A}} \mu(E).$$

(e) If $\mathcal{A}$ is a countable subset of $\mathcal{B}$ then

$$\mu\left(\bigcup_{E \in \mathcal{A}} E\right) \leq \sum_{E \in \mathcal{A}} \mu(E).$$

Proof. Measures are contents so 7.6.6a, 7.6.6b and 7.6.6e follow from 7.3.3a, 7.3.3b and 7.3.3c. 7.6.6d is part of the definition of a measure.

If $\mathcal{A}$ is a countable subset of $\mathcal{B}$. The finite case was covered by Proposition 7.3.3 so we need only consider the case where $\mathcal{A}$ is countably infinite, in which case we can write it as

$$\mathcal{A} = \{E_0, E_1, E_2, \ldots\}$$

for distinct $E_0, E_1, E_2, \in \mathcal{B}$. Set

$$G_j = E_j \setminus \bigcup_{i<j} E_i.$$

Then

$$G_j \subseteq E_j.$$

Also, for every $x \in \bigcup_{j=0}^{\infty} E_j$ there is a first value of j for which $x \in E_j$ and so $x \in G_j$ for this j. It follows that

$$\bigcup_{j=0}^{\infty} E_j \subseteq \bigcup_{j=0}^{\infty} G_j$$

These inclusions, together with 7.6.6b, give

$$\mu(G_j) \leq \mu(E_j)$$

and

$$\mu\left(\bigcup_{j=0}^{\infty} E_j\right) \leq \mu\left(\bigcup_{j=0}^{\infty} G_j\right).$$

Now $G_j \cap G_k = \varnothing$ if $j \neq k$ so

$$\mu\left(\bigcup_{j=0}^{\infty} G_j\right) = \sum_{j=0}^{\infty} \mu(G_j).$$

Combining this with the two inequalities which precede it gives

$$\mu\left(\bigcup_{j=0}^{\infty} E_j\right) \leq \sum_{j=0}^{\infty} \mu(E_j),$$

which is 7.6.6e □

Proposition 7.6.7. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $E\colon \mathbf{N} \to \mathcal{B}$ is a sequence of sets which is monotone increasing in the sense that $E_j \subseteq E_k$ if $j \leq k$. Then*

$$\mu\left(\bigcup_{j=0}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

Proof. The fact that $\bigcup_{j=0}^{\infty} E_j \in \mathcal{B}$, and hence that $\mu\left(\bigcup_{j=0}^{\infty} E_j\right)$ is meaningful, follows from the assumption that $\mathcal{B}$ is a σ-algebra.

As in the proof of 7.6.6e, define

$$G_j = E_j \setminus \bigcup_{i<j} E_i.$$

As before

$$\bigcup_{j=0}^{\infty} G_j = \bigcup_{j=0}^{\infty} E_j$$

and the G's are disjoint so

$$\mu\left(\bigcup_{j=0}^{\infty} E_j\right) = \sum_{j=0}^{\infty} \mu(G_j).$$

The same argument gives the corresponding properties of the partial sums.

$$\bigcup_{j=0}^{m} G_j = \bigcup_{j=0}^{m} E_j$$

and

$$\mu\left(\bigcup_{j=0}^{m} E_j\right) = \sum_{j=0}^{m} \mu(G_j).$$

The monotonicity assumption on E means that

$$\bigcup_{j=0}^{m} E_j = E_m.$$

So

$$\mu(E_m) = \sum_{j=0}^{m} \mu(G_k)$$

and

$$\begin{aligned}\mu\left(\bigcup_{j=0}^{\infty} E_j\right) &= \sum_{j=0}^{\infty} \mu(G_j)\\ &= \lim_{m\to\infty} \sum_{j=0}^{m} \mu(G_j)\\ &= \lim_{m\to\infty} \mu(E_m).\end{aligned}$$

□

There is a corresponding result for intersections of decreasing sequences, but it requires and additional hypothesis.

Proposition 7.6.8. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $E\colon \mathbf{N} \to \mathcal{B}$ is a sequence of sets which is monotone decreasing in the sense that $E_j \supseteq E_k$ if $j \leq k$. If $\mu(E_m) < +\infty$ for some m then*

$$\mu\left(\bigcap_{j=0}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

Proof. Let

$$F_j = E_m \setminus E_{j+m}.$$

If $j \leq k$ then $j + m \leq k + m$ so $E_{j+m} \supseteq E_{k+m}$ and $F_j \subseteq F_k$. It follows from the preceding proposition that

$$\mu\left(\bigcup_{j=0}^{\infty} F_j\right) = \lim_{j\to\infty} \mu(F_k).$$

Now

$$E_m = F_j \cup E_{j+m}$$

and

$$F_j \cap E_{j+m} = \varnothing$$

so

$$\mu(E_m) = \mu(F_j) + \mu(E_{j+m}).$$

Taking limits,

$$\begin{aligned}\mu(E_m) &= \lim_{j\to\infty} \mu(F_j) + \lim_{j\to\infty} \mu(E_{j+m})\\ &= \mu\left(\bigcup_{j=0}^{\infty} F_j\right) + \lim_{j\to\infty} \mu(E_j).\end{aligned}$$

Now

$$\bigcup_{j=0}^{\infty} F_j = \bigcup_{j=0}^{\infty} (E_m \setminus E_{j+m}) = E_m \setminus \left(\bigcap_{j=0}^{\infty} E_{j+m} \right)$$

and

$$\bigcap_{j=0}^{\infty} E_{j+m} \subseteq E_m$$

so

$$E_m = \left(\bigcap_{j=0}^{\infty} E_{j+m} \right) \cup \left(\bigcup_{j=0}^{\infty} F_j \right)$$

and

$$\left(\bigcap_{j=0}^{\infty} E_{j+m} \right) \cap \left(\bigcup_{j=0}^{\infty} F_j \right) = \varnothing.$$

Therefore

$$\mu(E_m) = \mu \left(\bigcap_{j=0}^{\infty} E_{j+m} \right) + \mu \left(\bigcup_{j=0}^{\infty} F_j \right).$$

If $x \in \bigcap_{j=0}^{\infty} E_{j+m}$ then $x \in E_m$ and, by the monotonicity assumption on E, $x \in E_k$ for all $k < m$. But also $x \in E_k$ for all $k \geq m$ because such k can be written as $j + m$ for some $m \in \mathbf{N}$. It follows that $x \in \bigcap_{k=0}^{\infty} E_k$. Therefore

$$\bigcap_{j=0}^{\infty} E_{j+m} \subseteq \bigcap_{k=0}^{\infty} E_k.$$

The reverse inclusion holds as well because if $k = j + m$ and $j \geq 0$ then $k \geq 0$, so

$$\bigcap_{j=0}^{\infty} E_{j+m} = \bigcap_{k=0}^{\infty} E_k.$$

We therefore have

$$\mu(E_m) = \mu \left(\bigcap_{k=0}^{\infty} E_k \right) + \mu \left(\bigcup_{j=0}^{\infty} F_j \right),$$

or, changing the indices,

$$\mu(E_m) = \mu \left(\bigcap_{j=0}^{\infty} E_j \right) + \mu \left(\bigcup_{j=0}^{\infty} F_j \right),$$

We combine this with the equation

$$\mu(E_m) = \mu \left(\bigcup_{j=0}^{\infty} F_j \right) + \lim_{j \to \infty} \mu(E_j)$$

obtained earlier. Either of these equations, together with the fact that $\mu(E_m) < +\infty$, gives $\mu \left(\bigcup_{j=0}^{\infty} F_j \right) < +\infty$ so we can subtract it from both sides to obtain

$$\mu \left(\bigcap_{j=0}^{\infty} E_j \right) = \lim_{j \to \infty} \mu(E_j).$$

□

Without the hypothesis that $\mu(E_m) < +\infty$ for some m the statement would not be true. To see this, consider $\mathbf{N}$ with counting measure and

$$E_j = \{k \in \mathbf{N} \colon j \leq k\}.$$

This gives

$$\mu \left(\bigcap_{j=0}^{\infty} E_j \right) = \mu(\varnothing) = 0$$

and

$$\lim_{j \to \infty} \mu(E_j) = \lim_{j \to \infty} +\infty = +\infty.$$

Definition 7.6.9. A measure space $(X, \mathcal{B}, \mu)$ is called *finite* if $\mu(X) < +\infty$ and is called σ-finite if there is a countable subset $\mathcal{A} \subseteq \mathcal{B}$ such that $X = \bigcup_{E \in \mathcal{A}} E$ and $\mu(E) < +\infty$ for all $E \in \mathcal{A}$.

Note that $(X, \mathcal{B}, \mu)$ does not mean that X is a finite set. In fact neither of these statements implies the other.

Definition 7.6.10. Suppose $(X, \mathcal{T})$ is a locally compact σ-compact Hausdorff topological space. Let $\mathcal{B}$ be the Borel σ-algebra on X. A measure μ on $(X, \mathcal{B})$ is called a *Borel measure*. If it also satisfies the following conditions then it is called a *Radon measure*:

(a) If K is a compact subset of X then $\mu(K) < +\infty$.

(b) If $E \in \mathcal{B}$ then $\mu(E) = \sup \mu(K)$, where the supremum is over all compact subsets K of E.

(c) If $E \in \mathcal{B}$ then $\mu(E) = \inf \mu(U)$, where the infimum is over all open supersets U of E.

Lebesgue measure, which we will define in a later chapter, is a Radon measure on $\mathbf{R}$.

The following theorem is the analogue for measures of Theorem 7.3.4 for contents.

Theorem 7.6.11. *Suppose $\mathcal{B}$ is a σ-algebra on a set X and μ is a measure on $(X, \mathcal{B})$. Let $\mathcal{B}^\dagger$ be the set of $F \in \wp(X)$ such that for every $\epsilon > 0$ there are $D, H \in \mathcal{B}$ such that*

$$F \triangle H \subseteq D$$

and

$$\mu(D) < \epsilon.$$

Then $\mathcal{B}^\dagger$ is a σ-algebra on X and $\mathcal{B} \subseteq \mathcal{B}^\dagger$. For $F \in \mathcal{B}^\dagger$ we define

$$\mu^-(F) = \sup_{\substack{E \in \mathcal{B} \\ E \subseteq F}} \mu(E)$$

and

$$\mu^+(F) = \inf_{\substack{G \in \mathcal{B} \\ F \subseteq G}} \mu(G).$$

Then $\mu^-(F) = \mu^+(F)$ for all $F \in \mathcal{B}^\dagger$. Let $\mu^\dagger(F)$ be their common value. Then $\mu^\dagger$ is a measure on $(X, \mathcal{B}^\dagger)$ and

$$\mu^\dagger(F) = \mu(F)$$

for all $F \in \mathcal{B}$.

$(X, \mathcal{B}^\dagger, \mu^\dagger)$ is called the *completion* of $(X, \mathcal{B}, \mu)$.

Proof. $\mathcal{B}$ is a Boolean algebra and μ is a content so we can use Theorem 7.3.4 to conclude that $\mathcal{B}^\dagger$ is a Boolean algebra on X, that $\mathcal{B} \subseteq \mathcal{B}^\dagger$, that $\mu^+(F) = \mu^-(F)$ for all $F \in \mathcal{B}^\dagger$, that $\mu^\dagger$ is a content on $(X, \mathcal{B}^\dagger)$ and that $\mu^\dagger(F) = \mu(F)$ for all $F \in \mathcal{B}$. The only things which remain to be proved are that $\mathcal{B}^\dagger$ is a σ-algebra rather than just a Boolean algebra and that $\mu^\dagger$ is a measure rather than just a content. In other words, we need to show that

$$\bigcup_{F \in \mathcal{A}} F \in \mathcal{B}^\dagger$$

if $\mathcal{A}$ is a countable subset of $\mathcal{B}^\dagger$ and that

$$\mu^\dagger\left(\bigcup_{F \in \mathcal{A}} F\right) = \sum_{F \in \mathcal{A}} \mu^\dagger(F)$$

if, in addition, the F's are disjoint. Only the countably infinite case is needed because for finite $\mathcal{A}$ we already have both statements. We can therefore assume that

$$\mathcal{A} = \{F_0, F_1, \ldots\}$$

for some sequence of distinct F's and prove that

$$\bigcup_{j=0}^{\infty} F_j \in \mathcal{B}^\dagger$$

and

$$\mu^\dagger\left(\bigcup_{j=0}^{\infty} F_j\right) = \sum_{j=0}^{\infty} \mu^\dagger(F_j).$$

$F_i \in \mathcal{B}^\dagger$ so for any $\delta_i > 0$ there are $D_i, H_i \in \mathcal{B}$ such that $F_i \triangle H_i \subseteq D_i$ and $\mu(D_i) < \delta_i$. If $\epsilon > 0$ then

$$\delta_i = \frac{\epsilon}{2^{i+1}} > 0$$

we can choose D_i and H_i such that

$$F_i \triangle H_i \subseteq D_i$$

and

$$\mu(D_i) < \frac{\epsilon}{2^{i+1}}.$$

Let

$$D = \bigcup_{i=0}^{\infty} D_i,$$

$$F = \bigcup_{i=0}^{\infty} F_i$$

and

$$H = \bigcup_{i=0}^{\infty} H_i.$$

If $x \in F \triangle H$ then $x \in F$ and $x \notin H$ or $x \in H$ and $x \notin F$. In the former case $x \in F_i$ for some i but $x \notin H_j$ for any j. In particular $x \notin H_i$ so $x \in F_i \triangle H_i$ and therefore $x \in D_i$ and $x \in D$. The same argument works in the latter case, with the roles of F_i and H_i reversed. So

$$F \triangle H \subseteq D.$$

Also,

$$\mu(D) = \mu\left(\bigcup_{i=0}^{\infty} D_i\right) \leq \sum_{i=0}^{\infty} \mu(D_i) < \sum_{i=0}^{\infty} \frac{\epsilon}{2^{i+1}} = \epsilon.$$

So $F \in \mathcal{B}^\dagger$. Therefore $\mathcal{B}^\dagger$ is a σ-algebra.

$$F_i \subseteq \bigcup_{j=0}^{\infty} F_i$$

so

$$\mu^-(F_i) \leq \mu^-\left(\bigcup_{j=0}^{\infty} F_j\right)$$

and hence

$$\mu^\dagger(F_i) \leq \mu^\dagger\left(\bigcup_{j=0}^{\infty} F_j\right).$$

If $\mu^\dagger(F_i) = +\infty$ for some i then $\mu^\dagger\left(\bigcup_{j=0}^{\infty} F_j\right) = +\infty$ and so

$$\mu^\dagger\left(\bigcup_{j=0}^{\infty} F_j\right) = \sum_{j=0}^{\infty} \mu^\dagger(F_j).$$

We may therefore restrict our attention to the case where $\mu^\dagger(F_i) < +\infty$ for all i.

$$\mu^+(F_i) = \inf_{\substack{G_i \in \mathcal{B} \\ F_i \subseteq G_i}} \mu(G_i)$$

If $\epsilon > 0$ then $\mu^+(F_i) + \epsilon/2^{i+1}$ is greater than the infimum so there is a $G_i \in \mathcal{B}$ such that $F_i \subseteq G_i$ and

$$\mu(G_i) < \mu^+(F_i) + \frac{\epsilon}{2^{i+1}}.$$

Let

$$G = \bigcup_{i=0}^{\infty} G_i.$$

Then

$$\begin{aligned}
\mu(G) &\leq \sum_{i=0}^{\infty} \mu(G_i) \\
&< \sum_{i=0}^{\infty} \left(\mu^+(F_i) + \frac{\epsilon}{2^{i+1}}\right) \\
&= \sum_{i=0}^{\infty} \mu^+(F_i) + \epsilon.
\end{aligned}$$

Now $F \subseteq G$ and $G \in \mathcal{B}$ so $\mu^+(F) \leq \mu(G)$, and therefore

$$\mu^+(F) < \sum_{i=0}^{\infty} \mu^+(F_i) + \epsilon.$$

This holds for all $\epsilon > 0$ so

$$\mu^+(F) \leq \sum_{i=0}^{\infty} \mu^+(F_i).$$

Similarly,

$$\mu^-(F_i) = \sup_{\substack{E_i \in \mathcal{B} \\ E_i \subseteq F_i}} \mu(E_i)$$

If $\epsilon > 0$ then $\mu^-(F_i) - \epsilon/2^{i+1}$ is less than the supremum so there is an $E_i \in \mathcal{B}$ such that $E_i \subseteq F_i$ and

$$\mu(E_i) < \mu^-(F_i) - \frac{\epsilon}{2^{i+1}}.$$

Let

$$E = \bigcup_{i=0}^{\infty} E_i.$$

Then

$$\begin{aligned}
\mu(E) &= \sum_{i=0}^{\infty} \mu(E_i) \\
&> \sum_{i=0}^{\infty} \left(\mu^-(F_i) - \frac{\epsilon}{2^{i+1}}\right) \\
&= \sum_{i=0}^{\infty} \mu^-(F_i) + \epsilon.
\end{aligned}$$

In the first line above we've used the fact that the F's are disjoint so the E's, which are subsets of the

F's, are also disjoint. Now $E \subseteq F$ and $F \in \mathcal{B}$ so $\mu^-(F) \geq \mu(E)$, and therefore

$$\mu^-(F) > \sum_{i=0}^{\infty} \mu^-(F_i) - \epsilon.$$

This holds for all $\epsilon > 0$ so

$$\mu^-(F) \geq \sum_{i=0}^{\infty} \mu^-(F_i).$$

But $F_i \in \mathcal{B}^\dagger$ for each i and $F \in \mathcal{B}^\dagger$

$$\mu^-(F_i) = \mu^\dagger(F_i) = \mu^+(F_i)$$

and

$$\mu^-(F) = \mu^\dagger(F) = \mu^+(F).$$

From

$$\mu^+(F) \leq \sum_{i=0}^{\infty} \mu^+(F_i)$$

and

$$\mu^-(F) \geq \sum_{i=0}^{\infty} \mu^-(F_i).$$

it therefore follows that

$$\mu^\dagger(F) = \sum_{i=0}^{\infty} \mu^\dagger(F_i).$$

Thus $\mu^\dagger$ is a measure on $(X, \mathcal{B}^\dagger)$. $\square$

Proposition 7.6.12. *Suppose that $(X, \mathcal{B}, \mu)$ and $(X, \mathcal{B}^\dagger, \mu^\dagger)$ are as in the preceding theorem. Then $F \in \mathcal{B}^\dagger$ if and only if there are $D, H \in \mathcal{B}$ such that*

$$F \triangle H \subseteq D$$

and

$$\mu(D) = 0.$$

Then

$$\mu^\dagger(F) = \mu(H).$$

Proof. Suppose that there are $D, H \in \mathcal{B}$ such that

$$F \triangle H \subseteq D$$

and

$$\mu(D) = 0.$$

For any $\epsilon > 0$ we have $\mu(D) < \epsilon$ so $F \in \mathcal{B}^\dagger$.

Suppose, conversely, that $F \in \mathcal{B}^\dagger$. $1/2^{k+1} > 0$ so there are $D_k, H_k \in \mathcal{B}$ such that

$$F \triangle H_k \subseteq D_k$$

and

$$\mu(D_k) < \frac{1}{2^{k+1}}.$$

Let

$$D = \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} D_j$$

and

$$H = \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} H_j.$$

Note that $D, H \in \mathcal{B}$. Now

$$F \triangle H = \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} F \triangle H_i.$$

$F \triangle H_i \subseteq D_i$ so

$$\bigcup_{j=i}^{\infty} F \triangle H_i \subseteq \bigcup_{j=i}^{\infty} D_i$$

and

$$\bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} F \triangle H_i \subseteq \bigcap_{i=0}^{\infty} \bigcup_{j=i}^{\infty} D_i,$$

i.e.

$$F \triangle H \subseteq D.$$

Also,

$$\mu(\bigcup_{j=i}^{\infty} D_j) \leq \sum_{j=i}^{\infty} \mu(D_j) \leq \sum_{j=i}^{\infty} \frac{1}{2^{j+1}} = \frac{1}{2^i}.$$

The sequence of sets $\bigcup_{j=i}^{\infty} D_j$ is monotone decreasing

$\mu(D_0) < +\infty$ so

$$\mu(D) = \mu\left(\bigcap_{i=0}^{\infty}\bigcup_{j=i}^{\infty} D_j\right)$$
$$= \lim_{i\to\infty} \mu\left(\bigcup_{j=i}^{\infty} D_j\right)$$
$$\le \lim_{i\to\infty} \frac{1}{2^i} = 0.$$

So there are $D, H \in \mathcal{B}$ such that $F \triangle H \subseteq D$ and $\mu(D) = 0$.

$$\mu^\dagger(H \cup D) + \mu^\dagger(H \cap D) = \mu^\dagger(H) + \mu^\dagger(D)$$

and $\mu^\dagger(D) = \mu(D) = 0$, from which it follows that $\mu^\dagger(H \cap D) = 0$ as well. Therefore

$$\mu^\dagger(H \cup D) = \mu^\dagger(H).$$

Now $F \subseteq H \cup D$ so

$$\mu^\dagger(F) \le \mu^\dagger(H \cup D) = \mu^\dagger(H) = \mu(H).$$

On the other hand,

$$\mu^\dagger(F \cup D) + \mu^\dagger(F \cap D) = \mu^\dagger(F) + \mu^\dagger(D),$$

$\mu^\dagger(D) = 0$ and $\mu^\dagger(H \cap D) = 0$ so

$$\mu^\dagger(F \cup D) = \mu^\dagger(F).$$

Now $H \subseteq F \cup D$ so

$$\mu(H) = \mu^\dagger(H) \le \mu^\dagger(F \cup D) = \mu^\dagger(F).$$

Therefore

$$\mu^\dagger(F) = \mu(H).$$

□

Proposition 7.6.13. *Suppose that $(X, \mathcal{B}, \mu)$ and $(X, \mathcal{B}^\dagger, \mu^\dagger)$ are as in the preceding theorem. The following two statements are equivalent:*

(a) $F \in \mathcal{B}^\dagger$ and $\mu^\dagger(F) = 0$.

(b) There is a $G \in \mathcal{B}$ such that $F \subseteq G$ and $\mu(G) = 0$.

Proof. Suppose $F \in \mathcal{B}^\dagger$ and $\mu^\dagger(F) = 0$. By the preceding proposition there are $D, H \in \mathcal{B}$ such that $F \triangle H \subseteq D$, $\mu(D) = 0$ and $\mu(H) = \mu(F) = 0$. Let $G = D \cup H$. Then $G \in \mathcal{B}$ and

$$\mu(G) = \mu(D \cup H) \le \mu(D) + \mu(H) = 0$$

and hence $\mu(G) = 0$. Also $F \subseteq G$.

Suppose conversely that there is a $G \in \mathcal{B}$ such that $F \subseteq G$ and $\mu(G) = 0$. Let $D = H = G$. Then

$$F \triangle H = G \setminus F \subseteq G = D$$

and $\mu(D) = \mu(G) = 0$. So $F \in \mathcal{B}^\dagger$ by the preceding proposition. Also, $F \subseteq G$ so

$$\mu^\dagger(F) \le \mu^\dagger(G) = \mu(G) = 0$$

and hence $\mu^\dagger(F) = 0$. □

7.7 Atomic algebras

Definition 7.7.1. An *atomic algebra* on a set X is a $\mathcal{B} \in \wp(\wp(X))$ satisfying the following conditions.

(a) $\varnothing \in \mathcal{B}$.

(b) If $E \in \mathcal{B}$ then $X \setminus E \in \mathcal{B}$.

(c) If $\mathcal{A} \subseteq \mathcal{B}$ then $\bigcup_{E \in \mathcal{A}} E \in \mathcal{B}$.

Proposition 7.7.2. *Every atomic algebra is a σ-algebra and a Boolean algebra.*

Proof. The first two conditions in the definition are identical. Suppose that $\mathcal{B}$ is an atomic algebra, so that if $\mathcal{A} \subseteq \mathcal{B}$ then $\bigcup_{E \in \mathcal{A}} E \in \mathcal{B}$. Then this holds in particular for all countable $\mathcal{A}$ or for all finite $\mathcal{A}$. The former shows that $\mathcal{B}$ is a σ-algebra while the latter shows that it is a Boolean algebra. □

Not every Boolean algebra or σ-algebra is an atomic algebra. For example $\mathcal{I}$ is a Boolean algebra which is not an atomic algebra. $\mathcal{I}$ contains the set $\{x\}$ for each $x \in \mathbf{R}$. If it were an atomic algebra it would contain every union of such sets and therefore every subset of $\mathbf{R}$, but not every subset is a finite union of intervals. Similarly, the Borel σ-algebra on $\mathbf{R}$ is not an atomic algebra because it contains $\{x\}$ for every $x \in \mathbf{R}$ but not every subset of $\mathbf{R}$ is a Borel set. This last statement is more complicated to prove however.

Proposition 7.7.3. *Suppose $\mathcal{B}$ is an atomic algebra and $\mathcal{A} \subseteq \mathcal{B}$. Then $\bigcap_{E\in\mathcal{A}} E \in \mathcal{B}$.*

Proof.

$$\bigcap_{E\in\mathcal{A}} E = X \setminus \bigcap_{E\in\mathcal{A}} (X \setminus E).$$

□

We can construct atomic algebras via partitions or equivalence relations.

Definition 7.7.4. Suppose X is a set. A *partition* of X is a $\mathcal{P} \in \wp(\wp(X))$ such that

(a) If $E \in \mathcal{B}$ then $E \neq \varnothing$

(b) If $E, F \in \mathcal{P}$ then $E = F$ or $E \cap F = \varnothing$.

(c) $X = \bigcup_{E\in\mathcal{P}} E$.

Proposition 7.7.5. *If $\sim$ is an equivalence relation on X then the set of equivalence classes with respect to $\sim$ is a partition of X.*

Proof. For every $x \in X$ the set $\{y \in X : x \sim y\}$ is an equivalence class by definition and $x \in \{y \in X : x \sim y\}$ so every element of X belongs to some equivalence class and every equivalence class is non-empty. Suppose E and F are equivalence classes, i.e. that $E = \{y \in X : w \sim y\}$ and $F = \{y \in X : x \sim y\}$ for some $w, x \in X$. If $E \cap F \neq \varnothing$ then there a $z \in E \cap F$. Then $w \sim z$ and $x \sim z$. Also therefore $z \sim w$ and $z \sim x$ If $y \in F$ then $x \sim y$. From this and $z \sim x$ it follows that $z \sim y$. From that and $w \sim z$ it follows that $w \sim y$, i.e. that $y \in E$. So $F \subseteq E$. The same argument with the roles of w and x reversed along with those of E and F gives $E \subseteq F$. So $E = F$. We've just seen that for any equivalence classes E and F, $E \cap F = \varnothing$ implies $E = F$. In other words, $E = F$ or $E \cap F = \varnothing$. □

Proposition 7.7.6. *Suppose $\mathcal{P}$ is a partition of X. Let $\mathcal{B}$ be set of all sets of the form $\bigcup_{F\in\mathcal{Q}} F$ where $\mathcal{Q} \subseteq \mathcal{P}$. Then $\mathcal{B}$ is an atomic algebra.*

Proof. $\varnothing \subseteq \mathcal{P}$ and $\varnothing = \bigcup_{E\in\varnothing} E$ so $\varnothing \in \mathcal{B}$.

If $E \in \mathcal{B}$ then $E = \bigcup_{F\in\mathcal{Q}} F$ for some $\mathcal{Q} \subseteq \mathcal{P}$. Then

$$X \setminus E = X \setminus \bigcup_{F\in\mathcal{Q}} F = \bigcap_{F\in\mathcal{Q}} (X \setminus F).$$

Suppose $x \in X \setminus E$. Then $x \notin F$ for any $F \in \mathbf{X}$. On the other hand, $x \in X = \bigcup_{F\in\mathcal{P}} F$, so $x \in F$ for some $F \in \mathcal{P}$. Therefore $x \in F$ for some $F \in \mathcal{P} \setminus \mathcal{Q}$, i.e. $x \in \bigcup_{F\in\mathcal{P}\setminus\mathcal{Q}} F$. Suppose, conversely, that $x \in \bigcup_{F\in\mathcal{P}\setminus\mathcal{Q}} F$, i.e. that there is an $F \in \mathcal{P} \setminus \mathcal{Q}$ such that $x \in F$. If $G \in \mathcal{Q}$ then $F \neq G$ so $F \cap G = \varnothing$. Therefore $x \notin G$, i.e. $x \in X \setminus G$. This holds for all $G \in \mathcal{Q}$ so $x \in \bigcap_{G\in\mathcal{Q}} (X \setminus G)$. Therefore $x \in X \setminus E$. So we've now seen that $x \in X \setminus E$ if and only if $x \in \bigcup_{F\in\mathcal{P}\setminus\mathcal{Q}} F$, i.e. that

$$X \setminus E = \bigcup_{F\in\mathcal{P}\setminus\mathcal{Q}} F.$$

But $\mathcal{P} \setminus \mathcal{Q} \subseteq \mathcal{P}$ so the set on the left belongs to $\mathcal{B}$.

If $A \subseteq B$ then for each $E \in \mathcal{A}$ there is a $\mathcal{Q}_E$ such that $E = \bigcup_{F\in\mathcal{Q}_E} F$. Then

$$\bigcup_{E\in\mathcal{A}} E = \bigcup_{E\in\mathcal{A}} \bigcup_{F\in\mathcal{Q}_E} F = \bigcup_{f\in\bigcup_{E\in\mathcal{A}} \mathcal{Q}_E} F$$

so $\bigcup_{E\in\mathcal{A}} E \in \mathcal{B}$. □

Proposition 7.7.7. *Suppose $\mathcal{B}$ is an atomic algebra on a set X. Define a relation $\sim$ on X by $x \sim y$ if $x \in E \Leftrightarrow y \in E$ for all $E \in \mathcal{B}$. Then $\sim$ is an equivalence relation.*

Proof. Trivially $x \in E \Leftrightarrow x \in E$, so $x \sim x$. Also, if $x \in E \Leftrightarrow y \in E$ then $y \in E \Leftrightarrow x \in E$ so if $x \sim y$ then $y \sim x$. Finally, if $x \in E \Leftrightarrow y \in E$ and $y \in E \Leftrightarrow z \in E$ then $x \in E \Leftrightarrow z \in E$, so if $x \sim y$ and $y \sim z$ then $x \sim z$. □

Proposition 7.7.8. *Suppose $\mathcal{B}$ is an atomic algebra on a set X, $\sim$ is the equivalence relation defined by $x \sim y$ if and only if for all $E \in \mathcal{B}$ we have $x \in E \Leftrightarrow y \in E$, $\mathcal{P}$ is the set of equivalence classes for the relation $\sim$ and $\mathcal{B}'$ is the set of unions of these equivalence classes. Then $\mathcal{B}' = \mathcal{B}$.*

Proof. Suppose $E \in \mathcal{B}$. For each $x \in E$ define

$$C_x = \{y \in X : x \sim y\}.$$

In other words, C_x is the equivalence class of x with respect to the equivalence relation $\sim$. Let

$$F = \bigcup_{x\in E} C_x.$$

This is a union of equivalence classes so $F \in \mathcal{B}'$. Suppose $w \in E$. Then $w \in C_w$ because $w \sim w$ so $w \in F$. Suppose, conversely, that $w \in F$. Then $w \in C_x$ for some $x \in E$ and therefore $x \sim w$. So $w \in E$, by the definition of the relation $\sim$. So $w \in E$ if and only if $w \in F$ and therefore $E = F$. So $E \in \mathcal{B}'$. So if $E \in \mathcal{B}$ then $E \in \mathcal{B}'$.

Suppose $E \in \mathcal{B}'$. In other words, there is some set $\mathcal{Q}$ of equivalence classes such that

$$E = \bigcup_{C \in \mathcal{Q}} C.$$

If $C \in \mathcal{Q}$ then C is an equivalence class so

$$C = \{y \in X : x \sim y\}$$

for some $x \in X$. By the definition of $\sim$ we have $y \in C$ if and only if $y \in E$ for all $E \in \mathcal{B}$ such that $x \in E$. In other words,

$$C = \bigcap_{\substack{E \in \mathcal{B} \\ x \in E}} E.$$

This is an intersection of elements of B and so is an element of $\mathcal{B}$. This holds for each $C \in \mathcal{Q}$ so E is a union of elements of $\mathcal{B}$ and therefore also an element of $\mathcal{B}$. So if $E \in \mathcal{B}'$ then $E \in \mathcal{B}$. We already proved the reverse implication so $E \in \mathcal{B}$ if and only if $E \in \mathcal{B}'$. In other words, $\mathcal{B} = \mathcal{B}'$. ☐

We started with an atomic algebra and then went through an equivalence relation and a partition to get back to an atomic algebra but we could equally well have started with the equivalence relation or the partition. The point of the propositions above is that atomic algebras, equivalence relations and partitions are largely equivalent concepts.

Definition 7.7.9. A *system of weights* for a set X is a function $w : X \to [0, +\infty]$. It is called *finite* if the set

$$\{x \in X : w(x) > 0\}$$

is finite and is called *countable* if the set is countable.

For any set X the set $\wp(X)$ is an atomic algebra on X. We've already seen that for system of weights $w : X \to [0, +\infty]$ the function $\mu : \wp(X) \to [0, +\infty]$ defined by

$$\mu(E) = \sum_{x \in E} w(x)$$

is a measure. That means in particular that it is countably additive, i.e. that if $\mathcal{A} \subseteq \wp(X)$ is a countable set of disjoint subsets then

$$\mu\left(\bigcup_{E \in \mathcal{A}} E\right) = \sum_{E \in \mathcal{A}} \mu(E).$$

This equation in fact holds even without the assumption that $\mathcal{A}$ is countable. Indeed,

$$\begin{aligned} \mu\left(\bigcup_{E \in \mathcal{A}} E\right) &= \sum_{x \in \bigcup_{E \in \mathcal{A}} E} w(x) \\ &= \sum_{E \in \mathcal{A}} \sum_{x \in E} w(x) \\ &= \sum_{E \in \mathcal{A}} \mu(E). \end{aligned}$$

The first and last equations are definitions while the middle one is a consequence of Theorem 6.4.1. Note that these observations apply to any system of weights, where or not they are finite or countable.

Definition 7.7.10. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are partitions of X. Then $\mathcal{Q}$ is said to be a *refinement* of $\mathcal{P}$ if for every $E \in \mathcal{Q}$ there is an $F \in \mathcal{P}$ such that $E \subseteq F$.

Proposition 7.7.11. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are partitions of X. Let $\mathcal{B}_\mathcal{P}$ be the set of unions of elements of $\mathcal{P}$ and let $\mathcal{B}_\mathcal{Q}$ be the set of unions of elements of $\mathcal{Q}$. Then $\mathcal{Q}$ is a refinement of $\mathcal{P}$ if and only if $\mathcal{B}_\mathcal{P} \subseteq \mathcal{B}_\mathcal{Q}$.*

Proof. Suppose $\mathcal{B}_\mathcal{P} \subseteq \mathcal{B}_\mathcal{Q}$. Suppose also that $E \in \mathcal{Q}$. Then $E \neq \varnothing$ so there is an $x \in E$. $X = \bigcup_{F \in \mathcal{P}} F$ so there is an $F \in \mathcal{P}$ such that $x \in F$. Now $F \in \mathcal{B}_\mathcal{P}$ and $\mathcal{B}_\mathcal{P} \subseteq \mathcal{B}_\mathcal{Q}$ so $F \in \mathcal{B}_\mathcal{Q}$. In other words, F is a union of elements of $\mathcal{Q}$, i.e. $F = \bigcup_{G \in \mathcal{A}} G$ for some $\mathcal{A} \subseteq \mathcal{Q}$. $x \in F$ so $x \in G$ for some $G \in \mathcal{A}$. Now $x \in E \cap G$ so $E \cap G \neq \varnothing$. $E, G \in \mathcal{Q}$ and $\mathcal{Q}$ is a partition so it follows that $E = G$, i.e. that $E \in \mathcal{A}$. But then $E \subseteq F$. For every $E \in \mathcal{Q}$ there is therefore an $F \in \mathcal{P}$ such that $E \subseteq F$. In other words, $\mathcal{Q}$ is a refinement of $\mathcal{P}$.

Suppose, conversely, that $\mathcal{Q}$ is a refinement of $\mathcal{P}$. Suppose also that $G \in \mathcal{B}_{\mathcal{P}}$, i.e. that there is some $\mathcal{A} \subseteq \mathcal{P}$ such that

$$G = \bigcup_{E \in \mathcal{A}} E.$$

Let

$$H = \bigcup_{\substack{F \in \mathcal{Q} \\ F \subseteq G}} F$$

Each such F is a subset of G so their union is as well. Therefore

$$H \subseteq G.$$

Suppose $x \in G$. $X = \bigcup_{F \in \mathcal{Q}} F$ because $\mathcal{Q}$ is a partition so $x \in F$ for some $F \in \mathcal{Q}$. $\mathcal{Q}$ is a refinement of $\mathcal{P}$ so $F \subseteq E'$ for some $E' \in \mathcal{P}$. Also $x \in G$ so $x \in E$ for some $E \in \mathcal{A}$. $x \in E \cap E'$ so $E \cap E' \neq \varnothing$ and hence $E = E'$. Therefore $F \subseteq E$ for some $E \in \mathcal{A}$. But then $F \subseteq G$, so $x \in H$. We've shown that if $x \in G$ then $x \in H$, so

$$G \subseteq H$$

and, since we already have the reverse inclusion,

$$G = H.$$

Now $H \in \mathcal{B}_{\mathcal{Q}}$ because it's a union of elements of $\mathcal{Q}$. We've now shown that if $G \in \mathcal{B}_{\mathcal{P}}$ then $G \in \mathcal{B}_{\mathcal{Q}}$. So

$$\mathcal{B}_{\mathcal{P}} \subseteq \mathcal{B}_{\mathcal{Q}}$$

if $\mathcal{Q}$ is a refinement of $\mathcal{P}$. □

Definition 7.7.12. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are partitions of a set X. Their *common refinement* is the set of non-empty subsets of X of the form $E \cap F$ for $E \in \mathcal{P}$ and $F \in \mathcal{Q}$.

This name is justified by the following proposition.

Proposition 7.7.13. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are partitions of a set X and $\mathcal{R}$ is their common refinement. Then $\mathcal{R}$ is a partition of X and $\mathcal{R}$ is a refinement of $\mathcal{P}$ and of $\mathcal{Q}$. If $\mathcal{P}$ and $\mathcal{Q}$ are finite then so is $\mathcal{R}$. Similarly, if $\mathcal{P}$ and $\mathcal{Q}$ are countable then so is $\mathcal{R}$.*

Proof. Suppose $G_1, G_2 \in \mathcal{R}$ and $G_1 \cap G_2 \neq \varnothing$. Then there is an $x \in G_1 \cap G_2$. Also $G_1 = E_1 \cap F_1$ for some $E_1 \in \mathcal{P}$ and $F_1 \in \mathcal{Q}$ and $G_2 = E_2 \cap F_2$ for some $E_2 \in \mathcal{P}$ and $F_1 \in \mathcal{Q}$. But then $x \in E_1 \cap E_2$ and $x \in F_1 \cap F_2$. Therefore $E_1 \cap E_2 \neq \varnothing$ and $F_1 \cap F_2 \neq \varnothing$. $E_1, E_2 \in \mathcal{P}$ and $\mathcal{P}$ is a partition so $E_1 \cap E_2 \neq \varnothing$ implies $E_1 = E_2$. Similarly, $F_1, F_2 \in \mathcal{Q}$ and $\mathcal{Q}$ is a partition so $F_1 \cap F_2 \neq \varnothing$ implies $F_1 = F_2$. But then

$$G_1 = E_1 \cap F_1 = E_2 \cap F_2 = G_2.$$

So if $G_1, G_2 \in \mathcal{R}$ and $G_1 \cap G_2 \neq \varnothing$ then $G_1 = G_2$. The elements of $\mathcal{R}$ are non-empty by definition, so $\mathcal{R}$ is a partition.

For every $G \in \mathcal{R}$ there is an $E \in \mathcal{P}$ and an $F \in \mathcal{Q}$ such that $G = E \cap F$ and hence $G \subseteq E$ and $G \subseteq F$. Therefore $\mathcal{R}$ is a refinement of $\mathcal{P}$ and $\mathcal{R}$ is a refinement of $\mathcal{Q}$.

If $\mathcal{P}$ and $\mathcal{Q}$ are finite then so is $\mathcal{P} \times \mathcal{Q}$. Subsets of finite sets are finite so

$$S = \{(E, F) \in \mathcal{P} \times \mathcal{Q} \colon E \cap F \neq \varnothing\}$$

is finite. Define $h \colon S \to \mathcal{R}$ by $h(E, F) = E \cap F$. The definition of $\mathcal{R}$ means that h is a surjection. The image of a finite set under a surjection is finite, so $\mathcal{R}$ is finite.

The argument above applies without change if the word "finite" is replaced by "countable". □

Proposition 7.7.14. *If $\mathcal{P}$ and $\mathcal{Q}$ are partitions of a set X, $\mathcal{R}$ is their common refinement, and $\mathcal{B}$ is a Boolean algebra on X such that $\mathcal{P} \subseteq \mathcal{B}$ and $\mathcal{Q} \subseteq \mathcal{B}$ then $\mathcal{R} \subseteq \mathcal{B}$.*

Proof. If $G \in \mathcal{R}$ then $G = E \cap F$ for some $E \in \mathcal{P}$ and $F \in \mathcal{Q}$. $\mathcal{P} \subseteq \mathcal{B}$ and $\mathcal{Q} \subseteq \mathcal{B}$ so $E \in \mathcal{B}$ and $F \in \mathcal{B}$. Therefore $E \cap F \in \mathcal{B}$, i.e. $G \in \mathcal{B}$. So if $G \in \mathcal{R}$ then $G \in \mathcal{B}$, i.e. $\mathcal{R} \subseteq \mathcal{B}$. □

Proposition 7.7.15. *Suppose $\mathcal{B}$ is a Boolean algebra on a set X. Then the set $\mathbf{P}(X, \mathcal{B})$ of partitions $\mathcal{Q}$ of X such that $\mathcal{Q} \subseteq \mathcal{B}$, with the order relation $\mathcal{P} \preccurlyeq \mathcal{Q}$ if $\mathcal{Q}$ is a refinement of $\mathcal{P}$, is a non-empty directed set. So is the set of finite partitions, or the set of countable partitions.*

Proof. $\{X\} \in \mathbf{P}(X, \mathcal{B})$ so $P(X, \mathcal{B})$ is non-empty.

If $\mathcal{B}_{\mathcal{Q}} \subseteq \mathcal{B}_{\mathcal{Q}}$ so $\mathcal{Q}$ is a refinement of itself, i.e. $\mathcal{Q} \preccurlyeq \mathcal{Q}$ for all $\mathcal{Q} \in \mathbf{P}(X, \mathcal{B})$.

If $\mathcal{Q}_1 \preccurlyeq \mathcal{Q}_2$ and $\mathcal{Q}_2 \preccurlyeq \mathcal{Q}_3$ then $\mathcal{Q}_2$ is a refinement of $\mathcal{Q}_1$ and $\mathcal{Q}_3$ is a refinement of $\mathcal{Q}_2$. Therefore $\mathcal{B}_{\mathcal{Q}_1} \subseteq \mathcal{B}_{\mathcal{Q}_2}$ and $\mathcal{B}_{\mathcal{Q}_2} \subseteq \mathcal{B}_{\mathcal{Q}_3}$ so and $\mathcal{B}_{\mathcal{Q}_1} \subseteq \mathcal{B}_{\mathcal{Q}_3}$. Thus $\mathcal{Q}_3$ is a refinement of $\mathcal{Q}_1$ and $\mathcal{Q}_1 \preccurlyeq \mathcal{Q}_3$.

Given any $\mathcal{P}, \mathcal{Q} \in \mathbf{P}(X, \mathcal{B})$ their common refinement $\mathcal{R}$ belongs to $\mathbf{P}(X, \mathcal{B})$ and is a refinement of both. In other words, $\mathcal{P} \preccurlyeq \mathcal{R}$ and $\mathcal{Q} \preccurlyeq \mathcal{R}$. Thus $\mathbf{P}(X, \mathcal{B})$ satisfies all the requirements for a directed set. □

8 Integration

8.1 Refinements of content spaces

Definition 8.1.1. Suppose $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ are content spaces. A *morphism* of content spaces is a function $f \colon X \to Y$ satisfying the following conditions:

(a)
$$\mathcal{C} \subseteq f^{**}(\mathcal{B}).$$

(b)
$$\nu(E) = \mu(f^*(E))$$
for all $E \in \mathcal{C}$.

Proposition 8.1.2. *Suppose $(X, \mathcal{B}, \mu)$, $(Y, \mathcal{C}, \nu)$ and $(Z, \mathcal{D}, \xi)$ are content spaces and $f \colon X \to Y$ and $g \colon Y \to Z$ are morphisms. Then $g \circ f$ is also a morphism.*

Proof. The hypotheses mean that
$$\mathcal{C} \subseteq f^{**}(\mathcal{B}),$$
$$\nu(E) = \mu(f^*(E))$$
for all $E \in \mathcal{C}$,
$$\mathcal{D} \subseteq g^{**}(\mathcal{C}),$$
and
$$\xi(F) = \nu(g^*(F)),$$
for all $F \in \mathcal{D}$. Then
$$g^{**}(\mathcal{C}) \subseteq g^{**}(f^{**}(\mathcal{B})) = (g \circ f)^{**}(\mathcal{B})$$
and hence
$$\mathcal{D} \subseteq (g \circ f)^{**}(\mathcal{B}).$$
Also, if $F \in \mathcal{D}$ then $g^*(F) \in \mathcal{C}$ and
$$\xi(F) = \nu(g^*(F)) = \mu(f^*(g^*(F))) = \mu((g \circ f)^*(F)).$$
□

Most of the time we're interested in morphisms f from $(X, \mathcal{B}, \mu)$ to $(Y, \mathcal{C}, \nu)$ where $X = Y$ and f is the identity function.

Definition 8.1.3. Suppose $(X, \mathcal{B}', \mu')$ and $(X, \mathcal{B}', \mu')$ are content spaces. We say that $(X, \mathcal{B}', \mu')$ is a *refinement* of $(X, \mathcal{B}, \mu)$ if $\mathcal{B} \subseteq \mathcal{B}'$ and $\mu'(F) = \mu(F)$ for every $F \in \mathcal{B}$.

Proposition 8.1.4. *$(X, \mathcal{B}', \mu')$ is a refinement of $(X, \mathcal{B}, \mu)$ if and only if the identity function $i \colon X \to X$ is a morphism of content spaces from $(X, \mathcal{B}', \mu')$ to $(X, \mathcal{B}, \mu)$.*

Proof. Suppose $(X, \mathcal{B}', \mu')$ is a refinement of $(X, \mathcal{B}, \mu)$. If $E \in \mathcal{B}$ then $i^*(E) = E \in \mathcal{B}'$ so $E \in i^{**}(\mathcal{B}')$. Also, $\mu(E) = \mu(i^*(E)) = \mu'(i^*(E))$. So if $E \in \mathcal{B}$ then $E \in i^{**}(\mathcal{B}')$, i.e.
$$\mathcal{B} \subseteq i^{**}(\mathcal{B}'),$$
and
$$\mu(E) = \mu'(i^*(E))$$
for all $E \in \mathcal{B}$. Therefore i is a morphism of content spaces from $(X, \mathcal{B}, \mu)$ to $(X, \mathcal{B}', \mu')$.

Suppose, conversely, that i is a morphism of content spaces from $(X, \mathcal{B}, \mu)$ to $(X, \mathcal{B}', \mu')$, i.e. that
$$\mathcal{B} \subseteq i^{**}(\mathcal{B}'),$$
and
$$\mu(E) = \mu'(i^*(E))$$
for all $E \in \mathcal{B}$. If $E \in \mathcal{B}$ then $E \in i^{**}(\mathcal{B}')$, so $E = i^*(E) \in \mathcal{B}'$. So
$$\mathcal{B} \subseteq \mathcal{B}'.$$
Also
$$\mu(E) = \mu'(i^*(E)) = \mu'(E)$$
for all $E \in \mathcal{B}$. So $(X, \mathcal{B}', \mu')$ is a refinement of $(X, \mathcal{B}, \mu)$. □

Proposition 8.1.5. *Suppose $(X,\mathcal{B}',\mu')$ is a refinement of $(X,\mathcal{B},\mu)$ and $(X,\mathcal{B}'',\mu'')$ is a refinement of $(X,\mathcal{B}',\mu')$. Then $(X,\mathcal{B}'',\mu'')$ is a refinement of $(X,\mathcal{B},\mu)$.*

Proof. $(X,\mathcal{B}',\mu')$ is a refinement of $(X,\mathcal{B},\mu)$ so the identity function on X is a morphism from $(X,\mathcal{B}',\mu')$ to $(X,\mathcal{B},\mu)$. Similarly, $(X,\mathcal{B}'',\mu'')$ is a refinement of $(X,\mathcal{B}',\mu')$ so the identity is a morphism from $(X,\mathcal{B}'',\mu'')$ to $(X,\mathcal{B}',\mu')$. The composition of two morphism is a morphism so the identity is a morphism from $(X,\mathcal{B}'',\mu'')$ to $(X,\mathcal{B},\mu)$. Therefore $(X,\mathcal{B}'',\mu'')$ is a refinement of $(X,\mathcal{B},\mu)$. ☐

The notion of a refinement generalises properties we saw for completions of content spaces and measure spaces in Theorems 7.3.4 and 7.6.11.

Proposition 8.1.6. *Suppose $(X,\mathcal{B},\mu)$ and $(X,\mathcal{B}^\dagger,\mu^\dagger)$ are as in Theorem 7.3.4. Then $(X,\mathcal{B}^\dagger,\mu^\dagger)$ is a refinement of $(X,\mathcal{B},\mu)$.*

Proof. The statements that $\mathcal{B}\subseteq\mathcal{B}^\dagger$ and that $\mu^\dagger(F)=\mu(F)$ for every $F\in\mathcal{B}$ were part of the conclusions of Theorem 7.3.4. ☐

As an example $(\mathbf{R},\mathcal{J},\mu)$ is a refinement of $(\mathbf{R},\mathcal{I},\mu)$. Strictly speaking we should write $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{I}}$ for the two content functions. They have different domains, $\mathcal{J}$ and $\mathcal{I}$, respectively and so are different functions. But one generally uses the same symbol for both, which causes no ambiguity because $\mu_{\mathcal{J}}(E)=\mu_{\mathcal{I}}(E)$ whenever both are defined, i.e. whenever $E\in\mathcal{I}$. In fact there's no ambiguity precisely because $(\mathbf{R},\mathcal{J},\mu_{\mathcal{J}})$ is a refinement of $(\mathbf{R},\mathcal{I},\mu_{\mathcal{I}})$.

Proposition 8.1.7. *Suppose $(X,\mathcal{B},\mu)$ and $(X,\mathcal{B}^\dagger,\mu^\dagger)$ are as in Theorem 7.6.11. Then $(X,\mathcal{B}^\dagger,\mu^\dagger)$ is a refinement of $(X,\mathcal{B},\mu)$.*

Proof. The statements that $\mathcal{B}\subseteq\mathcal{B}^\dagger$ and that $\mu^\dagger(F)=\mu(F)$ for every $F\in\mathcal{B}$ were part of the conclusions of Theorem 7.6.11. ☐

The notions of refinement of partitions and refinement of content spaces are closely related.

Proposition 8.1.8. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are partitions of a set X and that $\mathcal{Q}$ is a refinement of $\mathcal{P}$. Let $\mathcal{B}_{\mathcal{P}}$ and $\mathcal{B}_{\mathcal{Q}}$ be the sets of unions of elements of $\mathcal{P}$ and unions of elements of $\mathcal{Q}$ respectively. Suppose $\mu_{\mathcal{Q}}$ is a content on $(X,\mathcal{B}_{\mathcal{Q}})$. Define $\mu_{\mathcal{P}}\colon\mathcal{B}_{\mathcal{P}}\to[0,+\infty]$ by*

$$\mu_{\mathcal{P}}(E)=\mu_{\mathcal{Q}}(E)$$

for all $E\in\mathcal{B}_{\mathcal{P}}$. Then $(X,\mathcal{Q},\mu_{\mathcal{Q}})$ is a refinement of $(X,\mathcal{P},\mu_{\mathcal{P}})$.

Proof. We've already seen that $\mathcal{B}_{\mathcal{P}}\subseteq\mathcal{B}_{\mathcal{Q}}$. Also, we defined $\mu_{\mathcal{P}}$ such that $\mu_{\mathcal{P}}(E)=\mu_{\mathcal{Q}}(E)$ for all $E\in\mathcal{B}_{\mathcal{P}}$, so all the requirements for a refinement are met. ☐

8.2 Definition of the integral

Definition 8.2.1. Suppose $(X,\mathcal{B},\mu)$ is a content space, $\mathcal{P}$ is a partition of X and w is a system of weights on X. The three are said to be *compatible* if $\mathcal{P}\subseteq\mathcal{B}$ and

$$\mu(E)=\sum_{x\in E}w(x)$$

for all $E\in\mathcal{P}$.

Proposition 8.2.2. *Suppose $(X,\mathcal{B}',\mu')$ is a refinement of $(X,\mathcal{B},\mu)$, $\mathcal{P}$ is a partition of X and w is a system of weights on X. Then $(X,\mathcal{B}',\mu')$, $\mathcal{P}$ and w are compatible if $(X,\mathcal{B},\mu)$, $\mathcal{P}$ and w are. $(X,\mathcal{B},\mu)$, $\mathcal{P}$ and w are compatible if $(X,\mathcal{B}',\mu')$, $\mathcal{P}$ and w are and $\mathcal{P}\subseteq\mathcal{B}$.*

Proof. Suppose that $(X,\mathcal{B},\mu)$, $\mathcal{P}$ and w are compatible. $\mathcal{P}\subseteq\mathcal{B}$ and $\mathcal{B}\subseteq\mathcal{B}'$ so

$$\mathcal{P}\subseteq\mathcal{B}'.$$

Also,

$$\mu(E)=\sum_{x\in E}w(x)$$

for each $E\in\mathcal{P}$ since $(X,\mathcal{B},\mu)$, $\mathcal{P}$ and w are compatible. But $\mu'(E)=\mu(E)$ for $E\in\mathcal{P}$ so

$$\mu'(E)=\sum_{x\in E}w(x)$$

Therefore $(X,\mathcal{B}',\mu')$, $\mathcal{P}$ and w are compatible.

Suppose $(X, \mathcal{B}', \mu')$, $\mathcal{P}$ and w are compatible and

$$\mathcal{P} \subseteq \mathcal{B}.$$

We have

$$\mu'(E) = \sum_{x \in E} w(x)$$

for each $E \in \mathcal{P}$ since $(X, \mathcal{B}', \mu')$, $\mathcal{P}$ and w are compatible. But $\mu(E) = \mu'(E)$ for all $E \in \mathcal{B}$ and hence for all $E \in \mathcal{P}$, so

$$\mu(E) = \sum_{x \in E} w(x)$$

Therefore $(X, \mathcal{B}, \mu)$, $\mathcal{P}$ and w are compatible. □

Proposition 8.2.3. *Suppose $(X, \mathcal{B}, \mu)$ is a content space, $\mathcal{P}$ is a finite partition of X and w is a system of weights on X. Let $\mathcal{B}_{\mathcal{P}}$ be the set of unions of elements of $\mathcal{P}$. Let $\mu_{\mathcal{P}}(E) = \sum_{x \in E} w(x)$ for $E \in \mathcal{B}_{\mathcal{P}}$ and let $\mu_w(E) = \sum_{x \in E} w(x)$ for $E \in \wp(X)$. Then $(X, \mathcal{B}, \mu)$, $\mathcal{P}$ and w are compatible if and only if $(X, \mathcal{B}, \mu)$ and $(X, \wp(X), \mu_w)$ are both refinements of $(X, \mathcal{B}_{\mathcal{P}}, \mu_{\mathcal{P}})$.*

Proof. Suppose $(X, \mathcal{B}, \mu)$, $\mathcal{P}$ and w are compatible, i.e. that

$$\mathcal{P} \subseteq \mathcal{B}$$

and

$$\mu(E) = \sum_{x \in X} w(x)$$

for all $E \in \mathcal{P}$. The elements of $\mathcal{B}_{\mathcal{P}}$ are finite unions of elements of $\mathcal{P}$, and therefore finite unions of elements of $\mathcal{B}$, and $\mathcal{B}$ is a Boolean algebra so

$$\mathcal{B}_{\mathcal{P}} \subseteq \mathcal{B}.$$

If $F \in \mathcal{B}_{\mathcal{P}}$ then $F = \bigcup_{E \in \mathcal{A}} E$ for some finite $\mathcal{A} \subseteq \mathcal{P}$. $\mathcal{P}$ is a partition so $\mathcal{A}$ is a disjoint collection and

$$\begin{aligned} \mu_{\mathcal{P}}(F) &= \sum_{x \in F} w(x) = \sum_{E \in \mathcal{A}} \sum_{x \in E} w(x) \\ &= \sum_{E \in \mathcal{A}} \mu(E) = \mu\left(\bigcup_{E \in \mathcal{A}} E\right) \\ &= \mu(F). \end{aligned}$$

So $\mathcal{B}_{\mathcal{P}} \subseteq \mathcal{B}$ and $\mu_{\mathcal{P}}(F) = \mu(F)$ for all $F \in \mathcal{B}_{\mathcal{P}}$. In other words, $(X, \mathcal{B}, \mu)$ is a refinement of $(X, \mathcal{B}_{\mathcal{P}}, \mu_{\mathcal{P}})$. Also, $\mathcal{B}_{\mathcal{P}} \subseteq \wp(X)$ and

$$\mu_{\mathcal{P}}(F) = \sum_{x \in F} w(x) = \mu_w(F)$$

if $F \in \mathcal{B}_{\mathcal{Q}}$ so $(X, \wp(X), \mu_w)$ is a refinement of $(X, \mathcal{B}_{\mathcal{P}}, \mu_{\mathcal{P}})$.

Suppose, conversely, that $(X, \mathcal{B}, \mu)$ and $(X, \wp(X), \mu_w)$ are refinements of $(X, \mathcal{B}_{\mathcal{P}}, \mu_{\mathcal{P}})$, i.e. that

$$\mathcal{B}_{\mathcal{P}} \subseteq \mathcal{B},$$

$$\mathcal{B}_{\mathcal{P}} \subseteq \wp(X),$$

and

$$\mu(E) = \mu_{\mathcal{P}}(E) = \mu_w(E)$$

for all $E \in \mathcal{B}_{\mathcal{P}}$. Then

$$\mathcal{P} \subseteq \mathcal{B}$$

since $\mathcal{P} \subseteq \mathcal{B}_{\mathcal{P}}$. Also,

$$\mu_{\mathcal{P}}(E) = \mu_w(E) = \sum_{x \in E} \mu(x)$$

for all $E \in \mathcal{P}$. Therefore $(X, \mathcal{B}, \mu)$, $\mathcal{P}$ and w are compatible. □

There is a similar result for measure spaces and countable partitions.

Proposition 8.2.4. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space, $\mathcal{P}$ is a countable partition of X and w is a system of weights on X. Let $\mathcal{B}_{\mathcal{P}}$ be the set of unions of elements of $\mathcal{P}$. Let $\mu_{\mathcal{P}}(E) = \sum_{x \in E} w(x)$ for $E \in \mathcal{B}_{\mathcal{P}}$ and let $\mu_w(E) = \sum_{x \in E} w(x)$ for $E \in \wp(X)$. Then $(X, \mathcal{B}, \mu)$, $\mathcal{P}$ and w are compatible if and only if $(X, \mathcal{B}, \mu)$ and $(X, \wp(X), \mu_w)$ are both refinements of $(X, \mathcal{B}_{\mathcal{P}}, \mu_{\mathcal{P}})$.*

Proof. The proof is the same as for the previous proposition, with "finite" replaced by "countable", "Boolean algebra" replaced by "σ-algebra" and "content" replaced by "measure" everywhere. □

Proposition 8.2.5. *Suppose $(X, \mathcal{B}, \mu)$ is a content space, $\mathcal{P}$ and $\mathcal{Q}$ are finite partitions of X and w is a system of weights on X. Suppose also that $\mathcal{Q}$ is a refinement of $\mathcal{P}$. If $(X, \mathcal{B}, \mu)$, $\mathcal{Q}$ and w are compatible then so are $(X, \mathcal{B}, \mu)$, $\mathcal{P}$ and w.*

Proof. $(X, \mathcal{B}, \mu)$, $\mathcal{Q}$ and w are compatible so $(X, \mathcal{B}, \mu)$ and $(X, \wp(X), \mu_w)$ are refinements of $(X, \mathcal{B}_\mathcal{Q}, \mu_\mathcal{Q})$. $\mathcal{Q}$ is a refinement of $\mathcal{P}$ so $(X, \mathcal{B}_\mathcal{Q}, \mu_\mathcal{Q})$ is a refinement of $(X, \mathcal{B}_\mathcal{P}, \mu_\mathcal{P})$. Therefore $(X, \mathcal{B}, \mu)$ and $(X, \wp(X), \mu_w)$ are refinements of $(X, \mathcal{B}_\mathcal{P}, \mu_\mathcal{P})$. So $(X, \mathcal{B}, \mu)$, $\mathcal{P}$ and w are compatible. $\square$

Again, there's a version for measure spaces and countable partitions.

Proposition 8.2.6. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space, $\mathcal{P}$ and $\mathcal{Q}$ are countable partitions of X and w is a system of weights on X. Suppose also that $\mathcal{Q}$ is a refinement of $\mathcal{P}$. If $(X, \mathcal{B}, \mu)$, $\mathcal{Q}$ and w are compatible then so are $(X, \mathcal{B}, \mu)$, $\mathcal{P}$ and w.*

Proof. The proof is word for word identical to that of the preceding proposition. $\square$

Definition 8.2.7. Suppose $(X, \mathcal{B}, \mu)$ is a content space Y is either $[0, +\infty]$ or $\mathbf{R}$, and $f\colon X \to Y$ is a function. Let $\mathbf{P}$ be the set of finite subsets of $\mathcal{B}$ which are partitions of X and let U be the set of systems of weights w on X such that $\sum_{x\in X} w(x)f(x)$ converges. Define $R_f\colon U \to Y$ by

$$R_f(w) = \sum_{x\in X} w(x)f(x).$$

Define $\alpha\colon \mathbf{P} \to \wp(U)$ by

$$\alpha(\mathcal{Q}) = \{w \in U\colon (X, \mathcal{B}, \mu),\ \mathcal{Q} \text{ and } w \text{ are compatible}\}.$$

Let $\mathcal{E}$ be the upward closure of $\alpha_*(\mathbf{P})$. f is said to be *integrable* with respect to the content space $(X, \mathcal{B}, \mu)$ if $\alpha(\mathcal{Q}) \neq \varnothing$ for each $\mathcal{Q} \in \mathbf{P}$ and the filter

$$R_g^{**}(\mathcal{E})$$

converges. Its limit is then known as the *integral* of f with respect to $(X, \mathcal{B}, \mu)$ and is denoted

$$\int_{x\in X} f(x)\, d\mu(x).$$

$\mathbf{P}$ is a non-empty directed set. $\alpha_*(\mathbf{P})$ is therefore also a non-empty directed subset of $\wp(X)$, the ordering being given by the superset relation. If f is integrable then $\varnothing \notin \alpha_*(\mathbf{P})$. $\alpha_*(\mathbf{P})$ is therefore a prefilter and $\mathcal{E}$ is a filter. So the reference to convergence is meaningful. Also Y is Hausdorff, so the limit, i.e. the integral, is unique if it exists.

Similarly we can define the integral of a function on a measure space.

Definition 8.2.8. Suppose $(X, \mathcal{B}, \mu)$ is a measure space, Y is either $[0, +\infty]$ or $\mathbf{R}$ and $f\colon X \to Y$ is a function. Let $\mathbf{P}$ be the set of countable subsets of $\mathcal{B}$ which are partitions of X and let U be the set of systems of weights w on X such that $\sum_{x\in X} w(x)f(x)$ converges and define $R_f\colon U \to Y$ by

$$R_f(w) = \sum_{x\in X} w(x)f(x).$$

Define $\alpha\colon \mathbf{P} \to \wp(U)$ by

$$\alpha(\mathcal{Q}) = \{w \in U\colon (X, \mathcal{B}, \mu),\ \mathcal{Q} \text{ and } w \text{ are compatible}\}.$$

Let $\mathcal{E}$ be the upward closure of $\alpha_*(\mathbf{P})$. f is said to be *integrable* with respect to the measure space $(X, \mathcal{B}, \mu)$ if $\alpha(\mathcal{Q}) \neq \varnothing$ for each $\mathcal{Q} \in \mathbf{P}$ and the filter

$$R_f^{**}(\mathcal{E})$$

converges. Its limit is then known as the *integral* of f with respect to $(X, \mathcal{B}, \mu)$ and is denoted

$$\int_{x\in X} f(x)\, d\mu(x).$$

Every measure space is a content space, so we appear to have defined the integral twice in this case. Are these definitions compatible? Yes, in the sense that f is integrable when considered as a function on the measure space if it's integrable when considered as a function on the content space, and the two integrals then agree. This will follow from Proposition 8.2.11. It's possible for a f to be integrable as a function on the measure space but not on the content space however.

The advantage of defining integrals as limits is that we can immediately see that they have the usual properties of limits.

Proposition 8.2.9. *Suppose $(X, \mathcal{B}, \mu)$ is a content space or a measure space and f and g are integrable functions on X such that $f(x) \leq g(x)$ for all $x \in X$. Then*

$$\int_{x \in X} f(x)\, d\mu(x) \leq \int_{x \in X} g(x)\, d\mu(x).$$

Proof. This follows from Theorem 1.16.3b. □

Proposition 8.2.10. *Suppose $(X, \mathcal{B}, \mu)$ is a content space or a measure space $c_1, \ldots, c_m \in \mathbf{R}$ and $f_1, \ldots, f_m \colon X \to [-\infty, +\infty]$ are integrable functions. Define $g \colon X \to [-\infty, +\infty]$ by*

$$g(x) = \sum_{i=1}^{m} c_i f_i(x),$$

assuming this is possible. Then g is an integrable function and

$$\int_{x \in X} g(x)\, d\mu(x) = \sum_{i=1}^{m} c_i \int_{x \in X} f_i(x)\, d\mu(x).$$

Proof. This follows from Theorem 1.16.3c. □

Proposition 8.2.11. *Suppose that $(X, \mathcal{B}', \mu')$ is a refinement of $(X, \mathcal{B}, \mu)$. If f is integrable with respect to $(X, \mathcal{B}, \mu)$ then it is integrable with respect to $(X, \mathcal{B}', \mu')$ and the integrals are equal.*

We haven't specified whether $(X, \mathcal{B}, \mu)$ and $(X, \mathcal{B}', \mu')$ are content spaces or measure spaces. In fact the proposition holds if both are content space or if both are measure spaces. It also holds if $(X, \mathcal{B}, \mu)$ is a content space and $(X, \mathcal{B}', \mu')$ is a measure space, although it can fail if $(X, \mathcal{B}, \mu)$ is a measure space and $(X, \mathcal{B}', \mu')$ is a content space.

Proof. To avoid giving three separate, but nearly identical, proofs for the three different cases listed above we adopt the following convention for the remainder of the proof. The word "tiny" will mean "finite" if $(X, \mathcal{B}, \mu)$ is a content space and will mean "countable" if $(X, \mathcal{B}, \mu)$ is a measure space. The word "small" will mean "finite" if $(X, \mathcal{B}', \mu')$ is a content space and will mean "countable" if $(X, \mathcal{B}', \mu')$ is a measure space. In each of the three cases allowed above it's true that every tiny set is small, though this is not true in the excluded case where $(X, \mathcal{B}, \mu)$ is a measure space and $(X, \mathcal{B}', \mu')$ is a content space.

Let $\mathbf{P}$ be the set of tiny subsets of $\mathcal{B}$ which are partitions of X and let $\mathbf{P}'$ be the set of small subsets of $\mathcal{B}'$ which are partitions of X. $\mathcal{B} \subseteq \mathcal{B}'$ and every tiny subset is small so $\mathbf{P} \subseteq \mathbf{P}'$. Let U be the set of systems of weights w on X such that $\sum_{x \in X} w(x) f(x)$ converges and define $R_f \colon U \to Y$ by

$$R_f(w) = \sum_{x \in X} w(x) f(x).$$

Define $\alpha \colon \mathbf{P} \to \wp(U)$ by

$$\alpha(\mathcal{Q}) = \{w \in U \colon (X, \mathcal{B}, \mu),\ \mathcal{Q} \text{ and } w \text{ are compatible}\}$$

and $\alpha' \colon \mathbf{P}' \to \wp(U)$ by

$$\alpha'(\mathcal{Q}) = \{w \in U \colon (X, \mathcal{B}', \mu'),\ \mathcal{Q} \text{ and } w \text{ are compatible}\}.$$

If $\mathcal{Q} \in \mathbf{P}$ then $\alpha(\mathcal{Q}) = \alpha'(\mathbf{Q})$ since $(X, \mathcal{B}', \mu')$, $\mathcal{Q}$ and w are compatible if and only if $(X, \mathcal{B}, \mu)$, $\mathcal{Q}$ and w are compatible, by Proposition 8.2.2. Let $\mathcal{E}$ be the upward closure of $\alpha_*(\mathbf{P})$. and let $\mathcal{E}'$ be the upward closure of $\alpha'_*(\mathbf{P}')$. Suppose $Z \in \mathcal{E}$, i.e. that there is a $\mathcal{Q} \in \mathbf{P}$ such that $\alpha(\mathcal{Q}) \subseteq Z$. Then $\mathcal{Q} \in \mathbf{P}'$ and $\alpha'(\mathbf{Q}) \subseteq Z$. So $\mathcal{Q} \in \mathcal{E}'$. Therefore $\mathcal{E} \subseteq \mathcal{E}'$ so $\mathcal{E}'$ converges if $\mathcal{E}$ does and the limits are the same. □

The characterisation of integrals as limits is useful for proving theorem but it's helpful to have a more explicit description for examples. This will require some preliminary definitions.

Definition 8.2.12. Suppose $\mathcal{B}$ is a Boolean algebra on a set X and Y is a set. A function $f \colon X \to Y$ is called a *simple function* if there is a finite partition $\mathcal{Q} \subseteq \mathcal{B}$ such that $\wp(Y) \subseteq f^{**}(\mathcal{B}_{\mathcal{Q}})$ where $\mathcal{B}_{\mathcal{Q}}$ is the atomic algebra of unions of elements of $\mathcal{Q}$.

Proposition 8.2.13. *Suppose $\mathcal{B}$ is a Boolean algebra on a set X, Y is a set and $f \colon X \to Y$ is a function. The following statements are equivalent.*

(a) f is a simple function.

(b) There is a finite partition $\mathcal{Q} \subseteq \mathcal{B}$ such that $\wp(Y) = f^{**}(\mathcal{B}_{\mathcal{Q}})$.

(c) *There is a finite partition $\mathcal{Q} \subseteq \mathcal{B}$ such that*

$$\{x \in X \colon f(x) = y\} \in \mathcal{B}_{\mathcal{Q}}$$

for each $y \in Y$.

(d) *There is a finite partition $\mathcal{Q} \subseteq \mathcal{B}$ of X and a function $\varphi\colon \mathcal{Q} \to Y$ such that $f(x) = \varphi(E)$ whenever $E \in \mathcal{Q}$.*

Proof. For any function $f\colon X \to Y$ we have

$$f^{**}(\mathcal{B}_{\mathcal{Q}}) \subseteq \wp(Y)$$

so (a) implies (b).

Suppose (b) holds, i.e. that

$$\wp(Y) = f^{**}(\mathcal{B}_{\mathcal{Q}}).$$

$\{y\} \in \wp(Y)$ so $\{y\} \in f^{**}(\mathcal{B}_{\mathcal{Q}})$ and $f^{*}(\{y\}) \in \mathcal{B}_{\mathcal{Q}}$. Therefore (c) holds.

Suppose (c) holds. Define an equivalence relation $\sim$ by $w \sim x$ if and only if $f(w) = f(x)$. Let $\mathcal{P}$ be the set of equivalence classes for this equivalence relation. As with any set of equivalence classes, these form a partition. If $E \in \mathcal{P}$ then E is the equivalence class of some $w \in X$. Then

$$\begin{aligned} E &= \{x \in X \colon w \sim x\} \\ &= \{z \in X \colon f(w) = f(x)\} \\ &= f^{*}(\{f(w)\}) \end{aligned}$$

so $E \in \mathcal{B}_{\mathcal{Q}}$, because of (c). We define $\varphi(E) = f(w)$. This is independent of which $w \in E$ is chosen since $f(w) = f(x)$ for all $w \in E$. Then

$$f(x) = \varphi(E),$$

so (d) holds.

Suppose (d) holds. Let

$$\mathcal{A} = \{E \in \mathcal{Q} \colon \exists w \in E \colon f(w) \in V\}$$

Suppose $x \in f^{*}(V)$, i.e. that $f(x) \in V$. $\mathcal{Q}$ is a partition so there is an $E \in \mathcal{Q}$ such that $x \in E$ and therefore $E \in \mathcal{A}$ and

$$x \in \bigcup_{E \in \mathcal{A}} E.$$

Suppose, conversely, that $x \in \bigcup_{E \in \mathcal{A}} E$. Then there is an $E \in \mathcal{A}$ such that $x \in E$. By the definition of $\mathcal{A}$ there is a $w \in E$ such that $f(w) \in V$. Then

$$f(x) = \varphi(E) = f(w) \in V$$

so $f(x) \in V$, i.e. $x \in f^{*}(V)$. We've just seen that $x \in f^{*}(V)$ if and only if $x \in \bigcup_{E \in \mathcal{A}} E$ so

$$f^{*}(V) = \bigcup_{E \in \mathcal{A}} E.$$

Now $E \in \mathcal{Q}$ and $Q \subseteq \mathcal{B}_{\mathcal{Q}}$ so $E \in \mathcal{B}_{\mathcal{Q}}$ and therefore

$$\bigcup_{E \in \mathcal{A}} E \in \mathcal{B}_{\mathcal{Q}}.$$

Then $f^{*}(V) \in \mathcal{B}_{\mathcal{Q}}$, i.e. $V \in f^{**}(\mathcal{B}_{\mathcal{Q}})$. This holds for all $V \in \wp(Y)$ so

$$\wp(Y) \subseteq f^{**}(\mathcal{B}_{\mathcal{Q}}),$$

which is (a). ∎

Simple functions are easy to integrate.

Proposition 8.2.14. *Suppose $(X, \mathcal{B}, \mu)$ is a content space, $Y = [0, +\infty]$ or $Y = \mathbf{R}$ and $f\colon X \to Y$ is a simple function. Then*

$$\int_{x \in X} f(x)\, d\mu = \sum_{E \in \mathcal{Q}} \varphi(E)\mu(E)$$

where $\mathcal{Q}$ and φ are as in the preceding proposition.

Proof. Let

$$z = \sum_{E \in \mathcal{Q}} \varphi(E)\mu(E).$$

If $(X, \mathcal{B}, \mu)$, $\mathcal{Q}$ and w are compatible then

$$\begin{aligned} R_f(w) &= \sum_{x \in X} w(x) f(x) = \sum_{E \in \mathcal{Q}} \sum_{x \in E} w(x) f(x) \\ &= \sum_{E \in \mathcal{Q}} \sum_{x \in E} w(x)\varphi(E) = \sum_{E \in \mathcal{Q}} \varphi(E) \sum_{x \in E} w(x) \\ &= \sum_{E \in \mathcal{Q}} \varphi(E)\mu(E) = z. \end{aligned}$$

In other words, $R_f(w) = \{z\}$ for all $w \in \alpha(\mathcal{Q})$, or, equivalently,

$$\alpha(\mathcal{Q}) \subseteq R_f^{*}(\{z\}).$$

But $\mathcal{Q} \in \mathbf{P}$ so
$$\alpha(\mathcal{Q}) \in \alpha_*(\mathbf{P})$$
and therefore
$$R_f^*(\{z\}) \in \mathcal{E}$$
and so
$$\{z\} \in R_f^{**}(\mathcal{E}).$$
If $V \in \mathcal{N}(z)$ then $z \in V$ so $\{z\} \subseteq V$. $R_f^{**}(\mathcal{E})$ is a filter, hence is upward closed, so $V \in R_f^{**}(\mathcal{E})$. This holds for all $V \in \mathcal{N}(z)$ so
$$\mathcal{N}(z) \subseteq R_f^{**}(\mathcal{E}).$$
In other words, $R_f^{**}(\mathcal{E})$ converges to z. Thus $z = \sum_{E\in\mathcal{Q}} \varphi(E)\mu(E)$ is the integral of f with respect to $(X, \mathcal{B}, \mu)$. □

Definition 8.2.15. Suppose $\mathcal{B}$ is a σ-algebra on a set X and Y is a set. A function $f\colon X \to Y$ is called a *semisimple function* if there is a countable partition $\mathcal{Q} \subseteq \mathcal{B}$ such that $\wp(Y) \subseteq f^{**}(\mathcal{B}_{\mathcal{Q}})$ where $\mathcal{B}_{\mathcal{Q}}$ is the atomic algebra of unions of elements of $\mathcal{Q}$.

The term "simple function" is nearly universal. There is less unanimity on whether to require $\mathcal{Q} \subseteq \mathcal{B}$ but the the usual convention is to require it, as I have above. The term "semisimple function" is not new, but it is rare. There doesn't seem to be any other term which is less rare though.

Proposition 8.2.16. *Suppose $\mathcal{B}$ is a σ-algebra on a set X, Y is a set and $f\colon X \to Y$ is a function. The following statements are equivalent.*

(a) f is a semisimple function.

(b) There is a countable partition $\mathcal{Q} \subseteq \mathcal{B}$ such that $\wp(Y) = f^{**}(\mathcal{B}_{\mathcal{Q}})$.

(c) There is a countable partition $\mathcal{Q} \subseteq \mathcal{B}$ such that
$$\{x \in X : f(x) = y\} \in \mathcal{B}_{\mathcal{Q}}$$
for each $y \in Y$.

(d) There is a countable partition $\mathcal{Q} \subseteq \mathcal{B}$ of X and a function $\varphi\colon \mathcal{Q} \to Y$ such that
$$f(x) = \varphi(E)$$
whenever $E \in \mathcal{Q}$.

Proof. The proof is word for word identical with that of Proposition 8.2.13. □

Semisimple functions are also easy to integrate.

Proposition 8.2.17. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space, $Y = [0, +\infty]$ or $Y = \mathbf{R}$ and $f\colon X \to Y$ is a function. Then*
$$\int_{x\in X} f(x)\,d\mu = \sum_{E\in\mathcal{Q}} \varphi(E)\mu(E)$$
where $\mathcal{Q}$ and φ are as in the preceding proposition.

Proof. The proof is word for word the same as that of Proposition 8.2.14. □

8.3 Alternate characterisations of the integral

Definition 8.3.1. Suppose $(X, \mathcal{B}, \mu)$ is a content space. A statement is said to hold for *almost all* $x \in X$ if there is a set E such that $\mu(E) = 0$ and the statement holds for all $x \in X \setminus E$.

In other words, the statement holds except for x in a null set. Of course measure spaces are content spaces, so the definition above applies to them to.

Proposition 8.3.2. *Suppose $(X, \mathcal{B}, \mu)$ is a content space, $Y \subseteq [-\infty, +\infty]$ and $g, h\colon X \to Y$ are functions such that $g(x) = h(x)$ for almost all $x \in X$. Then g is integrable if and only if h is, in which case*
$$\int_{x\in X} g(x)\,d\mu(x) = \int_{x\in X} h(x)\,d\mu(x).$$

Proof. The hypothesis that $g(x) = h(x)$ almost everywhere means that there is an $E \in \mathcal{B}$ with $\mu(E) = 0$ such that $g(x) = h(x)$ for all $x \in X \setminus E$. With notation as in the definition of the integral, suppose $T \in R_g^{**}(\mathcal{E})$, i.e. that $R_g^*(T) \in \mathcal{E}$. Let $V = R_g^*(T)$. Then $V \in \mathcal{E}$ so there is $\mathcal{Q} \in \mathbf{P}$ such that $\alpha(\mathcal{Q}) \subseteq V$. Let $\mathcal{R}$ be the common refinement of $\mathcal{Q}$ and $\{E, X\setminus E\}$. If $w \in \alpha(\mathcal{R})$ then $w \in \alpha(\mathcal{Q})$ and so $w \in V$. So $\alpha(\mathcal{R}) \subseteq V$ and hance $\alpha(\mathcal{R}) \in \mathcal{E}$. For $w \in \alpha(\mathcal{R})$ we have $E \in \mathcal{B}_{\mathcal{R}}$ so
$$\sum_{x\in E} w(x) = \mu(E) = 0$$

so $w(x) = 0$ for all $x \in E$ and hence

$$\sum_{x \in E} g(x)w(x) = 0 = \sum_{x \in E} h(x)w(x).$$

On the other hand, we have $g(x) = h(x)$ for $x \in X \setminus E$ so

$$\sum_{x \in X \setminus E} g(x)w(x) = \sum_{x \in X \setminus E} h(x)w(x).$$

Then

$$\begin{aligned} R_g(w) &= \sum_{x \in X} g(x)w(x) \\ &= \sum_{x \in E} g(x)w(x) + \sum_{x \in X \setminus E} g(x)w(x) \\ &= \sum_{x \in E} h(x)w(x) + \sum_{x \in X \setminus E} h(x)w(x) \\ &= \sum_{x \in X} h(x)w(x) = R_h(w). \end{aligned}$$

So $R_g(w) = R_h(w)$ for all $w \in \alpha(\mathcal{R})$. Let $S = R_{h*}(\alpha(\mathcal{R}))$. If $w \in \alpha(\mathcal{R})$ then $R_h(w) \in S$ so $w \in R_h^*(S)$. Therefore

$$\alpha(\mathcal{R}) \subseteq R_h^*(S).$$

$\alpha(\mathcal{R}) \in \mathcal{E}$ and $\mathcal{E}$ is upward closed so

$$R_h^*(S) \in \mathcal{E}$$

and hence

$$S \in R_h^{**}(\mathcal{E}).$$

If $z \in S$ then $z = R_h(w)$ for some $w \in \alpha(\mathcal{R})$. Then $z = R_g(w)$. $\alpha(\mathcal{R}) \subseteq V$ so $w \in V$ and hence $z = R_g(w) \in T$. This holds for all $z \in S$ so $S \subseteq T$. From this and $S \in R_h^{**}(\mathcal{E})$ it follows that

$$T \in R_h^{**}(\mathcal{E}),$$

since $R_h^{**}(\mathcal{E})$ is upward closed. T was an arbitrary element of $R_g^{**}(\mathcal{E})$ so

$$R_g^{**}(\mathcal{E}) \subseteq R_h^{**}(\mathcal{E}).$$

The same argument with the roles of g and h reversed gives

$$R_h^{**}(\mathcal{E}) \subseteq R_g^{**}(\mathcal{E}),$$

so

$$R_g^{**}(\mathcal{E}) = R_h^{**}(\mathcal{E}).$$

Therefore $R_g^{**}(\mathcal{E})$ converges if and only if $R_h^{**}(\mathcal{E})$ converges, in which case the limits are the same. In terms of integrals this means that g is integrable if and only if h is, in which case

$$\int_{x \in X} g(x)\, d\mu(x) = \int_{x \in X} h(x)\, d\mu(x).$$

□

Proposition 8.3.3. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space, $Y \subseteq [-\infty, +\infty]$ and $g, h \colon X \to Y$ are functions such that $g(x) = h(x)$ for almost all $x \in X$. Then g is integrable if and only if h is, in which case*

$$\int_{x \in X} g(x)\, d\mu(x) = \int_{x \in X} h(x)\, d\mu(x).$$

Proof. The proof is word for word the same as for the previous proposition. □

Corollary 8.3.4. *Suppose that $(X, \mathcal{B}, \mu)$ is a measure space or a content space, $Y \subseteq [-\infty, +\infty]$ and $g \colon X \to Y$ is a function such that $g(x) = 0$ for almost all $x \in X$. Then g is integrable and*

$$\int_{x \in X} f(x)\, d\mu(x) = 0.$$

Proof. We just take $h = 0$. This is a simple function so its integral was computed in the last section and is, as expected, zero. □

The propositions show that as far as integration is concerned there is not much point in distinguishing functions which take the same values at almost all points. Similarly, it generally makes more sense to impose a condition at almost all points rather than all of them. That's why the inequalities $f(x) \leq g(x) \leq h(x)$ in the following definition are required to hold for almost all $x \in X$ rather than for all $x \in X$.

Definition 8.3.5. Suppose $(X, \mathcal{B}, \mu)$ is a content space and $g \colon X \to Y$ is a function, where $Y =$

$[0,+\infty]$ or $Y = \mathbf{R}$. The *lower integral* of g with respect to $(X, \mathcal{B}, \mu)$, denoted

$$\underline{\int}_{x \in (X,\mathcal{B},\mu)} g(x)\, d\mu(x),$$

is the supremum of all $\int_{x\in X} f(x)\, d\mu(x)$ where f ranges over all simple functions such that $f(x) \leq g(x)$ for almost all $x \in X$. The *upper integral* of g with respect to $(X, \mathcal{B}, \mu)$, denoted is the infimum of all

$$\overline{\int}_{x \in (X,\mathcal{B},\mu)} g(x)\, d\mu(x),$$

$\int_{x\in X} h(x)\, d\mu(x)$ where h ranges over all simple functions such that $g(x) \leq h(x)$ for almost all $x \in X$.

Definition 8.3.6. Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $g\colon X \to Y$ is a function, where $Y = [0,+\infty]$ or $Y = \mathbf{R}$. The *lower integral* of g with respect to $(X, \mathcal{B}, \mu)$, denoted

$$\underline{\int}_{x \in (X,\mathcal{B},\mu)} g(x)\, d\mu(x),$$

is the supremum of all $\int_{x\in X} f(x)\, d\mu(x)$ where f ranges over all semisimple functions such that $f(x) \leq g(x)$ for all x. The *upper integral* of g with respect to $(X, \mathcal{B}, \mu)$, denoted

$$\overline{\int}_{x \in (X,\mathcal{B},\mu)} g(x)\, d\mu(x),$$

is the infimum of all $\int_{x\in X} h(x)\, d\mu(x)$ where h ranges over all semisimple functions such that $g(x) \leq h(x)$ for all x.

Every measure space is a content space so we've defined upper and lower integrals twice for measure spaces. The values do not, in general, agree. There is therefore an unfortunate ambiguity. In general, if the space is a measure space and we refer to upper and lower integrals then we mean the versions for measure spaces unless otherwise stated. From the fact that every simple function is a semisimple function it follows that the lower integral of f, viewed as a function on a measure space, is no smaller than its lower integral when viewed as a function on a content space and that the upper integral of f, viewed as a function on a measure space, is no larger than its upper integral when viewed as a function on a content space.

The following is the analogue for filters of Lemma 6.3.5 for nets, which, in turn, was a generalisation of a familiar convergence criterion for sequences.

Proposition 8.3.7. *Suppose $\mathcal{E}$ is a filter on a set X, $Y \subseteq [-\infty,+\infty]$, and $r\colon X \to Y$ is a function. Then*

$$\sup_{V\in\mathcal{E}} \inf_{w\in V} r(w) \leq \inf_{V\in\mathcal{E}} \sup_{w\in V} r(w)$$

in $[-\infty,+\infty]$. $r^{**}(\mathcal{E})$ is convergent in Y if and only if

$$\inf_{V\in\mathcal{E}} \sup_{w\in V} r(w) \in Y,$$

$$\sup_{V\in\mathcal{E}} \inf_{w\in V} r(w) \in Y,$$

and

$$\inf_{V\in\mathcal{E}} \sup_{w\in V} r(w) \leq \sup_{V\in\mathcal{E}} \inf_{w\in V} r(w).$$

In this case their common value is the limit of $r^{**}(\mathcal{E})$

We are, of course, primarily interested in the case $Y = [0,+\infty]$ and $Y = \mathbf{R}$.

Proof. Note that all the infima and suprema are understood to be in $[-\infty,+\infty]$, and so definitely exist. That they belong to Y is an additional hypothesis though, which we make only in the "if" part of the statement.

If $Z \in \mathcal{E}$ then $Z \neq \varnothing$ so

$$\inf_{w\in Z} r(w) \leq \sup_{w\in Z} r(w).$$

Therefore

$$\sup_{S\in\mathcal{E}} \inf_{w\in Z} r(w) \leq \sup_{w\in Z} r(w)$$

and

$$\sup_{S\in\mathcal{E}} \inf_{w\in Z} r(w) \leq \inf_{T\in\mathcal{E}} \sup_{w\in T} r(w)$$

so

$$\sup_{V\in\mathcal{E}} \inf_{w\in V} r(w) \leq \inf_{V\in\mathcal{E}} \sup_{w\in V} r(w).$$

Suppose that

$$\inf_{V\in\mathcal{E}} \sup_{w\in V} r(w) \in Y,$$

$$\sup_{V\in\mathcal{E}} \inf_{w\in V} r(w) \in Y,$$

and

$$\inf_{V\in\mathcal{E}} \sup_{w\in V} r(w) \le \sup_{V\in\mathcal{E}} \inf_{w\in V} r(w).$$

Then the two sides of the inequality are equal, since we already have the reverse inequality, so there is a $y \in Y$ such that

$$\sup_{V\in\mathcal{E}} \inf_{w\in V} r(w) = y = \inf_{V\in\mathcal{E}} \sup_{w\in V} r(w).$$

Suppose $x < y$ and let $I = Y \cap (x, +\infty]$. Then

$$x < \sup_{V\in\mathcal{E}} \inf_{w\in V} r(w)$$

so there is a $V \in \mathcal{E}$ such that

$$\inf_{w\in V} r(w) > x.$$

So if $w \in V$ then $r(w) > x$, i.e. $r(w) \in I$.

Similarly, if $z > y$ and $J = Y \cap [-\infty, y)$ Then

$$z > \inf_{V\in\mathcal{E}} \sup_{w\in V} r(w)$$

so there is a $V \in \mathcal{E}$ such that

$$\sup_{w\in V} r(w) < z.$$

So if $w \in V$ then $r(w) < z$, i.e. $r(w) \in J$.

If $x < y < z$ let $K = Y \cap (x, z) = I \cap J$. We've just seen that there is a $V_I \in \mathcal{E}$ such that $r(w) \in I$ for all $w \in V_I$ and a $V_J \in \mathcal{E}$ such that $r(w) \in J$ for all $w \in V_J$. Let $V = V_I \cap V_J$. Then $V \in \mathcal{E}$ and $r(w) \in K$ for all $w \in V$.

Every neighbourhood of y in Y contains a set of the form I, J, of K, so if N is a neighbourhood of y then there is a $V \in \mathcal{E}$ such that $r(w) \in N$ if $w \in V$. In other words, $V \subseteq r^*(N)$. $V \in \mathcal{E}$ and $\mathcal{E}$ is upward closed so $r^*(N) \in \mathcal{E}$, i.e. $N \in r^{**}(\mathcal{E})$. This holds for all $N \in \mathcal{N}(y)$ so

$$\mathcal{N}(y) \subseteq r^{**}(\mathcal{E}).$$

Therefore $\mathcal{E}$ converges to y. Thus if

$$\inf_{V\in\mathcal{E}} \sup_{w\in V} r(w) \le \sup_{V\in\mathcal{E}} \inf_{w\in V} r(w)$$

then they are equal and $\mathcal{E}$ converges to their common value.

Suppose, conversely, that $\mathcal{E}$ converges to some $y \in Y$. If $x < y$ then $I = Y \cap (x, +\infty]$ is a neighbourhood of y in Y so $r^*(I) \in \mathcal{E}$. If $w \in r^*(I)$ then $r(w) \in I$ and hence $r(w) > x$. This holds for all $w \in r^*(I)$ so $\inf_{w\in r^*(I)} r(w) \ge x$. $r^*(I) \in \mathcal{E}$ so

$$\sup_{V\in\mathcal{E}} \inf_{w\in V} r(w) \ge x.$$

This holds for all $x < y$ so

$$\sup_{V\in\mathcal{E}} \inf_{w\in V} r(w) \ge y.$$

Strictly speaking, the argument above fails if $y = -\infty$ because then there is no $x < y$, but

$$\sup_{V\in\mathcal{E}} \inf_{w\in V} r(w) \ge y$$

holds trivially if $y = -\infty$, since every element of $[-\infty, +\infty]$ is greater than or equal to $-\infty$.

Similarly, If $z > y$ then $J = Y \cap [-\infty, z)$ is a neighbourhood of y in Y so $r^*(J) \in \mathcal{E}$. If $w \in r^*(J)$ then $r(w) \in J$ and hence $r(w) < z$. This holds for all $w \in r^*(J)$ so $\sup_{w\in r^*(J)} r(w) \ge x$. $r^*(J) \in \mathcal{E}$ so

$$\inf_{V\in\mathcal{E}} \sup_{w\in V} r(w) \le z.$$

This holds for all $z > y$ so

$$\inf_{V\in\mathcal{E}} \sup_{w\in V} r(w) \le y.$$

This time the argument fails for $y = +\infty$ but again the inequality holds trivially in that case. From

$$\inf_{V\in\mathcal{E}} \sup_{w\in V} r(w) \le y$$

and

$$\sup_{V\in\mathcal{E}} \inf_{w\in V} r(w) \ge y$$

we get

$$\inf_{V\in\mathcal{E}} \sup_{w\in V} r(w) \le \sup_{V\in\mathcal{E}} \inf_{w\in V} r(w).$$

The reverse inequality holds without any assumptions so in fact

$$\inf_{V \in \mathcal{E}} \sup_{w \in V} r(w) = \sup_{V \in \mathcal{E}} \inf_{w \in V} r(w).$$

But we've already seen that the left hand side is less than or equal to y while the right hand side is greater than or equal to y so

$$\inf_{V \in \mathcal{E}} \sup_{w \in V} r(w) = y = \sup_{V \in \mathcal{E}} \inf_{w \in V} r(w).$$

$y \in Y$ so

$$\inf_{V \in \mathcal{E}} \sup_{w \in V} r(w) \in Y$$

and

$$\sup_{V \in \mathcal{E}} \inf_{w \in V} r(w) \in Y.$$

□

Proposition 8.3.8. *Suppose that $(X, \mathcal{B}, \mu)$ is a measure space, $Y \subseteq [-\infty, +\infty]$ and $g\colon X \to Y$ is a function. Then*

$$\inf_{V \in \mathcal{E}} \sup_{w \in V} R_g(w) = \overline{\int}_{x \in X} g(x)\, d\mu(x)$$

and

$$\sup_{V \in \mathcal{E}} \inf_{w \in V} R_g(w) = \underline{\int}_{x \in X} g(x)\, d\mu(x).$$

All the suprema, and infima, including those in the definition of $\overline{\int}_{x \in X} g(x)\, d\mu(x)$ and $\underline{\int}_{x \in X} g(x)\, d\mu(x)$ are to be understood in $[-\infty, +\infty]$.

Proof. We'll prove only the first of these equations. The second can proved by swapping infima and suprema and the directions of inequalities in the proof of the first, or, more simply, by applying the first equation to $-g$ in place of g.

Suppose

$$\inf_{V \in \mathcal{E}} \sup_{w \in V} R_g(w) < \overline{\int}_{x \in X} g(x)\, d\mu(x).$$

Then there are $y_1, y_2 \in \mathbf{R}$ such that

$$\inf_{V \in \mathcal{E}} \sup_{w \in V} R_g(w) < y_1 < y_2 < \overline{\int}_{x \in X} g(x)\, d\mu(x).$$

By the definition of the infimum there is then a $V \in \mathcal{E}$ such that

$$\sup_{w \in V} R_g(w) < y_1.$$

$V \in \mathcal{E}$ so there is a countable partition $\mathcal{Q}$ of X such that $\alpha(\mathcal{Q}) \subseteq V$. Then

$$\sup_{w \in \alpha(\mathcal{Q})} R_g(w) < y_1.$$

Define $\varphi\colon \mathcal{Q} \to Y$ by

$$\varphi(E) = \sup_{y \in E} g(y)$$

for $E \in \mathcal{Q}$ and $h\colon E \to Y$ by

$$h(x) = \varphi(E),$$

where E is the unique element of $\mathcal{Q}$ such that $x \in E$. Then $h \in \mathcal{B}_{\mathcal{Q}}$ and $g(x) \leq h(x)$ for all x so h is one of the functions appearing in the definition of $\overline{\int}_{x \in X} g(x)\, d\mu(x)$. Therefore

$$\int_{x \in X} h(x)\, d\mu(x) \geq \overline{\int}_{x \in X} g(x)\, d\mu(x) > y_2.$$

Let

$$\epsilon = y_2 - y_1.$$

Then $\epsilon > 0$.

$\mathcal{Q}$ is countable. Label its elements E_0, E_1, etc. The sequence may or may not terminate after finitely many terms. For each j choose a sequence $m_{j,0}$, $m_{j,1}$, ...such that $m_{j,k} < +\infty$ for each j and k but

$$\sum_k m_{j,k} = \mu(E_j).$$

This sequence also may or may not terminate after finitely many terms. There definitely is a sequence satisfying these requirements. If $\mu(E_j) < +\infty$ then we can just take

$$m_{j,0} = \mu(E_j)$$

and stop there. If $\mu(E_j) = +\infty$ then we can take

$$m_{j,k} = 1$$

for all k. Let

$$\delta_{j,k} = \frac{\epsilon}{2^{j+k+2}(1+m_{j,k})} > 0.$$

We then choose, for each j, a sequence of points such that $x_{j,k} \in E_j$ and

$$g(x_{j,k}) > \varphi(E_j) - \delta_{j,k}.$$

There must exist such an $x_{j,k}$ because $\varphi(E_j)$ is the supremum of g on E_j and so $\varphi(E_j) - \delta_{j,k}$ is less than the supremum. Define a system of weights w by

$$w(x) = \sum_{j,k\colon x=x_{j,k}} m_{j,k}.$$

Then

$$\begin{aligned}\sum_{x\in E_l} w(x) &= \sum_{x\in E_l} \sum_{j,k\colon x=x_{j,k}} m_{j,k} \\ &= \sum_{j,k\colon x_{j,k}\in E_l} m_{j,k} = \sum_k m_{l,j} = \mu(E_l).\end{aligned}$$

The second to last equation holds because $x_{j,k} \in E_j$ and $E_j \cap E_l = \varnothing$ if $j \neq l$. So $(X, \mathcal{B}, \mu)$, $\mathcal{Q}$ and w are compatible and therefore $w \in U$.

$$\begin{aligned}R_g(w) &= \sum_{x\in X} g(x)w(x) \\ &= \sum_{x\in X} g(x) \sum_{j,k\colon x=x_{j,k}} m_{j,k} \\ &= \sum_{j,k} g(x_{j,k})m_{j,k} \\ &> \sum_{j,k} \varphi(E_j)m_{j,k} - \sum_{j,k} \delta_{j,k}m_{j,k} \\ &> \sum_j \sum_k \varphi(E_j)m_{j,k} - \sum_{j,k} \frac{\epsilon}{2^{j+k+2}} \\ &= \sum_j \varphi(E_j) \sum_k m_{j,k} - \sum_j \sum_k \frac{\epsilon}{2^{j+k+2}} \\ &\geq \sum_j \varphi(E_j)w(E_j) - \sum_j \frac{\epsilon}{2^{j+1}} \\ &\geq y_2 - \epsilon = y_1.\end{aligned}$$

But this contradicts the inequality

$$R_g(w) < y_1$$

from earlier, so the assumption that

$$\inf_{V\in\mathcal{E}} \sup_{w\in V} R_g(w) < \overline{\int}_{x\in X} g(x)\, d\mu(x)$$

can't hold.

Suppose now that

$$\overline{\int}_{x\in X} g(x)\, d\mu(x) < \inf_{V\in\mathcal{E}} \sup_{w\in V} R_g(w).$$

Then there is a $y \in \mathbf{R}$ such that

$$\overline{\int}_{x\in X} g(x)\, d\mu(x) < y < \inf_{V\in\mathcal{E}} \sup_{w\in V} R_g(w).$$

$\overline{\int}_{x\in X} g(x)\, d\mu(x)$ is the infimum of $\int_{x\in X} h(x), d\mu(x)$ over all semisimple functions h such that $g(x) \leq h(x)$ for all $x \in X$. There must therefore be such an h such that

$$\int_{x\in X} h(x)\, d\mu(x) < y.$$

In other words, there is a countable partition $\mathcal{Q}$ of X and a function $\varphi\colon \mathcal{Q} \to Y$ such that

$$h(x) = \varphi(E)$$

for each $x \in X$, where E is the unique element of $\mathcal{Q}$ such that $x \in E$. Let $V = \alpha(\mathcal{Q})$. If $w \in V$ then

$$\begin{aligned}R_g(w) &= \sum_{x\in X} g(x)w(x) \leq \sum_{x\in X} h(x)w(x) \\ &= \sum_{E\in\mathcal{Q}} \sum_{x\in E} h(x)w(x) = \sum_{E\in\mathcal{Q}} \sum_{x\in E} \varphi(E)w(x) \\ &= \sum_{E\in\mathcal{Q}} \varphi(E) \sum_{x\in E} w(x) = \sum_{E\in\mathcal{Q}} \varphi(E)\mu(E) \\ &= \int_{x\in X} h(x)\, d\mu(x)\, dx < y\end{aligned}$$

so

$$\sup_{w\in V} R_g(w) \leq y,$$

where $V = \alpha(\mathcal{Q})$, and $\alpha(\mathcal{Q}) \in \mathcal{E}$, so

$$\inf_{V\in\mathcal{E}} \sup_{w\in V} R_g(w) \leq y.$$

But this contradicts the inequality

$$\inf_{V \in \mathcal{E}} \sup_{w \in V} R_g(w) > y.$$

from earlier, so the assumption

$$\overline{\int}_{x \in X} g(x)\, d\mu(x) < \inf_{V \in \mathcal{E}} \sup_{w \in V} R_g(w).$$

cannot hold either. Therefore

$$\overline{\int}_{x \in X} g(x)\, d\mu(x) = \inf_{V \in \mathcal{E}} \sup_{w \in V} R_g(w).$$

□

Proposition 8.3.9. *Suppose that $(X, \mathcal{B}, \mu)$ is a content space, $Y \subseteq [-\infty, +\infty]$ and $g \colon X \to Y$ is a function. Then*

$$\inf_{V \in \mathcal{E}} \sup_{w \in V} R_g(w) = \overline{\int}_{x \in X} g(x)\, d\mu(x)$$

and

$$\sup_{V \in \mathcal{E}} \inf_{w \in V} R_g(w) = \underline{\int}_{x \in X} g(x)\, d\mu(x).$$

Proof. The proof is almost identical to that of the preceding proposition. The partitions $\mathcal{Q}$ are finite though, and the range of the index j is therefore finite as well, although k may still need to be countable unless $\mu(X) < +\infty$. Some further simplifications are possible, but none are necessary. □

Theorem 8.3.10. *Suppose $(X, \mathcal{B}, \mu)$ is a content space, Y is a subset of $[-\infty, +\infty]$ and $g \colon X \to Y$ is a function. Then*

$$\underline{\int}_{x \in X} g(x)\, d\mu(x) \le \overline{\int}_{x \in X} g(x)\, d\mu(x).$$

Also, g is integrable if and only if

$$\underline{\int}_{x \in X} g(x)\, d\mu(x) \in Y,$$

$$\overline{\int}_{x \in X} g(x)\, d\mu(x) \in Y,$$

and

$$\overline{\int}_{x \in X} g(x)\, d\mu(x) \le \underline{\int}_{x \in X} g(x)\, d\mu(x),$$

in which case the integral is their common value.

We are, of course, primarily interested in the cases $Y = [0, +\infty]$ and $Y = \mathbf{R}$. In the former case the hypotheses

$$\underline{\int}_{x \in X} g(x)\, d\mu(x) \in Y$$

and

$$\overline{\int}_{x \in X} g(x)\, d\mu(x) \in Y$$

are always fulfilled.

Proof. We apply Proposition 8.3.7 to $r = R_g$ and $\mathcal{E}$ the filter from the definition of the integral. Proposition 8.3.9 allows us to identify the supremum of the infima and the infimum of the suprema with the lower and upper intervals, respectively. □

Theorem 8.3.11. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space, Y is a subset of $[-\infty, +\infty]$ and $g \colon X \to Y$ is a function. Then*

$$\underline{\int}_{x \in X} g(x)\, d\mu(x) \le \overline{\int}_{x \in X} g(x)\, d\mu(x).$$

Also, g is integrable if and only if

$$\underline{\int}_{x \in X} g(x)\, d\mu(x) \in Y,$$

$$\overline{\int}_{x \in X} g(x)\, d\mu(x) \in Y,$$

and

$$\overline{\int}_{x \in X} g(x)\, d\mu(x) \le \underline{\int}_{x \in X} g(x)\, d\mu(x),$$

in which case the integral is their common value.

Again, we are primarily interested in the cases $Y = [0, +\infty]$ and $Y = \mathbf{R}$ and in the former case the hypotheses

$$\underline{\int}_{x\in X} g(x)\, d\mu(x) \in Y$$

and

$$\overline{\int}_{x\in X} g(x)\, d\mu(x) \in Y$$

are always fulfilled.

Proof. The proof is the same as in the preceding theorem, except with Proposition 8.3.8 in place of Proposition 8.3.9. □

The characterisation of integrability in terms of upper and lower integrals can be used to show that a function is integrable over a content or measure space if and only if it is integrable over the completion of that space.

Theorem 8.3.12. *Suppose $(X, \mathcal{B}, \mu)$ is a content space and $(X, \mathcal{B}^\dagger, \mu^\dagger)$ is its completion. Suppose $Y \subseteq [-\infty, +\infty]$. Then $g\colon X \to Y$ is integrable with respect to $(X, \mathcal{B}^\dagger, \mu^\dagger)$ if and only if it is integrable with respect to $(X, \mathcal{B}, \mu)$. The two integrals are then equal.*

Proof. The "if" part and the equality of the two integrals follow immediately from Proposition 8.2.11 but the "only if" part requires more work.

Suppose that g is integrable with respect to $(X, \mathcal{B}^\dagger, \mu^\dagger)$. By Theorem 8.3.11 then

$$\underline{\int}_{x\in X} g(x)\, d\mu^\dagger(x) = \overline{\int}_{x\in X} g(x)\, d\mu^\dagger(x)$$

and both sides belong to Y.

The lower integral, i.e. the left hand side, was defined to be the supremum of $\int_{x\in X} f(x), d\mu^\dagger(x)$ where f ranges over simple functions such that $f(x) \le g(x)$ for all $x \in X$. Simple here means that there is a finite partition $\mathcal{Q} \subseteq \mathcal{B}^\dagger$ of X such that $\wp(Y) \subseteq f^{**}(\mathcal{B}_{\mathcal{Q}})$. It's $\mathcal{B}^\dagger$ rather than $\mathcal{B}$ since this is the lower integral with respect to $(X, \mathcal{B}^\dagger, \mu^\dagger)$ rather than with respect to $(X, \mathcal{B}, \mu)$.

Suppose $y_1, y_2 \in \mathbf{R}$ are such that

$$y_1 < y_2 < \underline{\int}_{x\in X} g(x)\, d\mu^\dagger(x).$$

We can do this unless $\underline{\int}_{x\in X} g(x)\, d\mu^\dagger(x) = -\infty$, a case which we will consider separately later. By the definition of the supremum there's an f as above and a corresponding $\mathcal{Q}$ such that

$$\int_{x\in X} f(x)\, d\mu^\dagger(x) > y_2.$$

Let

$$\epsilon = y_2 - y_1.$$

As usual we have a $\varphi\colon \mathcal{Q} \to Y$ such that

$$f(x) = \varphi(E)$$

where E is the unique element of $\mathcal{Q}$ such that $x \in E$. We can write f as

$$f = \sum_{E\in\mathcal{Q}} \varphi(E)\chi_E,$$

where χ_E is the characteristic function of E. Define

$$\mathcal{Q}_+ = \{E \in \mathcal{Q}\colon \varphi(E) > 0\},$$

$$\mathcal{Q}_0 = \{E \in \mathcal{Q}\colon \varphi(E) = 0\},$$

and

$$\mathcal{Q}_- = \{E \in \mathcal{Q}\colon \varphi(E) < 0\}.$$

These are finite, so

$$Q_+ = \{F_0, F_1, \dots, F_m\}$$

for some distinct F_0, F_1, …, F_m and

$$Q_+ = \{G_0, G_1, \dots, G_n\}$$

for some distinct G_0, G_1, …, G_n. These are all elements of $\mathcal{Q}$ and hence of $\mathcal{B}^\dagger$.

Let

$$\delta_j = \frac{\epsilon}{2^{j+2}|\varphi(E_j)|} > 0.$$

$F_j \in \mathcal{B}^\dagger$. By the definition of $\mathcal{B}^\dagger$ there are $D_j, H_j \in \mathcal{B}$ such that

$$F_j \triangle G_j \subseteq D_j$$

and

$$\mu(D_j) < \delta_j.$$

Then

$$H_j \setminus D_j \subseteq F_j \subseteq H_j \cup D_j$$

so

$$\mu^\dagger(H_j \setminus D_j) \leq \mu^\dagger(F_j) \leq \mu^\dagger(H_j \cup D_j).$$

But

$$H_j \cup D_j = (H_j \setminus D_j) \cup D_j$$

and

$$(H_j \setminus D_j) \cap D_j = \varnothing$$

so

$$\begin{aligned} \mu^\dagger(H_j \cup D_j) &= \mu^\dagger(H_j \setminus D_j) + \mu^\dagger(D_j) \\ &= \mu^\dagger(H_j \setminus D_j) + \mu(D_j) \\ &< \mu^\dagger(H_j \setminus D_j) + \delta_j \end{aligned}$$

It follows that

$$\mu^\dagger(\tilde{F}_j) < \mu(F_j) + \delta_j$$

where

$$\tilde{F}_j = H_j \setminus D_j.$$

$\varphi(F_j) > 0$ so

$$\varphi(F_j)\mu(\tilde{F}_j) < \varphi(F_j)\mu(F_j) + \frac{\epsilon}{2^{j+2}}.$$

Also $\tilde{F}_j \subseteq F_j$ so

$$\chi_{\tilde{F}_j}(x) \leq \chi_{F_j}(x)$$

and hence

$$\varphi(F_j)\chi_{\tilde{F}_j}(x) \leq \varphi(F_j)\chi_{F_j}(x)$$

for all $x \in X$.

Similarly, let

$$\theta_k = \frac{\epsilon}{2^{k+2}|\varphi(E_k)|} > 0.$$

$G_k \in \mathcal{B}^\dagger$. By the definition of $\mathcal{B}^\dagger$ there are $D_k, H_k \in \mathcal{B}$ such that

$$G_k \triangle G_k \subseteq D_k$$

and

$$\mu(D_k) < \theta_k.$$

Then

$$H_k \setminus D_k \subseteq G_k \subseteq H_k \cup D_k,$$

$$\mu^\dagger(H_k \setminus D_k) \leq \mu^\dagger(G_k) \leq \mu^\dagger(H_k \cup D_k),$$

$$H_k \cup D_k = (H_k \setminus D_k) \cup D_k,$$

$$(H_k \setminus D_k) \cap D_k = \varnothing,$$

and

$$\begin{aligned} \mu^\dagger(H_k \cup D_k) &= \mu^\dagger(H_k \setminus D_k) + \mu^\dagger(D_k) \\ &= \mu^\dagger(H_k \setminus D_k) + \mu(D_k) \\ &< \mu^\dagger(H_k \setminus D_k) + \theta_k. \end{aligned}$$

It follows that

$$\mu^\dagger(G_k) < \mu(\tilde{G}_k) + \theta_k$$

where

$$\tilde{G}_k = H_k \cup D_k.$$

Now $\varphi(G_k) < 0$ so

$$\varphi(G_k)\mu(\tilde{G}_k) < \varphi(G_k)\mu(G_k) + \frac{\epsilon}{2^{k+2}}.$$

Also $G_k \subseteq \tilde{G}_k$, so

$$\chi_{G_k}(x) \leq \chi_{\tilde{G}_k}(x)$$

and hence

$$\varphi(G_K)\chi_{\tilde{G}_k}(x) \leq \varphi(G_k)\chi_{G_k}(x)$$

for all $x \in X$.

Now let

$$\tilde{f} = \sum_j \varphi(F_j)\chi_{\tilde{F}_j} + \sum_k \varphi(G_k)\chi_{\tilde{G}_k}.$$

Then

$$\tilde{f}(x) \leq f(x) \leq g(x)$$

for all $x \in X$. Also $\tilde{F}_j \in \mathcal{B}$ and $\tilde{G}_k \in \mathcal{B}$. $\tilde{f}$ takes only finitely many values and takes each of them on an element of $\mathcal{B}$, so is a simple function with respect to $(X, \mathcal{B})$. Also,

$$\begin{aligned}
\int_{x\in X} f(x)\,d\mu^\dagger(x) &= \sum_j \varphi(F_j)\mu(F_j) + \sum_k \varphi(G_k)\mu(G_k) \\
&= \sum_j \varphi(F_j)\mu^\dagger(F_j) + \sum_k \varphi(G_k)\mu^\dagger(G_k) \\
&< \sum_j \varphi(\tilde{F}_j)\mu^\dagger(F_j) + \sum_j \frac{\epsilon}{2^{j+2}} \\
&\quad + \sum_k \varphi(\tilde{G}_k)\mu^\dagger(G_k) + \sum_k \frac{\epsilon}{2^{j+2}} \\
&= \int_{x\in X} \tilde{f}(x)\,d\mu(x) + \epsilon.
\end{aligned}$$

Now $\int_{x\in X} f(x)\,d\mu^\dagger(x) > y_2$ and $\epsilon = y_2 - y_1$ so

$$\int_{x\in X} \tilde{f}(x)\,d\mu(x) > y_1.$$

This holds for all $y_1 < \underline{\int}_{x\in X} g(x)\,d\mu^\dagger(x)$ so

$$\int_{x\in X} \tilde{f}(x)\,d\mu(x) \geq \underline{\int}_{x\in X} g(x)\,d\mu^\dagger(x),$$

and hence, since $\tilde{f}$ is one of the functions in the supremum defining $\underline{\int}_{x\in X} g(x)\,d\mu(x)$,

$$\underline{\int}_{x\in X} g(x)\,d\mu(x) \geq \underline{\int}_{x\in X} g(x)\,d\mu^\dagger(x).$$

The proof above assumed $\underline{\int}_{x\in X} g(x)\,d\mu^\dagger(x) \neq -\infty$ but the inequality holds even without this assumption because every element of $[-\infty, +\infty]$ is greater than or equal to $-\infty$.

A similar argument works for the upper integrals and gives

$$\overline{\int}_{x\in X} g(x)\,d\mu(x) \leq \overline{\int}_{x\in X} g(x)\,d\mu^\dagger(x).$$

We already have

$$\overline{\int}_{x\in X} g(x)\,d\mu^\dagger(x) \leq \underline{\int}_{x\in X} g(x)\,d\mu^\dagger(x)$$

by the integrability of g with respect to $(X, \mathcal{B}^\dagger, \mu^\dagger)$ and Theorem 8.3.11, so

$$\overline{\int}_{x\in X} g(x)\,d\mu(x) \leq \underline{\int}_{x\in X} g(x)\,d\mu(x).$$

Using Theorem 8.3.11 again we see that g is integrable with respect to $(X, \mathcal{B}, \mu)$. □

Theorem 8.3.13. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $(X, \mathcal{B}^\dagger, \mu^\dagger)$ is its completion. Suppose $Y \subseteq [-\infty, +\infty]$. Then $g\colon X \to Y$ is integrable with respect to $(X, \mathcal{B}^\dagger, \mu^\dagger)$ if and only if it is integrable with respect to $(X, \mathcal{B}, \mu)$. The two integrals are then equal.*

Proof. The proof is nearly identical to the proof of the preceding theorem, with the obvious changes, such as replacing the "finite" with "countable", "content" with "measure", "simple" with "semisimple", etc. There is one exception though. In the proof of the preceding theorem we used the fact a finite linear combination of characteristic functions takes only finitely many values to conclude that $\tilde{f}$ is simple. It is not true in general, however, that a countable linear combination of characteristic functions takes only countably many values. A convergent sum of positive numbers can have only countably many nonzero terms though, so we can modify $\tilde{f}$ by choosing its value to be an arbitrary element of Y on all of those sets which contribute nothing to the integral. The new f will be semisimple and will still satisfy the necessary inequality on the integral. □

8.4 Riemann integration

Definition 8.4.1. A function $g\colon \mathbf{R} \to \mathbf{R}$ is said to be *Riemann integrable* if it is integrable with respect to the content space $(\mathbf{R}, \mathcal{I}, \mu)$ where $\mathcal{I}$ is the set of finite unions of intervals and μ is the length content whose existence was proved in Proposition 7.4.2.

Proposition 8.4.2. *If f is Riemann integrable then it is integrable with respect to $(\mathbf{R}, \mathcal{J}, \mu)$, where $\mathcal{J}$ is the Jordan algebra and μ is its associated content. Conversely, if f is integrable with respect to $(\mathbf{R}, \mathcal{J}, \mu)$ then it is Riemann integrable.*

Proof. This follows from Theorem 8.3.12, since $(\mathbf{R}, \mathcal{J}, \mu)$ is the completion of $(\mathbf{R}, \mathcal{I}, \mu)$. $\square$

Definition 8.4.3. Suppose $(X, \mathcal{T})$ is a topological space and $f \colon X \to \mathbf{R}$ is a function. The *support* of f is the set

$$\overline{f^*(\mathbf{R} - \{0\})}.$$

f is said to be *compactly supported* if its support is compact.

Proposition 8.4.4. *Every compactly supported continuous function is Riemann integrable.*

Proof. Suppose g is compactly supported and continuous. Then the support of g is bounded since all compact subsets of a metric space are bounded. So

$$g^*(\mathbf{R} - \{0\}) \subseteq \overline{g^*(\mathbf{R} - \{0\})} \subseteq \bar{B}(0, r) = [-r, r]$$

for some $r > 0$. In other words, $g(x) = 0$ if $x \notin [-r, r]$. For each $n > 0$ we form a partition $\mathcal{Q}_n$ consisting of $(-\infty, -r)$, $(r, +\infty)$ and $2n$ intervals of length r/n which together form a partition of $[-r, r]$. Define

$$\varphi(E) = \inf_{x \in E} g(x),$$

$$\psi(E) = \sup_{x \in E} g(x),$$

$$f_n(x) = \varphi(E)$$

and

$$h_n(x) = \psi(E)$$

where E is the unique element of $\mathcal{Q}_n$ such that $x \in E$. Then f_n and h_n are simple functions and

$$f_n(x) \leq g(x) \leq h_n(x).$$

g is continuous on the compact set $[-r, r]$ and hence is uniformly continuous. For each $\theta > 0$ there is therefore a $\delta > 0$ $\delta > 0$ such that if $|x - y| < \delta$ then

$$|g(x) - g(y)| < \epsilon,$$

where

$$\theta = \frac{\epsilon}{4r + 1}.$$

Choose n sufficiently large that $r/n < \delta$. If $x \in E$ then

$$g(y) - \epsilon < g(x) < g(z) + \epsilon$$

for all $y, z \in E$ and, taking infima and suprema,

$$h_n(x) - \epsilon \leq g(x) \leq f_n(x) + \epsilon$$

and hence

$$h_n(x) \leq f_n(x) + 2\epsilon.$$

Then

$$\int_{x \in \mathbf{R}} h_n(x)\, d\mu(x) \leq \int_{x \in \mathbf{R}} f_n(x)\, d\mu(x) + 4r\epsilon < \theta.$$

Then

$$\overline{\int}_{x \in \mathbf{R}} g(x)\, d\mu(x) < \underline{\int}_{x \in \mathbf{R}} g(x)\, d\mu(x) + \theta.$$

This holds for all $\theta > 0$, so

$$\overline{\int}_{x \in \mathbf{R}} g(x)\, d\mu(x) \leq \underline{\int}_{x \in \mathbf{R}} g(x)\, d\mu(x).$$

Thus g is integrable by Theorem 8.3.11. $\square$

8.5 Measurable functions

Definition 8.5.1. Suppose $(X, \mathcal{B}, \mu)$ is a measure space, (Y, τ) is topological space and $f \colon X \to Y$ is a function. Then f is said to be *measurable* if $f^*(E) \in \mathcal{B}$ for all Borel subsets E of Y.

Lemma 8.5.2. *Suppose $(X, \mathcal{B}_X, \mu)$ is a measure space, $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ are topological spaces, $f \colon X \to Y$ is a measurable function and $g \colon Y \to Z$ is a continuous function. Then $g \circ f$ is a measurable function.*

Proof. Let $\mathcal{B}_Y$ and $\mathcal{B}_Z$ be the Borel σ-algebras on $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$, respectively. If $V \in \mathcal{B}_Z$ then $g^*(V) \in \mathcal{B}_Y$ by Proposition 7.2.9. Then $f^*(g^*(V)) \in \mathcal{B}_X$ because f is measurable. But

$$f^*(g^*(V)) = (f^* \circ g^*)(V) = (g \circ f)^*(V).$$

So $(g \circ f)^*(V) \in \mathcal{B}_X$ whenever $V \in \mathcal{B}_Z$. In other words, $g \circ f$ is measurable. $\square$

Proposition 8.5.3. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $f \colon X \to \mathbf{R}$ is a measurable function. Then $|f|$ is also measurable.*

Proof. The absolute value function is continuous, so this follows immediate from the preceding lemma. $\square$

Lemma 8.5.4. *Suppose $(X, \mathcal{B}_X, \mu)$ is a measure space, $(Y, \mathcal{T}_Y)$ is a topological space and $f\colon X \to Y$ is a function such that $f^*(V) \in \mathcal{B}_X$ for every $V \in \mathcal{T}_Y$. Then f is measurable.*

Proof. Let $\mathcal{B}_Y$ be the Borel σ-algebra on $(Y, \mathcal{T}_Y)$. The hypothesis that $f^*(V) \in \mathcal{B}_X$ for every $V \in \mathcal{T}_Y$ means that

$$\mathcal{T}_Y \subseteq f^{**}(\mathcal{B}_X).$$

$f^{**}(\mathcal{B}_X)$ is a σ-algebra by Proposition 7.2.4. Any σ-algebra which contains $\mathcal{T}_Y$ also contains the σ-algebra generated by $\mathcal{T}_Y$, i.e. $\mathcal{B}_Y$, so

$$\mathcal{B}_Y \subseteq f^{**}(\mathcal{B}_X).$$

In other words, if $E \in \mathcal{B}_Y$ then $f^*(E) \in \mathcal{B}_X$. So f is measurable. $\square$

Lemma 8.5.5. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $f\colon X \to \mathbf{R}$ and $g\colon X \to \mathbf{R}$ are measurable functions. Define $h\colon X \to \mathbf{R}^2$ by $h(x) = (f(x), g(x))$. Then h is measurable.*

Proof. Suppose that V is an open subset of $\mathcal{R}^2$. Let $\mathcal{A}$ be the set of sets of the form $(a, b) \times (c, d)$ such that $a, b, c, d \in \mathbf{Q}$ and $(a, b) \times (c, d) \subseteq V$. Then $\mathcal{A}$ is countable, because $\mathbf{Q}^4$ is countable. If $(x, y) \in V$ then there is an $r > 0$ such that $B((x, y), r) \subseteq V$. Every interval of positive length contains a rational number so choose a rational $\delta \in (0, r)$. Then

$$\begin{aligned}\left(x - \frac{\delta}{2}, x + \frac{\delta}{2}\right) \times \left(y - \frac{\delta}{2}, y + \frac{\delta}{2}\right) &\subseteq B((x, y), \delta) \\ &\subseteq B((x, y), r) \\ &\subseteq V.\end{aligned}$$

Using again the fact that every interval of positive length contains a rational number we choose

$$p \in \left(x - \frac{\delta}{6}, x + \frac{\delta}{6}\right), \quad q \in \left(y - \frac{\delta}{6}, y + \frac{\delta}{6}\right).$$

Then

$$\left(p - \frac{\delta}{3}, p + \frac{\delta}{3}\right) \subseteq \left(x - \frac{\delta}{2}, x + \frac{\delta}{2}\right)$$

and

$$\left(q - \frac{\delta}{3}, q + \frac{\delta}{3}\right) \subseteq \left(y - \frac{\delta}{2}, y + \frac{\delta}{2}\right)$$

so

$$\begin{aligned}(a, b) \times (c, d) &\subseteq \left(x - \frac{\delta}{2}, x + \frac{\delta}{2}\right) \times \left(y - \frac{\delta}{2}, y + \frac{\delta}{2}\right) \\ &\subseteq V,\end{aligned}$$

where

$$a = p - \frac{\delta}{3}, \quad b = p + \frac{\delta}{3}, \quad c = q - \frac{\delta}{3}, \quad d = q + \frac{\delta}{3}.$$

So $(a, b) \times (c, d) \in \mathcal{A}$. Also,

$$(x, y) \in (a, b) \times (c, d).$$

Therefore

$$(x, y) \in \bigcup_{(a,b)\times(c,d)\in\mathcal{A}} (a, b) \times (c, d).$$

This holds for all $(x, y) \in V$ so

$$V \subseteq \bigcup_{(a,b)\times(c,d)\in\mathcal{A}} (a, b) \times (c, d).$$

Every element of the union is a subset of V though, so

$$V = \bigcup_{(a,b)\times(c,d)\in\mathcal{A}} (a, b) \times (c, d).$$

But then

$$\begin{aligned}h^*(V) &= \bigcup_{(a,b)\times(c,d)\in\mathcal{A}} h^*((a, b) \times (c, d)) \\ &= \bigcup_{(a,b)\times(c,d)\in\mathcal{A}} f^*((a, b)) \cap g^*((c, d))\end{aligned}$$

(a, b) and (c, d) are Borel sets so $f^*((a, b))$ and $g^*((c, d))$ are elements of $\mathcal{B}$, as is their intersection. Any countable union of elements of $\mathcal{B}$ is an element of $\mathcal{B}$, so $h^*(V) \in \mathcal{V}$. This holds for all open subsets V of $\mathbf{R}^2$ so h is measurable, by the preceding lemma. $\square$

Proposition 8.5.6. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space, $f\colon X \to \mathbf{R}$ and $g\colon X \to \mathbf{R}$ are measurable functions and $\alpha, \beta \in \mathbf{R}$. Then $\alpha f + \beta g$ is measurable.*

Proof. Define $h\colon X \to \mathbf{R}^2$ by

$$h(x) = (g(x), h(x)),$$

as in the lemma. Then h is measurable. Define $k\colon \mathbf{R}^2 \to \mathbf{R}$ by

$$k(y, z) = \alpha y + \beta z.$$

Then k is continuous. By Lemma 8.5.2 then $k \circ h$ is measurable. But

$$k \circ h = \alpha f + \beta g.$$

$\square$

Proposition 8.5.7. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $f\colon X \to \mathbf{R}$ and $g\colon X \to \mathbf{R}$ are measurable functions. Then fg is measurable.*

Proof. Define $h\colon X \to \mathbf{R}^2$ as before and note that it's measurable. Define $k\colon \mathbf{R}^2 \to \mathbf{R}$ by

$$k(y, z) = yz.$$

Then k is continuous. By Lemma 8.5.2 then $k \circ h$ is measurable. But

$$k \circ h = fg.$$

$\square$

Proposition 8.5.8. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $f\colon \mathbf{N} \times X \to [-\infty, +\infty]$ is such that $f_n(x)$ is a measurable function of x for each n. Then*

(a) $\inf_{n\in\mathbf{N}} f_n$ *is measurable.*

(b) $\sup_{n\in\mathbf{N}} f_n$ *is measurable.*

(c) $\sup_{m\in\mathbf{N}} \inf_{n\geq m} f_n$ *is measurable.*

(d) $\inf_{m\in\mathbf{N}} \sup_{n\geq m} f_n$ *is measurable.*

(e) $\lim_{n\to\infty} f_n$ *is measurable, if it exists.*

Note that all of these infima and suprema exist as elements of $[-\infty, +\infty]$, though the limit might not. For functions with values in $[0, +\infty]$ the infima and suprema also exist as elements of $[0, +\infty]$, though the limit might not. For functions with values in $\mathbf{R}$ the infima and suprema exist as elements of $[-\infty, +\infty]$ but possibly not as elements of $\mathbf{R}$, and again the limit need not exist.

Proof. $y \geq \inf_{n\in\mathbf{N}} f_n(x)$ if and only if $y \geq f_n(x)$ for all x. In other words, $x \in g^*([y, +\infty])$ if and only if $x \in f_n^*([y, +\infty])$ for all n, where

$$g(x) = \inf_{n\in\mathbf{N}} f_n(x).$$

Thus

$$g^*([y, +\infty]) = \bigcap_{n\in\mathbf{N}} f_n^*([y, +\infty]).$$

From this it follows that

$$\begin{aligned}
g^*([-\infty, z)) &= g^*([-\infty, +\infty] \setminus [z, +\infty]) \\
&= g^*([-\infty, +\infty]) \setminus g^*([z, +\infty]) \\
&= X \setminus \bigcap_{n\in\mathbf{N}} f_n^*([z, +\infty]) \\
&= \bigcup_{n\in\mathbf{N}} (X \setminus f_n^*([z, +\infty])) \\
&= \bigcup_{n\in\mathbf{N}} (f_n^*([-\infty, +\infty]) \setminus f_n^*([z, +\infty])) \\
&= \bigcup_{n\in\mathbf{N}} f_n^*([-\infty, +\infty] \setminus [z, +\infty]) \\
&= \bigcup_{n\in\mathbf{N}} f_n^*([-\infty, z)).
\end{aligned}$$

Then

$$\begin{aligned}
g^*([y, z)) &= g^*([y, +\infty] \cap [-\infty, z)) \\
&= g^*([y, +\infty]) \cap g^*([-\infty, z)) \\
&= \left(\bigcap_{n\in\mathbf{N}} f_n^*([y, +\infty])\right) \cap \left(\bigcup_{n\in\mathbf{N}} f_n^*([-\infty, z))\right).
\end{aligned}$$

The sets $[y, +\infty]$ and $[-\infty, z)$ are Borel sets. f_n is measurable so $f_n^*([y, +\infty]) \in \mathcal{B}$ and $f_n^*([-\infty, z)) \in \mathcal{B}$ for each $n \in \mathcal{N}$. $\mathcal{B}$ is a σ-algebra and hence countable unions or intersections of elements of $\mathcal{B}$ are elements of $\mathcal{B}$. It follows that

$$g^*([y, z)) \in \mathcal{B}$$

for any $y, z \in [-\infty, +\infty]$.

Suppose V is an open subset of $\mathbf{R}$. Then

$$V = \bigcup_{\substack{(y,z)\in\mathbf{Q}^2 \\ [y,z)\subseteq V}} [y, z).$$

The proof is similar to, but simpler than the one given in Lemma 8.5.5 to show that every open subset of $\mathbf{R}^2$ is a union of countably many open rectangles with with rational coefficients. Then

$$g^*(V) = \bigcup_{\substack{(y,z)\in\mathbf{Q}^2\\ [y,z)\subseteq V}} g^*([y,z)) \in \mathcal{B}$$

because the union of countably many elements of $\mathcal{B}$ is an element of $\mathcal{B}$. In particular,

$$g^*(\mathbf{R}) \in \mathcal{B}$$

so

$$\begin{aligned} g^*(\{-\infty,+\infty\}) &= g^*([-\infty,+\infty] \setminus \mathbf{R}) \\ &= g^*([-\infty,+\infty]) \setminus g^*(\mathbf{R}) \\ &= X \setminus g^*(\mathbf{R}) \end{aligned}$$

is an element of $\mathcal{B}$. So are

$$\begin{aligned} g^*(\{-\infty\}) &= g^*(\{-\infty,+\infty\} \cap [-\infty,+\infty)) \\ &= g^*(\{-\infty,+\infty\}) \cap g^*([-\infty,+\infty)) \end{aligned}$$

and

$$\begin{aligned} g^*(\{+\infty\}) &= g^*(\{-\infty,+\infty\} \setminus \{-\infty\}) \\ &= g^*(\{-\infty,+\infty\}) \setminus g^*(\{-\infty\}). \end{aligned}$$

If W is an open set in $[-\infty,+\infty]$ then $W = V$, $W = V \cup \{-\infty,+\infty\}$, $W = V \cup \{-\infty\}$ or $W = V \cup \{+\infty\}$, where $V = W \cap \mathbf{R}$ is an open set in $\mathbf{R}$. so $g^*(W)$ is one of $g^*(V)$, $g^*(V) \cup g^*(\{-\infty,+\infty\})$, $g^*(V) \cup g^*(\{-\infty\})$ or $g^*(V) \cup g^*(\{+\infty\})$, each of when belongs to $\mathcal{B}$. So for any open $W \subseteq [-\infty,+\infty]$ we have $g^*(W) \in \mathcal{B}$. It follows from Lemma 8.5.4 that g is measurable. This completes the proof of 8.5.8a.

Fortunately the other parts are now easy to prove. If f_n is measurable for each n then so is $-f_n$, so

$$-\sup_{n\in\mathbf{N}} f_n(x) = \inf_{n\in\mathbf{N}} (-f_n(x))$$

is measurable and hence so is $\sup_{n\in\mathbf{N}} f_n$.

If f_n is measurable for each n then for any $m \in \mathcal{N}$ we have that $g_k = f_{m+k}$ is measurable. Then

$$\sup_{n\geq m} f_n = \sup_{k\in\mathbf{N}} g_k$$

is measurable, so

$$\inf_{m\in\mathbf{N}} \sup_{n\geq m} f_n$$

is measurable. Similarly,

$$\inf_{n\geq m} f_n = \inf_{k\in\mathbf{N}} g_k$$

is measurable, so

$$\sup_{m\in\mathbf{N}} \inf_{n\geq m} f_n$$

is measurable. $\lim_{n\to\infty}$, assuming it exists, is equal to $\sup_{m\in\mathbf{N}} \inf_{n\geq m} f_n$ and so is measurable. □

8.6 Integrability and measurability

Proposition 8.6.1. *Suppose $(X, \mathcal{B}, \mu)$ is a content space and $(X, \mathcal{B}^\dagger, \mu^\dagger)$ is its completion. If f is integrable with respect to $(X, \mathcal{B}, \mu)$ then it is measurable with respect of $(X, \mathcal{B}^\dagger, \mu^\dagger)$.*

Proposition 8.6.2. *If $\int |f| < +\infty$ and f is measurable then f is integrable.*

Proposition 8.6.3. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $f\colon X \to [0,+\infty]$ is a function. Then f is integrable if and only if its is measurable with respect to $(X, \mathcal{B}^\dagger, \mu)$, the completion of $(X, \mathcal{B}, \mu)$.*

Corollary 8.6.4. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $f\colon \mathbf{N} \times X \to [0,+\infty]$ is such that $f_n(x)$ is integrable as a function of x for each n and convergent as a sequence in n for each x. Then $\lim_{n\to\infty} f_n$ is integrable.*

Proof. Each f_n is integrable and hence measurable by the proposition above. The limit of a sequence of measurable functions is measurable by Proposition **??** so $\lim_{n\to\infty} f_n$ is measurable. Using the proposition above again we see that it is integrable. □

Note that this just says the integral of the limit exists, not that it is equal to the limit of the integrals. That's not true without further assumptions.

8.7 Convergence properties

The following theorem is known as the *Monotone Convergence Theorem* for integrals.

Theorem 8.7.1. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $f\colon \mathbf{N} \times X \to [0, +\infty]$ is a function such that*

(a) $f_n(x)$ is an integrable function of x for each $n \in \mathbf{N}$, and

(b) $f_n(x)$ is a monotone increasing sequence in n for each $x \in X$.

Then

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\, d\mu(x) = \int_{x\in X} \lim_{m\to\infty} f_m(x)\, d\mu(x).$$

Note the second condition above implies that $\lim_{m\to\infty} f_m(x)$ exists, so the integrand on the right hand side is well defined.

Proof.

$$f_m(x) \le \sup_{n\in\mathbf{N}} f_n(x) = \lim_{n\to\infty} f_n(x)$$

for all $m \in \mathbf{N}$. The equation between the supremum and the limit follows from the monotonicity assumption on f. Therefore

$$\int_{x\in X} f_m(x)\, d\mu(x) \le \int_{x\in X} \lim_{n\to\infty} f_n(x)\, d\mu(x).$$

The integral of the limit makes sense by Corollary 8.6.4. Taking the supremum over all $m \in \mathbf{N}$ we get

$$\sup_{m\in\mathbf{N}} \int_{x\in X} f_m(x)\, d\mu(x) \le \int_{x\in X} \lim_{n\to\infty} f_n(x)\, d\mu(x).$$

If $m \le n$ then $f_m(x) \le f_n(x)$ for all $x \in X$ by the monotonicity assumption so

$$\int_{x\in X} f_m(x)\, d\mu(x) \le \int_{x\in X} f_n(x)\, d\mu(x)$$

so $\int_{x\in X} f_m(x)\, d\mu(x)$ is a monotone sequence and therefore

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\, d\mu(x) = \sup_{m\in\mathbf{N}} \int_{x\in X} f_m(x)\, d\mu(x).$$

Thus

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\, d\mu(x) \le \int_{x\in X} \lim_{n\to\infty} f_n(x)\, d\mu(x).$$

The name of the variable in the limit is irrelevant, so

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\, d\mu(x) \le \int_{x\in X} \lim_{m\to\infty} f_m(x)\, d\mu(x).$$

Suppose g is a semisimple function such that

$$g(x) \le \lim_{m\to\infty} f_m(x)$$

for all $x \in X$. In other words, there is a countable partition $\mathcal{Q} \subseteq \mathcal{B}$ of X such that

$$\wp([0, +\infty]) \subseteq g^{**}(\mathcal{B}).$$

Then, as we've seen, there is a $\varphi\colon \mathcal{Q} \to [0, +\infty]$ such that $f(x) = \varphi(E)$ when $x \in E$. Suppose $\kappa \in (0, 1)$. Then

$$\lim_{m\to\infty} f_m(x) = \sup_{m\to\infty} f_m(x) \ge g(x) = \varphi(E) > \kappa\varphi(E)$$

for all $x \in E$. The last inequality requires $\varphi(E) \ne 0$, which we'll assume from now until further notice. Define

$$F_{m,E} = \{x \in E\colon f_m(x) > \kappa\varphi(E)\}.$$

Then

$$F_{m,E} \subseteq F_{n,E}$$

whenever $m \le n$ and

$$\bigcup_{m\in\mathbf{N}} F_{m,E} = E.$$

It follows from Theorem 7.6.7 that

$$\lim_{m\to\infty} \mu(F_{m,E}) = \mu(E).$$

Let

$$h_m(x) = \begin{cases} \kappa y & \text{if } x \in F_{m,E}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_m(x) \ge h_m(x)$ for all $x \in X$ so

$$\begin{aligned} \int_{x\in X} f_m(x)\, d\mu(x) &\ge \int_{x\in X} h_m(x)\, d\mu(x). \\ &= \sum_{E\in\mathcal{Q}} \kappa\varphi(E)\mu\left(F_{m,E}\right). \end{aligned}$$

The sum is over all E and we proved the inequality only for those E for which $\varphi(E) \neq 0$, but that's okay since the E's for which $\varphi(E) = 0$ don't contribute to any of the sums or integrals. Now

$$\begin{aligned}\lim_{m\to\infty} \sum_{E\in\mathcal{Q}} \kappa\varphi(E)\mu(F_{m,E}) &= \sum_{E\in\mathcal{Q}} \lim_{m\to\infty} \kappa\varphi(E)\mu(F_{m,E}) \\ &= \sum_{E\in\mathcal{Q}} \kappa\varphi(E)\mu(E) \\ &= \kappa \sum_{E\in\mathcal{Q}} \varphi(E)\mu(E) \\ &= \kappa \int_{x\in X} g(x)\, d\mu(x).\end{aligned}$$

The interchange of the sum and limit is justified by the Monotone Convergence Theorem for sums, Theorem 6.3.1. It follows that

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\, d\mu(x) \geq \kappa \int_{x\in X} g(x)\, d\mu(x)$$

for all $\kappa \in (0,1)$ and hence, taking the limit as κ tends to 1 from below,

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\, d\mu(x) \geq \int_{x\in X} g(x)\, d\mu(x).$$

This is true for all simple g such that

$$g(x) \leq \lim_{m\to\infty} f_m(x)$$

so the limit on the left is greater than or equal to the supremum over all such g, i.e.

$$\begin{aligned}\lim_{m\to\infty} \int_{x\in X} f_m(x)\, d\mu(x) &\geq \underline{\int}_{x\in X} \lim_{m\to\infty} f_m(x)\, d\mu(x). \\ &= \underline{\int}_{x\in X} \lim_{m\to\infty} f_m(x)\, d\mu(x).\end{aligned}$$

We already have the reverse inequality, so

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\, d\mu(x) = \int_{x\in X} \lim_{m\to\infty} f_m(x)\, d\mu(x).$$

□

The Monotone Convergence Theorem for sums applied to nets of functions, not just sequences. Unfortunately the Monotone Convergence Theorem for integrals does not apply to nets in general. To see this assume that there is a measure μ on the Borel σ-algebra on $\mathbf{R}$ such that $\mu(\{x\}) = 0$ for all $x \in \mathbf{R}$ and $\mu([0,1]) = 1$. We will see in the next chapter that there is a measure, the Lebesgue measure, with these properties. Let $\mathcal{D}$ be the set of finite subsets of $[0,1]$. We make this into a directed set by choosing $\subseteq$ as our order relation. Consider the function $f\colon \mathcal{D}\times\mathbf{R}$ defined by

$$f(G,x) = \begin{cases} 1 & \text{if } x \in G, \\ 0 & \text{if } x \notin G. \end{cases}$$

Then $f(G,x)$ is a monotone net in G for each x and a measurable function in x for each G. Also,

$$\int_{x\in\mathbf{R}} f(G,x)\, d\mu(x) = 0$$

for all $G \in \mathcal{D}$ and

$$\lim_{G\in\mathcal{D}} f(G,x) = \begin{cases} 1 & \text{if } x \in [0,1], \\ 0 & \text{if } x \notin [0,1]. \end{cases}$$

It follows that

$$\int_{x\in\mathbf{R}} \lim_{G\in\mathcal{D}} f(G,x)\, d\mu(x) = \mu([0,1]) = 1,$$

so

$$\lim_{G\in\mathcal{D}} \int_{x\in X} f(G,x)\, d\mu(x) \neq \int_{x\in X} \lim_{G\in\mathcal{D}} f(G,x)\, d\mu(x).$$

The following theorem is known as *Fatou's Lemma* for integrals.

Theorem 8.7.2. *Suppose $(X,\mathcal{B},\mu)$ is a measure space and $f\colon \mathbf{N}\times X \to [0,+\infty]$ is a function. Then*

$$\begin{aligned}&\int_{x\in X} \sup_{m\in\mathbf{N}} \inf_{n\geq m} f_n(s)\, d\mu(x) \\ &\qquad \leq \sup_{m\in\mathbf{N}} \inf_{n\geq m} \int_{x\in X} f_n(s)\, d\mu(x).\end{aligned}$$

Proof. Define $g\colon \mathbf{N}\times X \to [0,+\infty]$ by

$$g_m(x) = \inf_{n\geq m} f_n(x).$$

Also, if $m \leq n$ then

$$\{p \in \mathbf{N} \colon p \geq n\} \subseteq \{p \in \mathbf{N} \colon p \geq m\}$$

and so

$$\inf_{p \geq m} f_p(x) \leq \inf_{p \geq n} f_p(x).$$

In other words, if $m \leq n$ then

$$g_m(x) \leq g_n(x).$$

It follows from the Monotone Convergence Theorem that

$$\int_{x \in X} \lim_{m \to \infty} g_m(x)\, d\mu(x) = \lim_{m \to \infty} \int_{x \in X} g_m(x)\, d\mu(x).$$

These are monotone sequences so the limit is the same as the supremum and therefore

$$\int_{x \in X} \sup_{m \in \mathbf{N}} g_m(x)\, d\mu(x) = \sup_{m \in \mathbf{N}} \int_{x \in X} g_m(x)\, d\mu(x).$$

Now if $m \leq n$ then

$$g_m(x) = \inf_{p \geq m} f_p(x) \leq f_n(x)$$

so

$$\int_{x \in X} g_m(x)\, d\mu(x) \leq \int_{x \in X} f_n(x)\, d\mu(x).$$

This holds for all $n \geq m$ so

$$\int_{x \in X} g_m(x)\, d\mu(x) \leq \inf_{n \geq m} \int_{x \in X} f_n(x)\, d\mu(x)$$

and

$$\sup_{m \in \mathbf{N}} \int_{x \in X} g_m(x)\, d\mu(x) \leq \sup_{m \in \mathbf{N}} \inf_{n \geq m} \int_{x \in X} f_n(x)\, d\mu(x).$$

Combining this with the equation

$$\int_{x \in X} \sup_{x \in X} g_m(x)\, d\mu(x) = \sup_{m \in \mathbf{N}} \int_{x \in X} g_m(x)\, d\mu(x)$$

obtained earlier, we find that

$$\int_{x \in X} \sup_{m \in \mathbf{N}} g_m(x)\, d\mu(x) \leq \sup_{m \in \mathbf{N}} \inf_{n \geq m} \int_{x \in X} f_n(x)\, d\mu(x),$$

or, in view of how g was defined,

$$\int_{x \in X} \sup_{m \in \mathbf{N}} \inf_{n \geq m} f_n(x)\, d\mu(x) \leq \sup_{m \in \mathbf{N}} \inf_{n \geq m} \int_{x \in X} f_n(x)\, d\mu(x).$$

□

The following theorem is known as the *Dominated Convergence Theorem* for integrals.

Theorem 8.7.3. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space and $f \colon \mathbf{N} \times X \to \mathbf{R}$ is a function and $g \colon X \to [0, +\infty]$ is a function such that*

$$\lim_{m \to \infty} f_m(x)$$

exists for all $x \in X$,

$$\int_{x \in X} g(x)\, d\mu(x) < +\infty$$

and

$$|f_m(x)| \leq g(x)$$

for all $m \in \mathbf{N}$. Then

$$\lim_{m \to \infty} \int_{x \in X} f_m(x)\, d\mu(x) = \int_{x \in X} \lim_{m \to \infty} f_m(x)\, d\mu(x).$$

Proof. Define

$$h_m(x) = g(x) + f_m(x).$$

Then $h_m(x) \geq 0$ for all $m \in \mathbf{N}$ and $x \in X$. By Fatou's Lemma,

$$\int_{x \in X} \sup_{m \in \mathbf{N}} \inf_{n \geq m} h_n(x)\, d\mu(x) \leq \sup_{a \in \mathbf{N}} \inf_{n \geq m} \int_{x \in X} h_n(x)\, d\mu(x).$$

Now

$$\sup_{m \in \mathbf{N}} \inf_{n \geq m} h_n(x) = g(x) + \sup_{m \in \mathbf{N}} \inf_{n \geq m} f_n(x) = g(x) + \lim_{m \to \infty} f_m(x).$$

Also,

$$\int_{x\in X} h_m(x)\,d\mu(x) = \int_{x\in X} g(x)\,d\mu(x) + \int_{x\in X} f_m(x)\,d\mu(x)$$

so

$$\sup_{m\in\mathbf{N}} \inf_{n\geq m} \int_{x\in X} h_n(x)\,d\mu(x) = \int_{x\in X} g(x)\,d\mu(x) + \sup_{m\in\mathbf{N}} \inf_{n\geq m} \int_{x\in X} f_n(x)\,d\mu(x).$$

Therefore

$$\int_{x\in X} g(x)\,d\mu(x) + \int_{x\in X} \lim_{m\to\infty} f_m(x)\,d\mu(x) \leq \int_{x\in X} g(x)\,d\mu(X) + \sup_{m\in\mathbf{N}} \inf_{n\geq m} \int_{x\in X} f_n(x)\,d\mu(x)$$

Because

$$\int_{x\in X} g(x)\,d\mu(x) < +\infty$$

we can conclude that

$$\int_{x\in X} \lim_{m\to\infty} f_m(x)\,d\mu(x) \leq \sup_{m\in\mathbf{N}} \inf_{n\geq m} \int_{x\in X} f_n(x)\,d\mu(x)$$

We can apply the same argument with $-f_m(x)$ in place of $f_m(x)$ to get

$$\int_{x\in X} \lim_{m\to\infty} -f_m(x)\,d\mu(x) \leq \sup_{m\in\mathbf{N}} \inf_{n\geq m} \int_{x\in X} -f_m(x)\,d\mu(x),$$

or, equivalently,

$$\inf_{m\in\mathbf{N}} \sup_{n\geq m} \int_{x\in X} f_m(x)\,d\mu(x) \leq \int_{x\in X} \lim_{m\to\infty} f_m(x).$$

It follows that

$$\sup_{m\in\mathbf{N}} \inf_{n\geq m} \int_{x\in X} f_n(x)\,d\mu(x) \leq \inf_{m\in\mathbf{N}} \sup_{n\geq m} \int_{x\in X} f_n(x)\,d\mu(x)$$

and therefore

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\,d\mu(x)$$

exists and is equal to their common value. So

$$\lim_{m\to\infty} \int_{x\in X} f_m(x)\,d\mu(x) = \int_{x\in X} \lim_{m\to\infty} f_m(x)\,d\mu(x).$$

□

9 Constructing measures

9.1 Semicontinuity

Definition 9.1.1. Suppose $(X, \mathcal{T})$ is a topological space. A function $f\colon X \to \mathbf{R}$ is called *lower semicontinuous* if $f^*((a, +\infty)) \in \mathcal{T}$ for all $a \in \mathbf{R}$ and is called *upper semicontinuous* if $f^*((-\infty, b)) \in \mathcal{T}$ for all $b \in \mathbf{R}$.

Proposition 9.1.2. *Suppose $(X, \mathcal{T})$ is a topological space. $f\colon X \to \mathbf{R}$ is continuous if and only if it is both lower and upper semicontinuous.*

Proof. Suppose f is continuous. $(a, +\infty)$ and $(-\infty, b)$ are open so $f^*((a, +\infty))$ and $f^*((-\infty, b))$ are open. Therefore f is both lower and upper semicontinuous.

Suppose, conversely, that f is both lower and upper semicontinuous. Then $f^*((a, +\infty))$ and $f^*((-\infty, b))$ are open for all a and b, and therefore

$$\begin{aligned} f^*((a, b) &= f^*((a, +\infty) \cap (-\infty, b)) \\ &= f^*((a, +\infty)) \cap f^*((-\infty, b)) \end{aligned}$$

is open. Every open set is a union of open intervals and the preimage of a union is the union of the preimages, so the preimage of every open set is open. Therefore f is continuous. □

Proposition 9.1.3. *Suppose $(X, \mathcal{T})$ is a topological space, $E \in \wp(X)$ and χ_E is the characteristic function of E, i.e. $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. Then χ_E is lower semicontinuous if and only if E is open and χ_E is upper semicontinuous if and only if E is closed.*

Proof. Suppose χ_E is lower semicontinuous. Characteristic functions only take the values 0 and 1 so

$$E = \chi_E^*(\{1\}) = E = \chi_E^*((1/2, +\infty))$$

is open. Suppose, conversely, that E is open. Then

$$\chi_E^*((a, +\infty)) = \begin{cases} \varnothing & \text{if } a \geq 1. \\ E & \text{if } 0 \leq a < 1, \\ X & \text{if } a < 0 \end{cases}$$

In either case $\chi_E^*((a, +\infty))$ is open.

Suppose χ_E is upper semicontinuous. Characteristic functions only take the values 0 and 1 so

$$X \setminus E = \chi_E^*(\{0\}) = E = \chi_E^*((-\infty, 1/2))$$

is open and hence E is closed. Suppose, conversely, that E is closed. Then

$$\chi_E^*((-\infty, b)) = \begin{cases} \varnothing & \text{if } b \leq 0, \\ X \setminus E & \text{if } 0 < b \leq 1, \\ X & \text{if } b > 1. \end{cases}$$

In either case $\chi_E^*((-\infty, b))$ is open. ☐

Proposition 9.1.4. *Suppose $(X, \mathcal{T})$ is a topological space, $f_1, \ldots, f_m \colon X \to \mathbf{R}$ are functions and $g \colon X \to \mathbf{R}$ is defined by $g = \sum_{i=1}^m c_i f_i$. If $f_1, \ldots, f_m$ are lower semicontinuous then so is g. If $f_1, \ldots, f_m$ are upper semicontinuous then so is g.*

Proof. $g(x) > a$ if and only if there are $\alpha_1, \ldots, \alpha_m$ such that $f_i(x) > \alpha_i$ for each i and $\sum_{i=1}^m c_i \alpha_i = a$. In other words,

$$g^*((a, +\infty)) = \bigcup \bigcap_{i=1}^m f_i^*((\alpha_i, +\infty)),$$

where the union is over α's such that $\sum_{i=1}^m c_i \alpha_i = a$. If each f_i is lower semicontinuous then $f_i^*((\alpha_i, +\infty))$ is open. The intersection of finitely many open sets is open and the union of arbitrarily many open sets is open so $g^*((a, +\infty))$ is open. In other words, g is lower semicontinuous. The proof for upper semicontinuity is similar, except we use the identity

$$g^*((-\infty, b)) = \bigcup \bigcap_{i=1}^m f_i^*((-\infty, \beta_i)),$$

where the union is over β's such that $\sum_{i=1}^m c_i \beta_i = b$. ☐

9.2 The Riesz Representation Theorem, compact case

Proposition 9.2.1. *Suppose $(X, \mathcal{T})$ is a compact Hausdorff space. Suppose I is a linear transformation from the vector space of continuous functions from X to $\mathbf{R}$ such that $I(g) \geq 0$ whenever g is such that $g(x) \geq 0$ for all $x \in X$. Then there is a Radon measure μ on X such that*

$$I(g) = \int_{x \in X} g(x) \, d\mu(x)$$

for all continuous g.

Proof. This proof is very long, so it may be helpful to start with an overview of its structure.

(a) We prove a monotonicity property for I: If $f(x) \leq g(x)$ for all $x \in X$ then $I(f) \leq I(g)$.

(b) We define a function J from the set of bounded non-negative lower semicontinuous functions by $J(g) = \sup I(f)$, where the supremum is over all continuous functions f such that $0 \leq f(x) \leq g(x)$ for all $x \in X$.

(c) We show that $0 \leq J(g) < +\infty$ for all such g.

(d) We show that when $I(g)$ and $J(g)$ are defined, i.e. when g is continuous and non-negative, $I(g) = J(g)$.

(e) We prove a monotonicity property for J: If $g(x) \leq h(x)$ for all $x \in X$ then $J(g) \leq J(h)$.

(f) We prove that J is homogeneous in the sense that if $c \geq 0$ and g is bounded, non-negative and lower semicontinuous then $J(cg) = cJ(g)$.

(g) We prove that J is finitely superadditive in the sense that if $g_0, \ldots, g_m$ are bounded, non-negative and lower semicontinuous then

$$J\left(\sum_{k=0}^{m} g_k\right) \geq \sum_{k=0}^{m} J(g_k).$$

(h) We prove that J is countably subadditive in the sense that if $g_0, g_1, \ldots,$ are bounded, non-negative and lower semicontinuous then

$$J\left(\sum_{k=0}^{\infty} g_k\right) \leq \sum_{k=0}^{\infty} J(g_k).$$

This is in fact a special case of a slightly more general statement. If $g_0, g_1, \ldots,$ are bounded, non-negative and lower semicontinuous and

$$f(x) \leq \sum_{k=0}^{\infty} g_k(x)$$

for all $x \in X$, where f is also bounded, non-negative and lower semicontinuous, then

$$J(f) \leq \sum_{k=0}^{\infty} J(g_k).$$

(i) We define a function ν on the set of open subsets of X by

$$\nu(E) = J(\chi_E)$$

and a function ν on the set of closed subsets of X by

$$\nu(E) = J(\chi_X) - J(\chi_{X\setminus E})$$

and show that if E is both open and closed then the two definitions agree.

(j) We prove a monotonicity property of ν: If $V \subseteq U$ and $\nu(V)$ and $\nu(U)$ are both defined, i.e. if each of U and V is open or closed, then $\nu(V) \leq \nu(U)$.

(k) We define functions μ^- and μ^+ on $\wp(X)$ by

$$\mu^-(E) = \sup \nu(V)$$

and

$$\mu^+(E) = \inf \nu(U),$$

where the supremum is over closed V such that $V \subseteq E$ and the infimum is over open U such that $E \subseteq U$.

(l) We define $\mathcal{B}$ to be the set of $E \in \wp(X)$ such that $\mu^+(E) \leq \mu^-(E)$, which then implies $\mu^+(E) = \mu^-(E)$.

(m) We define a function μ on $\mathcal{B}$ by $\mu(E) = \mu^+(E) = \mu^-(E)$.

(n) We show that $\mathcal{B}$ is a σ-algebra on X and μ is a measure on $(X, \mathcal{B})$.

(o) We show that if E is closed then $E \in \mathcal{B}$ and $\mu(E) = \nu(E)$.

(p) We show that if E is open then $E \in \mathcal{B}$ and $\mu(E) = \nu(E)$.

(q) We show that $\mathcal{B}$ is a superset of the Borel σ-algebra.

(r) We show that if f is bounded, positive and lower semicontinuous then

$$\int_{x\in X} f(x)\, d\mu(x) = J(f).$$

(s) We show that if f is continuous then

$$\int_{x\in X} f(x)\, d\mu(x) = I(f).$$

Suppose that f and g are continuous functions from X to $\mathbf{R}$ such that

$$f(x) \leq g(x)$$

for all $x \in X$. Let $h = g - f$. Then

$$I(g) = I(f) + I(h)$$

and $h(x) \geq 0$ for all $x \in X$ so

$$I(h) \geq 0$$

and hence

$$I(f) \leq I(g).$$

So if $f(x) \leq g(x)$ for all $x \in X$ then $I(f) \leq I(g)$. This will be referred to below as the monotonicity property of I.

For each bounded non-negative lower semicontinuous function g on X we define $J(g)$ by

$$J(g) = \sup I(f),$$

where the supremum is over all continuous f such that $0 \leq f(x) \leq g(x)$ for all $x \in X$. Note that if $0 \leq f(x) \leq g(x)$ for all $x \in X$ then $0 \leq f(x) \leq h(x)$ for all $x \in X$, where $h(x) = \sup_{w \in X} g(w)$ so $0 = I(0) \leq I(f) \leq I(h)$. This holds for all f satisfying the conditions above so $0 \leq J(g) \leq I(h)$. In particular, $J(g)$ is finite and non-negative.

If g is continuous and non-negative then $f = g$ belongs to this set so $J(g) \geq I(g)$. On the other hand, for any continuous f such that $f(x) \leq g(x)$ for all x we have $I(f) \leq I(g)$ by the monotonicity property of I and hence $\sup I(f) \leq I(g)$. In other words, $J(g) \leq I(g)$. Since we already have the reverse inequality we conclude that

$$I(g) = J(g)$$

whenever both the left and right hand sides are defined, i.e. whenever g is non-negative and continuous.

If g and h are bounded non-negative lower semicontinuous functions on X and

$$g(x) \leq h(x)$$

for all $x \in X$ then any continuous f such that $0 \leq f(x) \leq g(x)$ for all $x \in X$ also satisfies $0 \leq f(x) \leq h(x)$ for all $x \in X$. The supremum over a larger set is at least as large as the supremum over a smaller set so

$$J(g) \leq J(h).$$

So if $g(x) \leq h(x)$ for all $x \in X$ then $J(g) \leq J(h)$. We'll refer to this as the monotonicity property of J.

Suppose that $c > 0$ and g is bounded, non-negative and lower semicontinuous. Then

$$J(cg) = \sup I(f)$$

where the supremum is over continuous f such that $0 \leq f(x) \leq cg(x)$ for all $x \in X$. For any such f we have $0 \leq c^{-1}f(x) \leq g(x)$ so $c^{-1}f$ is one of the functions appearing in the definition of $J(g)$ and hence

$$I(c^{-1}f) \leq J(g).$$

But

$$I(f) = I(cc^{-1}f) = cI(c^{-1}f)$$

by the linearity of I so

$$I(f) \leq cJ(g).$$

This holds for all continuous f such that $0 \leq f(x) \leq cg(x)$ for all $x \in X$, so it holds for their supremum, so

$$J(cg) \leq cJ(g).$$

The same argument works with c^{-1} in place of c and cg in place of g so

$$J(g) \leq c^{-1}J(cg)$$

and hence

$$cJ(g) \leq J(cg).$$

Since we already have the reverse inequality we conclude that

$$J(cg) = cJ(g)$$

for all $c > 0$ and all bounded non-negative lower semicontinuous functions g. In fact this applies for all $c \geq 0$, since the case $c = 0$ follows immediately from $J(0) = 0$.

If $g_0, \ldots, g_m$ are bounded non-negative lower semicontinuous functions then

$$J\left(\sum_{k=0}^{m} g_k\right) \geq \sum_{k=0}^{m} J(g_k).$$

To prove this we note that if $0 \leq f_k(x) \leq g_k(x)$ for each $x \in X$ and each k then $0 \leq f(x) \leq \sum_{k=1}^{m} g_k(x)$ for all $x \in X$, where $f = \sum_{k=1}^{m} f_k$. This f therefore belongs to the class over which we take the supremum of $I(f)$ to get $J\left(\sum_{k=0}^m g_k\right)$ so

$$I(f) \leq J\left(\sum_{k=0}^m g_k\right).$$

But

$$I(f) = I\left(\sum_{k=0}^m f_k\right) = \sum_{k=0}^m I(f_k)$$

by the linearity of I. So

$$\sum_{k=0}^m I(f_k) \leq J\left(\sum_{k=0}^m g_k\right).$$

Taking the supremum over all allowed $f_0, \ldots, f_m$ gives

$$\sum_{k=0}^m J(g_k) \leq J\left(\sum_{k=0}^m g_k\right).$$

We'll now prove the reverse inequality, but this is harder.

Suppose g is bounded, non-negative and lower semicontinuous, h_i is bounded, non-negative and lower semicontinuous for each $i \in \mathbf{N}$, and

$$g(x) \leq \sum_{i=0}^{\infty} h_i(x)$$

for all $x \in X$. Suppose $k \geq 1$. Suppose f is continuous and $0 \leq f(x) \leq g(x)$ for all $x \in X$. f is continuous on X and $\frac{1}{2^{2k}} > 0$ so for each $x \in X$ the set

$$U_x = \left\{y \in X \colon |f(x) - f(y)| < \frac{1}{2^{2k}}\right\}$$

is open. It contains x and so is a neighbourhood of x. By the definition of convergence there is an $n_x \in \mathbf{N}$ such that if $m \geq n_x$ then

$$\left|\sum_{i=0}^{\infty} h_i(x) - \sum_{i=0}^{m} h_i(x)\right| < \frac{1}{2^{2k}}.$$

In that case

$$\sum_{i=0}^{\infty} h_i(x) - \sum_{i=0}^{m} h_i(x) < \frac{1}{2^{2k}}$$

and so

$$\sum_{i=0}^{m} h_i(x) > g(x) - \frac{1}{2^{2k}}.$$

For each $j \leq n_x$ the function h_j is lower semicontinuous so the set

$$V_{x,j} = h_j^*\left(\left(h_j(x) - \frac{1}{2^{j+2k}}, +\infty\right)\right)$$

is open. It contains x so is an open neighbourhood of x. Therefore

$$W_x = U_x \cap \bigcap_{0 \leq j \leq n_x} V_{x,j}$$

is an open neighbourhood of x. So if $y \in W_x$ then

$$|f(x) - f(y)| < \frac{1}{2^{2k}}$$

and

$$h_j(y) > h_j(x) - \frac{1}{2^{j+2k}}$$

for $j \leq n_x$. X was assumed to be compact and Hausdorff, so is X is normal and there is an open neighbourhood Z_x of X such that

$$\overline{Z_x} \subseteq W_x.$$

Again because X was assumed to be compact, there is a finite set $x_1, \ldots, x_l$ such that

$$X = \bigcup_{i=1}^{l} Z_{x_i}.$$

By Proposition 3.13.4 there are continuous functions $\psi_1, \ldots, \psi_l$ from X to X to $[0, 1]$ such that $\psi_i(x) = 0$ if $x \notin Z_{x_i}$ and

$$\sum_{i=1}^{l} \psi_i(x) = 1$$

for all $x \in X$.

Suppose x is such that

$$f(x) \geq \frac{1}{2^k}.$$

Let

$$S_x = \{j \colon x \in Z_{x_j}\}.$$

Then

$$f(x) = 1f(x) = \sum_{j=1}^{m} \psi_j(x) f(x) \sum_{j \in S_x} \psi_j(x) f(x)$$

since the other summands are all zero. We have

$$|f(x) - f(x_j)| < \frac{1}{2^{2k}}$$

From this and $f(x) \geq \frac{1}{2^{2k}}$ is follows that

$$f(x_j) > \frac{1}{2^{k+1}}$$

and hence

$$g(x_j) > \frac{1}{2^{k+1}}.$$

Then

$$\sum_{i=0}^{n_{x_j}} h_i(x_j) \geq g(x_j) - \frac{1}{2^{2k}}$$

so

$$\sum_{i=0}^{n_{x_j}} \frac{h_i(x_j)}{g(x_j)} \geq \chi_X - \frac{1}{2^{k-1}}.$$

From this and

$$f(x) = \sum_{j \in S_x} \psi_j(x) f(x)$$

we get

$$\begin{aligned} f(x) &\leq \sum_{j \in S_x} \sum_{i=0}^{n_{x_j}} \frac{h_i(x_j)}{g(x_j)} \psi_j(x) f(x_j) + \frac{\sup f}{2^{k-1}}. \\ &= \sum_{i=0}^{\infty} \sum_{j \in T_{i,x}} \frac{h_i(x_j)}{g(x_j)} \psi_j(x) f(x_j) + \frac{\sup f}{2^{k-1}} \\ &= \sum_{i=0}^{\infty} \varphi_i(x) + \frac{\sup f}{2^{k-1}}, \end{aligned}$$

where

$$T_{i,x} = \{j \colon j \in S_x, i \leq n_{x_j}\}$$

and

$$\varphi_i(x) = \sum_{j \in T_{i,x}} \frac{h_i(x_j)}{g(x_j)} \psi_j(x) f(x_j).$$

If x is such that

$$f(x) \leq \frac{1}{2^k}$$

then

$$f(x) \leq \frac{1/2}{2^{k-1}} \leq \sum_{i=0}^{\infty} \varphi_i(x) + \frac{1/2}{2^{k-1}},$$

so for any $x \in X$ we have

$$f(x) \leq \frac{1/2}{2^{k-1}} \leq \sum_{i=0}^{\infty} \varphi_i(x) + \frac{C}{2^{k-1}},$$

where $C = \max(\sup f, 1/2)$.

$T_{i,x} = \varnothing$ if $i > \max_{1 \leq j \leq l} n_{x_j}$ so the sum over i is, despite appearances, finite. From the linearity of I we therefore have

$$I\left(\sum_{i=0}^{\infty} \varphi_i + \frac{C}{2^{k-1}}\right) = \sum_{i=0}^{\infty} I(\varphi_i) + \frac{C}{2^{k-1}} I(\chi_X).$$

From the monotonicity property of I it follows that

$$I(f) \leq \sum_{i=0}^{\infty} I(\varphi_i) + \frac{C}{2^{k-1}} I(\chi_X).$$

Now $f(x_j) \leq g(x_j)$ so

$$\varphi_i(x) \leq \sum_{j \in T_{i,x}} h_i(x_j) \psi_j(x) \leq \sum_{j=1}^{l} h_i(x_j) \psi_j(x).$$

Now

$$h_i(x_j) \leq h_i(x) + \frac{1}{2^{i+2k}}$$

so

$$\varphi_i(x) \leq h_i(x) + \frac{1}{2^{i+2k}}.$$

Using the linearity and monotonicity properties of I again,

$$I(\varphi_i) \leq I(h_i) + \frac{I(\chi_X)}{2^{i+2k}}.$$

Now h_i is non-negative and continuous so

$$I(h_i) = J(h_i).$$

Combining what we have so far,

$$\begin{aligned} I(f) &\leq \sum_{i=0}^{\infty} J(h_i) + \sum_{i=0}^{\infty} \frac{I(\chi_X)}{2^{i+2k}} + \frac{C}{2^{k-1}} I(\chi_X) \\ &= \sum_{i=0}^{\infty} J(h_i) + \frac{I(\chi_X)}{2^{2k-1}} + \frac{C}{2^{k-1}} I(\chi_X). \end{aligned}$$

This holds for all continuous f such that $0 \leq f(x) \leq g(x)$ for all $x \in X$ so, by the definition of J,

$$J(g) \leq \sum_{i=0}^{\infty} J(h_i) + \frac{I(\chi_X)}{2^{2k-1}} + \frac{C}{2^{k-1}} I(\chi_X).$$

This, in turn, holds for all $k \geq 1$ so

$$J(g) \leq \sum_{i=0}^{\infty} J(h_i).$$

This was proved under the assumption that $g(x) \leq \sum_{i=0}^{\infty} h_i(x)$ for all $x \in X$. In particular it holds if $h_0 = 0$ and $h_i = 0$ for $i > m$, in which case it gives

$$J\left(\sum_{i=1}^{m} h_i\right) \leq \sum_{i=1}^{m} J(h_i).$$

We already proved the reverse inequality, so

$$J\left(\sum_{i=1}^{m} h_i\right) = \sum_{i=1}^{m} J(h_i).$$

J is therefore finitely additive.

If $U \in \wp(X)$ is both closed and open then χ_U is upper and lower semicontinuous by Proposition 9.1.3 and so is continuous by Proposition 9.1.2. Therefore $\chi_X - \chi_U = \chi_{X \setminus U}$ is also continuous. Using Proposition 9.1.3 we see that it's lower semicontinuous. By the finite additivity property we just proved we therefore have

$$J(\chi_U) + J(\chi_{X \setminus U}) = J(\chi_U + \chi_{X \setminus U}) = J(\chi_X).$$

Define $\nu(U)$ for open $U \in \wp(X)$ by

$$\nu(U) = J(\chi_U)$$

and define $\nu(U)$ for closed $U \in \wp(X)$ by

$$\nu(U) = J(\chi_X) - J(\chi_{X \setminus U}).$$

This is consistent because we've just seen that if U is both open and closed then

$$J(\chi_U) = J(\chi_X) - J(\chi_{X \setminus U}).$$

Suppose

$$V \subseteq U.$$

Then

$$\nu(V) \leq \nu(U).$$

If U and V are both open then $\chi_V(x) \leq \chi_U(x)$ for all x so this follows from the monotonicity property of J. If U and V are both closed then $V \subseteq U$ implies $X \setminus U \subseteq X \setminus V$ so $\chi_{X \setminus U}(x) \leq \chi_{X \setminus V}(x)$ for all x and hence $J(\chi_{X \setminus U}) \leq J(\chi_{X \setminus V})$ and

$$J(\chi_X) - J(\chi_{X \setminus V}) \leq J(\chi_X) - J(\chi_{X \setminus U}),$$

i.e. $\nu(V) \leq \nu(U)$. If V is open and U is closed then $V \subseteq U$ implies

$$V \cap (X \setminus U) = \varnothing$$

so

$$\chi_V(x) + \chi_{X \setminus U}(x) \leq 1$$

for all $x \in X$ and so, by the finite additivity and monotonicity of J,

$$J(\chi_V) + J(\chi_{X \setminus U}) \leq J(\chi_X)$$

or

$$J(\chi_V) \leq J(\chi_X) - J(\chi_{X \setminus U})$$

or $\nu(V) \leq \nu(U)$. If V is closed and U is open then $V \subseteq U$ implies

$$\chi_U(x) + \chi_{X \setminus V}(x) = \chi_X(x) + \chi_{U \setminus V}(x).$$

Using the finite additivity of J we get

$$J(\chi_U) + J(\chi_{X \setminus V}) = J(\chi_X) + J(\chi_{U \setminus V})$$

so

$$J(\chi_U) = J(\chi_X) - J(\chi_{X \setminus V}) + J(\chi_{U \setminus V})$$

or

$$\nu(U) = \nu(V) + \nu(U \setminus V).$$

$\nu(U \setminus V) \geq 0$ so again $\nu(V) \leq \nu(U)$. In all cases we therefore have

$$\nu(V) \leq \nu(U)$$

if $V \subseteq U$. In particular,

$$0 = J(\varnothing) = \nu(\varnothing) \leq \nu(U) \leq \nu(X) = J(X) < +\infty$$

for all open or closed U.

Define $\mu^- \colon \wp(X) \to [0,+\infty]$ and $\mu^+ \colon \wp(X) \to [0,+\infty]$ by

$$\mu^-(E) = \sup \nu(V)$$

where the supremum is over all closed V such that $V \subseteq E$ and

$$\mu^+(E) = \inf \nu(U)$$

where the infimum is over all open U such that $E \subseteq U$. From $\varnothing \subseteq E \subseteq X$ it follows that

$$0 \le \mu^-(E) \le \mu^+(E) \le J(\chi_X)$$

for all $E \in \wp(X)$. Let $\mathcal{B}$ be the set of all $E \in \wp(X)$ such that

$$\mu^+(E) \le \mu^-(E).$$

For such E we define μ to be the common value of $\mu^-(E)$ and $\mu^+(E)$.

$\mathcal{B}$ is a σ-algebra and μ is a measure. We see this as follows. $\nu(\varnothing) = J(\varnothing) = 0$, $\varnothing \subseteq \varnothing$ and $\varnothing$ is open so $\mu^+(\varnothing) \le 0$. But $0 \le \mu^-(\varnothing) \le \mu^+(\varnothing)$ so $0 = \mu^-(\varnothing) = \mu^+(\varnothing)$. Therefore $\varnothing \in \mathcal{B}$ and $\mu(\varnothing) = 0$.

Suppose $E \in \mathcal{B}$. If V is closed and $V \subseteq E$ then $U = X \setminus V$ is open and $U \subseteq X \setminus E$. Conversely, if $U = X \setminus V$ is open and $U \subseteq X \setminus E$ then V is closed and $V \subseteq E$. So

$$\begin{aligned}\mu^-(E) &= \sup \nu(U) = \sup(\nu(X) - \nu(X \setminus U)) \\ &= \nu(X) - \inf \nu(V) = \mu(X) - \mu^+(X \setminus E).\end{aligned}$$

The same argument applied to $X \setminus E$ rather than E gives

$$\mu^-(X \setminus E) = \mu^+(X \setminus (X \setminus E)) = \mu^+(E).$$

But $E \in \mathcal{B}$ so $\mu^-(E) = \mu^+(E)$ and therefore

$$\mu^+(X \setminus E) = \mu^-(X \setminus E).$$

Then $X \setminus E \in \mathcal{B}$. So if $E \in \mathcal{B}$ then $X \setminus E \in \mathcal{B}$.

Suppose $E_0, E_1, \ldots \in \wp(X)$ and let

$$E = \bigcup_{i=0}^{\infty} E_i.$$

Then for each $i \in \mathbf{N}$ we have $E_i \in \mathcal{B}$ and

$$\mu^+(E_i) + \frac{\epsilon}{2^{i+2}} > \mu^+(E) = \inf \nu(U)$$

where the infimum is over open U such that $E_i \subseteq U$ so there is an open U_i such that $E_i \subseteq U_i$ and

$$\nu(U_i) < \mu^+(E_i) + \frac{\epsilon}{2^{i+2}}.$$

Similarly,

$$\mu^-(E_i) - \frac{\epsilon}{2^{i+2}} < \mu^-(E) = \sup \nu(V)$$

where the infimum is over closed V such that $V \subseteq U_i$ so there is a closed V_i such that $V_i \subseteq E_i$ and

$$\nu(V_i) > \mu^-(E_i) - \frac{\epsilon}{2^{i+2}}.$$

So

$$V_i \subseteq E_i \subseteq U_i$$

and

$$\nu(U_i) < \mu^+(E_i) + \frac{\epsilon}{2^{i+2}}.$$

We showed earlier that if g, h_i are non-negative, bounded and lower semicontinuous

$$g(x) \le \sum_{i=0}^{\infty} h_i(x)$$

for all x then

$$J(g) \le \sum_{i=0}^{\infty} J(h_i).$$

We apply this to $g\chi_U$ and $h_i = \chi_{U_i}$, where

$$U = \bigcup_{i=0}^{\infty} U_i.$$

This gives

$$\nu(U) = J(\chi_U) \le \sum_{i=0}^{\infty} J(\chi_{U_i}) = \sum_{i=0}^{\infty} \nu(U_i).$$

So

$$\nu(U) \le \sum_{i=0}^{\infty} \left(\mu^+(E_i) + \frac{\epsilon}{2^{i+2}}\right) = \sum_{i=0}^{\infty} \mu^+(E_i) + \frac{\epsilon}{2}.$$

But $E \subseteq U$ and U is open so

$$\mu^+(E) \le \nu(U).$$

Therefore

$$\mu^+(E) \leq \sum_{i=0}^{\infty} \mu^+(E_i) + \frac{\epsilon}{2}.$$

This holds for all $\epsilon > 0$ so

$$\mu^+(E) \leq \sum_{i=0}^{\infty} \mu^+(E_i).$$

Suppose now that $E_i \cap E_j = \varnothing$ if $i \neq j$. Then $V_i \cap V_j = \varnothing$ if $i \neq j$. Let

$$W_j = \bigcup_{i<j} V_i.$$

Then V_j and W_j is closed and $V_j \cap W_j = \varnothing$ so

$$\nu(V_j \cup W_j) = \nu(V_j) + \nu(W_j).$$

From this and

$$\nu(W_0) + \nu(\varnothing) = 0$$

it follows by induction that

$$\nu(W_j) = \sum_i < j\nu(V_i).$$

Now $\nu(W_j) \leq \nu(X) = J(\chi_X)$ so

$$\sum_i < j\nu(V_i) \leq J(\chi_X).$$

These partial sums form a bounded monotone sequence and so converge. There is therefore an n such that

$$\sum_{i<n} \nu(V_i) > \sum_{i=0}^{\infty} \nu(V_i) - \frac{\epsilon}{2}.$$

Let

$$V = W_n = \bigcap_{i<n} V_i.$$

Now

$$\begin{aligned}\nu(V) &= \sum_{i<n} \nu(V_i) \geq \sum_{i=0}^{\infty} \nu(V_i) - \frac{\epsilon}{2} \\ &\geq \sum_{i=0}^{\infty} \left(\mu^-(E_i) - \frac{\epsilon}{2^i}\right) - \frac{\epsilon}{2} \geq \sum_{i=0}^{\infty} \mu^-(E_i) - \epsilon.\end{aligned}$$

Also V is closed and $V \subseteq E$ so

$$\mu^-(E) \geq \nu(V)$$

and therefore

$$\mu^-(E) \geq \sum_{i=0}^{\infty} \mu^-(E_i) - \epsilon.$$

This holds for all $\epsilon > 0$ so

$$\mu^-(E) \geq \sum_{i=0}^{\infty} \mu^-(E_i).$$

If $E_i \in \mathcal{B}$ for each i then

$$\mu^-(E_i) = \mu(E_i) = \mu^+(E_i)$$

so

$$\mu^+(E) \leq \sum_{i=0}^{\infty} \mu(E_i) \leq \mu^-(E).$$

Therefore $E \in \mathcal{B}$ and

$$\mu(E) = \sum_{i=0}^{\infty} \mu(E_i).$$

So if $E_0, E_1, \ldots \in \mathcal{B}$ are disjoint then $E = \bigcup_{i=0}^{\infty} E_i \in \mathcal{B}$ and

$$\mu(E) = \sum_{i=0}^{\infty} \mu(E_i).$$

Suppose E is a closed subset of X. Let $\epsilon > 0$ and $\kappa \in (0, 1)$. Then

$$\nu(E) + \epsilon > \nu(E) = J(\chi_X) - J(\chi_{X \setminus E})$$

so

$$J(\chi_{X \setminus E}) > J(\chi_X) - \nu(E) - \epsilon.$$

Now

$$J(\chi_{X \setminus E}) = \sup I(f)$$

where the supremum is over all continuous f such that $0 \leq f(x) \leq \chi_{X \setminus E}(x)$. By the definition of the supremum there is such an f such that

$$I(f) > J(\chi_X) - \nu(E) - \epsilon.$$

Let $g = \chi_X - f$. Then

$$I(f) = I(\chi_X) - I(f) = J(\chi_X) - I(f) < \nu(E) + \epsilon$$

and

$$\chi_E(x) \le g(x) \le \chi_X(x)$$

for all $x \in X$. Define h by

$$h(x) = \begin{cases} 0 & \text{if } g(x) \le \kappa, \\ \frac{g(x)-\kappa}{1-\kappa} & \text{if } g(x) \ge \kappa. \end{cases}$$

Then h is continuous and non-negative. For all x we have $h(x) \le g(x)$ for all x, so

$$I(h) \le I(g)$$

by the monotonicity of I, so

$$I(h) \le \nu(E) + \epsilon.$$

Let

$$U = \{x \in X : h(x) > 0\} = \{x \in X : g(x) > \kappa\}$$

If $x \in U$ then $h(x)/\kappa \ge 1 = \chi_U(x)$. If $x \notin U$ then $h(x)/\kappa = 0 = \chi_U(x)$. So

$$\chi_U(x) \le h(x)/\kappa$$

for all $x \in X$ then therefore

$$I(\chi_U) \le I(h/\kappa) = I(h)/\kappa.$$

So

$$\nu(U) = J(\chi_U) = I(\chi_U) \le I(h)/\kappa \le \frac{\nu(E)+\epsilon}{\kappa}.$$

U is an open set such that $E \subseteq U$ so

$$\mu^+(E) \le \nu(E) \le \frac{\nu(E)+\epsilon}{\kappa}.$$

This holds for all $\epsilon > 0$ and all $\kappa \in (0,1)$ so

$$\mu^+(E) \le \nu(E).$$

$\mu^-(E)$ is the supremum of all closed sets such that $V \subseteq E$. E is closed so E itself is such a V. Therefore

$$\nu(E) \le \mu^-(E).$$

Therefore

$$\mu^+(E) \le \nu(E) \le \mu^-(E),$$

from which it follows that $E \in \mathcal{B}$ and $\mu(E) = \nu(E)$.

If E is open then $X \setminus E$ is open so $X \setminus E \in \mathcal{B}$. $\mathcal{B}$ is a σ-algebra so $E = X \setminus (X \setminus E) \in \mathcal{B}$. Also

$$\mu(E) = \mu(X) - \mu(X \setminus E) = \nu(X) - \nu(X \setminus E) = \nu(E).$$

So $\mu(E) = \nu(E)$ for all E for which $\nu(E)$ is defined.

Now $\mathcal{B}$ is a σ-algebra and contains every open set. The Borel σ-algebra is the smallest σ-algebra which contains the open sets, so $\mathcal{B}$ is a superset of the Borel σ-algebra.

If E in $\mathcal{B}$ then

$$\mu(E) = \mu^-(E) = \sup \nu(V) = \sup \mu(V),$$

where the supremum is over all closed V such that $V \subseteq E$. X is compact so closed subsets and compact sets are the same. So

$$\mu(E) = \sup \mu(V)$$

where the supremum is over all compact sets such that $V \subseteq E$. Also,

$$\mu(E) = \mu^+(E) = \inf \nu(U) = \inf \mu(U),$$

where the infimum is over all open U such that $E \subseteq U$. μ is therefore a Radon measure.

Suppose g is bounded, positive and lower semicontinuous. For any $\epsilon > 0$, let

$$\delta = \epsilon/\mu(X).$$

Define

$$h(x) = k\delta,$$

where k is the least integer such that

$$g(x) \le k\delta.$$

Then

$$0 \le g(x) \le h(x)$$

for all $x \in X$.

If $j = k - 1$ then $j < k$, so $g(x) > j\delta$, since otherwise k would not be the greatest integer with $g(x) \leq k\delta$, and therefore

$$h(x) = k\delta = j\delta + \delta < g(x) + \delta.$$

So

$$0 \leq g(x) \leq h(x) < g(x) + \delta.$$

g is bounded so h is bounded.

Suppose $a \in \mathbf{R}$. Then there is a greatest integer i such that $a \geq i\delta$. So

$$i\delta \leq a < (i+1)\delta.$$

Suppose

$$x \in h^*(a, +\infty).$$

Then

$$h(x) > a \geq i\delta.$$

Now $h(x)$ is an integer multiple of δ so if $h(x) > i\delta$ then $h(x) \geq (i+1)\delta$. Then

$$g(x) > h(x) - \delta > i\delta.$$

So

$$x \in g^*((i\delta, +\infty)).$$

Suppose, conversely, that

$$x \in g^*((i\delta, +\infty).$$

Then $g(x) > i\delta$ so $i < k$, $i + 1 \leq k$, and

$$a < (i+1)\delta \leq k\delta = h(x).$$

So $x \in h^*((a + \infty))$. We have $x \in h^*((a + \infty))$ if and only if $x \in g^*((i\delta, +\infty)$, so

$$h^*((a, +\infty)) = g^*((i\delta, +\infty)).$$

g is lower semicontinuous so the right hand side is an open set for all $a \in \mathbf{R}$ and therefore the left hand side is an open for all $a \in \mathbf{R}$. In other words, h is lower semicontinuous.

Define f by

$$f(x) = h(x) - \delta.$$

Then f and h are bounded, non-negative and lower semicontinuous and

$$0 \leq f(x) < g(x) \leq h(x) < f(x) + \delta$$

for all x. From the monotonicity property of J it follows that

$$J(f) \leq J(g) \leq J(h).$$

Using the additivity homogeneity properties as well,

$$J(h) \leq J(f + \delta\chi_X) = J(f) + \delta J(\chi_X) = J(f) + \epsilon.$$

Let

$$U_i = g^*((i\delta, +\infty)).$$

U_i is open because g is lower semi-continuous. Letting k be the least integer such that $g(x) \leq k\delta$, as before, we have $x \in U_i$ precisely when $i < k$, so

$$h(x) = k\delta = \delta \sum_i \chi_{U_i}(x).$$

The sum can be taken over all $i \in \mathbf{N}$, but we can also take it over all i such that $U_i \neq \varnothing$, of which there are only finitely many. We'll interpret it in the latter sense. Then

$$\begin{aligned} J(h) &= \delta \sum_i J(\chi_{U_i}) = \delta \sum_i \nu(U_i) \\ &= \delta \sum_i \mu(U_i) = \delta \sum_i \int_{x\in X} \chi_{U_i}(x)\, d\mu(x) \\ &= \int_{x\in X} \sum_i \delta\chi_{U_i}(x)\, d\mu(x) = \int_{x\in X} h(x)\, d\mu(x). \end{aligned}$$

Similarly,

$$J(f) = \int_{x\in X} f(x)\, d\mu(x).$$

So we have

$$\int_{x\in X} f(x)\, d\mu(x) \leq J(g) \leq \int_{x\in X} h(x)\, d\mu(x)$$

where f and h are simple functions such that

$$0 \leq f(x) \leq g(x) \leq h(x).$$

We also have

$$\begin{aligned}\int_{x\in X} f(x)\,d\mu(x) &\le \int_{x\in X} g(x)\,d\mu(x)\\ &\le \int_{x\in X} h(x)\,d\mu(x)\end{aligned}$$

by the monotonicity property of the integral. In addition we have

$$\int_{x\in X} h(x)\,d\mu(x) \le \int_{x\in X} f(x)\,d\mu(x) + \epsilon.$$

It follows from these that

$$\left|\int_{x\in X} g(x)\,d\mu(x) - J(g)\right| \le \epsilon.$$

This holds for all $\epsilon > 0$ so

$$\int_{x\in X} g(x)\,d\mu(x) = J(g).$$

Suppose g is continuous. Define

$$g_+(x) = 1 + \max(0, g(x))$$

and

$$g_-(x) = 1 - \min(0, g(x)).$$

Then g_+ and g_- are bounded positive continuous functions, hence also lower semicontinuous, and

$$g(x) = g_+(x) - g_-(x).$$

It follows that

$$\begin{aligned}I(g) = I(g_+) - I(g_-) &= J(g_+) - J(g_-)\\ &= \int_{x\in X} g_+(x)\,dx - \int_{x\in X} g_-(x)\,dx\\ &= \int_{x\in X} g(x)\,dx.\end{aligned}$$

□

9.3 Uniqueness

The proposition in the previous section just gives the existence of a measure with the stated properties. The uniqueness is, fortunately, easier to prove, even under weaker hypotheses. In the proof we'll need the following variant of Urysohn's Lemma, Lemma 3.13.2.

Lemma 9.3.1. *Suppose $(X, \mathcal{T})$ is a locally compact Hausdorff topological space and $K \in \wp(X)$ is compact, $U \in \wp(X)$ is open and $K \subseteq U$. Then there is a continuous compactly supported function $g\colon X \to [0,1]$ such that $g(x) = 1$ for all $x \in K$ and $g(x) = 0$ for all $x \in X \setminus U$.*

Proof. $(X, \mathcal{T})$ is locally compact by hypothesis. There is therefore a compact neighbourhood W_x of x for each $x \in X$. In other words, W_x is compact and there is an open V_x such that $x \in V_x$ and $V_x \subseteq W_x$. The V_x form an open cover of the compact set K so there is a finite subcover. In other words, there are $x_1, \ldots, x_m$ such that

$$K \subseteq V$$

where

$$V = \bigcup_{i=1}^{m} V_{x_i}.$$

Let

$$W = \bigcup_{i=1}^{m} W_{x_i}.$$

Then W is compact and

$$K \subseteq U \cap V \subseteq V \subseteq W.$$

Let

$$L = W \setminus (U \cap V).$$

$U \cap V$ is open so L is a closed subset of W. W is a compact Hausdorff space and so is normal. We can therefore apply Urysohn's Lemma, Lemma 3.13.2, to get a continuous $f\colon W \to [0,1]$ such that $f(x) = 0$ for $x \in L$ and $f(x) = 1$ for $x \in K$. Define $g\colon X \to [0,1]$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in V,\\ 0 & \text{if } x \in X \setminus W.\end{cases}$$

Suppose Z is an open subset of $[0,1]$. If $0 \notin Z$ then $g^*(Z) = f^*(Z)$ is open because f is continuous. If $0 \in Z$ then

$$g^*(Z) = f^*(Z) \cup (X \setminus W)$$

is open. So $g^*(Z)$ is open for all open Z. In other words, g is continuous. If $x \in K$ then $X \in V$ so

$g(x) = f(x) = 1$. If $x \in X \setminus U$. If $x \in X \setminus U$ then $x \in X \setminus W$ or $x \in W \setminus U \subseteq L$. In either case $f(x) = 0$. $g(x) = 0$ for $x \notin W$ so the support of g is contained in W. The support of any function is closed and a closed subset of a compact set is compact so the support of g is compact. □

Proposition 9.3.2. *Suppose $(X, \mathcal{T})$ is a locally compact Hausdorff space. Suppose I is a linear transformation from the vector space of continuous compactly supported functions from X to $\mathbf{R}$ such that $I(g) \geq 0$ whenever g is such that $g(x) \geq 0$ for all $x \in X$. Then there is at most one Radon measure μ on X such that*

$$I(g) = \int_{x \in X} g(x)\, d\mu(x)$$

for all continuous compactly supported g.

Proof. Suppose μ_1 and μ_2 are Radon measures such that

$$\int_{x \in X} g(x)\, d\mu_1(x) = I(g) = \int_{x \in X} g(x)\, d\mu_2(x)$$

for all continuous compactly supported g. Suppose K is compact, U is open and $K \subseteq U$. By Lemma 9.3.1 there is a continuous compactly supported function g such that such that $g(x) = 1$ for $x \in K$ and $g(x) = 0$ for $x \notin U$. In other words,

$$\chi_K(x) \leq g(x) \leq \chi_U(x)$$

for all $x \in U$. Therefore

$$\begin{aligned} \mu_1(K) = \int_{x \in X} \chi_K(x)\, d\mu_1(x) &\leq \int_{x \in X} g(x)\, d\mu_1(x) \\ &= I(g) = \int_{x \in X} g(x)\, d\mu_2(x) \\ &\leq \int_{x \in X} \chi_U(x)\, d\mu_2(x) = \mu_2(U). \end{aligned}$$

Thus $\mu_1(K) \leq \mu_2(U)$ whenever K is closed, U is open and $K \subseteq U$. μ_1 is a Radon measure so

$$\mu_1(U) = \sup \mu_1(K)$$

where the supremum is over all compact subsets K of U so

$$\mu_1(U) \leq \mu_2(U).$$

The same argument works with μ_1 and μ_2 swapped, so $\mu_2(U) \leq \mu_1(U)$ and hence

$$\mu_1(U) = \mu_2(U).$$

μ_1 and μ_2 are both Radon measures so

$$\mu_1(E) = \inf \mu_1(U)$$

and

$$\mu_2(E) = \inf \mu_2(U)$$

for any Borel set E, where the infimum in both cases is over open supersets U of E. The right hand sides are equal, so the left hand sides are equal:

$$\mu_1(E) = \mu_2(E).$$

□

9.4 The Riesz Representation Theorem

Proposition 9.4.1. *Suppose $(X, \mathcal{T})$ is a locally compact σ-compact Hausdorff topological space. Then there is a sequence $K_0, K_1, \ldots$ of compact subsets such that*

$$K_m \subseteq K_{m+1}^\circ$$

for all m and

$$\bigcup_{m=0}^{\infty} K_m = X.$$

Proof. By the definition of σ-compactness there is a sequence $A_0, A_1, \ldots$ of compact subsets such that

$$\bigcup_{m=0}^{\infty} A_m = X.$$

By the definition of local compactness there is, for each $x \in X$, a compact neighbourhood C_x of x, i.e. a compact set C_x such that there is an open set B_x with $x \in B_x$ and $B_x \subseteq C_x$. Define K_m inductively as follows. Let $A_0 = \varnothing$. The sets B_x for $x \in A_m \cup K_m$ form an open cover of $A_m \cup K_m$, which is a compact set, so there are is a finite subcover, i.e. there are $x_{m,0}, \ldots, x_{m,n_m}$ such that

$$A_m \cup K_m \subseteq U_m,$$

where
$$U_m = \bigcup_{j=0}^{n_m} B_{x_{m,j}}.$$
Let
$$K_{m+1} = \bigcup_{j=0}^{n_m} C_{x_{m,j}}.$$
This is a finite union of compact sets and so is compact. $U_m \subseteq K_{m+1}$, since $B_x \subseteq C_x$ for all x, and U_m is open so
$$U_m \subseteq K_{m+1}^\circ$$
and hence, since $K_m \subseteq U_m$,
$$K_m \subseteq K_{m+1}^\circ.$$
Also $A_m \subseteq K_{m+1}$ so
$$X = \bigcup_{m=0}^{\infty} A_m \subseteq \bigcup_{m=0}^{\infty} K_{m+1} = \bigcup_{m=1}^{\infty} K_m = \bigcup_{m=0}^{\infty} K_m.$$
But $K_m \subseteq X$ for all m so
$$\bigcup_{m=0}^{\infty} K_m \subseteq X$$
and hence
$$\bigcup_{m=0}^{\infty} K_m = X.$$
□

Theorem 9.4.2. *Suppose $\mathcal{B}$ is a σ algebra on a set X and μ_0, μ_1, ... are measures on $(X, \mathcal{B})$ which are monotone in the sense that for all $E \in \mathcal{B}$ then*
$$\mu_j(E) \leq \mu_k(E)$$
whenever $j \leq k$. Let
$$\mu(E) = \lim_{j\to\infty} \mu_j(E).$$
Then μ is a measure on $(X, \mathcal{B})$ and
$$\int_{x\in X} f(x)\, d\mu(x) = \lim_{x\in X} \int_{x\in X} f(x)\, d\mu_j(x).$$

This can be thought of as a Monotone Convergence Theorem for measures.

Proof.
$$\mu(\varnothing) = \lim_{j\to\infty} \mu_j(\varnothing) = \lim_{j\to\infty} = 0.$$
Suppose E_0, E_1, ... are disjoint elements of $\mathcal{B}$. Then
$$\begin{aligned}\mu\left(\bigcup_{k=0}^{\infty} E_k\right) &= \lim_{j\to\infty} \mu_j\left(\bigcup_{k=0}^{\infty} E_k\right)\\ &= \lim_{j\to\infty} \sum_{k=0}^{\infty} \mu_j(E_k)\\ &= \sum_{k=0}^{\infty} \lim_{j\to\infty} \mu_j(E_k)\\ &= \sum_{k=0}^{\infty} \mu(E_k).\end{aligned}$$
Here we've used the definition of μ, the fact that μ_j is a measure, the Monotone Convergence Theorem for sums, and the definition of μ again. So μ is a measure. If f is semisimple then there is a partition $\mathcal{Q}$ of X and a function $\varphi\colon \mathcal{Q}$ such that $f(x) = \varphi(E)$ if $x \in E$. Also,
$$\int_{x\in X} f(x)\, d\mu(x) = \sum_{E\in\mathcal{Q}} \varphi(E)\mu(E)$$
and
$$\int_{x\in X} f(x)\, d\mu_j(x) = \sum_{E\in\mathcal{Q}} \varphi(E)\mu_j(E).$$
Then
$$\begin{aligned}\int_{x\in X} f(x)\, d\mu(x) &= \sum_{E\in\mathcal{Q}} \varphi(E)\mu(E)\\ &= \sum_{E\in\mathcal{Q}} \varphi(E) \lim_{j\to\infty} \mu_j(E)\\ &= \lim_{j\to\infty} \sum_{E\in\mathcal{Q}} \varphi(E)\mu_j(E)\\ &= \lim_{j\to\infty} \int_{x\in X} f(x)\, \mu_j(x).\end{aligned}$$
Here we've used the Monotone Convergence Theorem for sums again. So we have
$$\int_{x\in X} f(x)\, d\mu(x) = \lim_{x\in X} \int_{x\in X} f(x)\, d\mu_j(x)$$

for semisimple f. From this we get the corresponding equation for the upper and lower integrals of any measurable function and therefore the same equation for the integrals of any measurable function. □

The following is the full version of the Riesz Representation Theorem.

Theorem 9.4.3. *Suppose $(X, \mathcal{T})$ is a locally compact σ-compact Hausdorff space. Suppose I is a linear transformation from the vector space of continuous functions from X to $\mathbf{R}$ such that $I(g) \geq 0$ whenever g is such that $g(x) \geq 0$ for all $x \in X$. Then there is a unique Radon measure μ on X such that*

$$I(g) = \int_{x \in X} g(x)\, d\mu(x)$$

for all continuous g.

Proof. The uniqueness follows from Proposition 9.3.2. For the existence we start by applying Proposition 9.4.1 to get a sequence K_0, K_1, ... of compact subsets such that

$$K_n \subseteq K_{n+1}^{\circ}$$

for all n and

$$\bigcup_{n=0}^{\infty} K_n = X.$$

We then apply Proposition 9.3.1 to get functions $h_n \colon X \to [0,1]$ such that $h_n(x) = 1$ if $x \in K_n$ and $h_n(x) = 0$ if $x \notin K_{n+1}^{\circ}$. For fixed x the property $K_n \subseteq K_{n+1}^{\circ}$ implies that $h_n(x)$ is a monotone sequence. Together with the fact that $\bigcup_{n=0}^{\infty} K_n = X$ we find that for any fixed x there is an m such that $h_n(x) = 1$ for all $n \geq m$.

For any continuous function g on X the function gh_n is a compactly supported continuous function on X and its restriction to K_{n+1} is also a compactly supported continuous function. gh_n depends only on the restriction of g to K_{n+1} since h_n is zero outside of K_{n+1}. It therefore makes sense to define

$$I_n(g) = I(gh_n)$$

and to consider it either as a function on compactly supported continuous functions on K_{n+1} or on compactly supported continuous functions on X.

By Proposition 9.2.1 there is a Radon measure μ_n on K_{n+1} such that

$$I_n(g) = \int_{x \in K_{n+1}} g(x)\, d\mu_n(x).$$

We can extend the measure μ_n from K_{n+1} to all of X by taking $\mu_n(E)$ where E is a Borel set on X to be $\mu_n(E \cap K_{n+1})$. Denoting both the measure on K_{n+1} and the measure on X by μ_n is an abuse of notation, but a harmless one since their value is the same for any set on which both are defined. The inclusion of K_{n+1} in X gives a morphism of measure spaces from $(K_{n+1}, \mathcal{B}_{K_{n+1}}, \mu_{n,K_{n+1}})$ to $(X, \mathcal{B}_X, \mu_{n,X})$, where I've temporarily added subscripts to distinguish the Borel algebras and measures on K_{n+1} from those on X. It follows that

$$\int_{x \in X} g(x)\, d\mu_n(x) = \int_{x \in K_{n+1}} g(x)\, d\mu_n(x)$$

for any integrable function g on X. For compactly supported continuous functions g on X we therefore have

$$I_n(g) = \int_{x \in X} g(x)\, d\mu_n(x).$$

Suppose g is a compactly supported continuous function from X to $\mathbf{R}$. Let L be its support. Then

$$L \subseteq X = \bigcup_{m=0}^{\infty} K_m \subseteq \bigcup_{m=0}^{\infty} K_{m+1}^{\circ}.$$

The sets K_{m+1}° are therefore an open cover of L and so there's a finite subcover. Since they form an increasing sequence there's a single m such that

$$L \subseteq K_{m+1}^{\circ}.$$

Also $L \subseteq K_{n+1}^{\circ}$ for all $n \geq m$. For such n we have $g(x) = g(x) h_n(x)$ For all such n we have

$$g(x) = g(x) h_n(x)$$

because $h_n(x) = 1$ when $x \in K_n$ and $g(x) = 0$ when $x \notin K_{m+1}^{\circ}$. It then follows that $I(g) = I_n(g)$ for $n \geq m$. But then

$$\begin{aligned} I(g) = \lim_{n \to \infty} I_n(g) &= \lim_{n \to \infty} \int_{x \in X} g(x)\, d\mu_n(x) \\ &= \int_{x \in X} g(x)\, d\mu(x). \end{aligned}$$

$\square$

We can now state the most important definition for this semester:

Definition 9.4.4. Let μ be the unique Radon measure on $\mathbf{R}$ such that

$$\int_{x\in X} g(x)\,dm(x) = \int g(x)\,dx$$

for all compactly supported continuous g, where the integral on the right hand side is the Riemann integral. Let $(\mathbf{R}, \mathcal{B}^\dagger, \mu^\dagger)$ be the completion of $(\mathbf{R}, \mathcal{B}, \mu)$ where $\mathcal{B}$ is the Borel σ-algebra on $\mathbf{R}$. $\mathcal{B}^\dagger$ is called the *Lebesgue algebra* on $\mathbf{R}$ and its elements are called *Lebesgue sets*. The measure $\mu^\dagger$ is called *Lebesgue measure*.

Note that this definition relies on a number of rather deep theorems. The existence of μ follows from the Riesz Representation Theorem, the linearity and monotonicity properties of the Riemann integral and the fact that $\mathbf{R}$ is locally compact and σ-compact. The existence of $\mu^\dagger$ relies on Theorem 7.6.11.

9.5 Subspace measure

This section makes precise the notions of extending a measure from a subset to a larger set and restricting a measure from a larger set to a smaller set.

The following is an extension of Proposition 8.2.11.

Proposition 9.5.1. *Suppose that $(X, \mathcal{B}_X, \mu_X)$ and $(Y, \mathcal{B}_Y,$
$mu_Y)$ are measure spaces and $j\colon X \to Y$ is a morphism. Suppose $Z \subseteq [-\infty, +\infty]$. If $f\colon Y \to Z$ is integrable with respect to $(Y, \mathcal{B}_Y, \mu_Y)$ then $f \circ j$ is integrable with respect to $(X, \mathcal{B}_X, \mu_X)$ and*

$$\int_{x\in X} f(j(x))\,d\mu_X(x) = \int_{y\in Y} f(y)\,d\mu_Y(y)$$

Proof. Let $\mathbf{P}_X$ be the set of countable subsets of $\mathcal{B}_X$ which are partitions of X and let $\mathbf{P}_Y$ be the set of countable subsets of $\mathcal{B}_Y$ which are partitions of Y. If $\mathcal{Q}$ is a partition of Y then $(j^*)_*(\mathcal{Q})$ is a partition of X, so

$$\mathbf{P}_Y \subseteq ((j^*)_*)^*(\mathbf{P}_X)$$

Let U_X be the set of systems of weights v on X such that $\sum_{x\in X} f(j(x))v(x)$ converges and let U_Y be the set of system of weights w on Y such that $\sum_{y\in Y} f(y)w(y)$ converges. Define $\varphi_j\colon U_X \to U_Y$ by

$$(\varphi_j(v))(y) = \sum_{x\in j^*(y)} v(x),$$

so that

$$\sum_{x\in X} f(j(x))v(x) = \sum_{y\in Y} f(y)(\varphi_j v)(y).$$

Define $R_{f\circ j}\colon U_X \to Z$ and $R_f\colon U_Y \to Z$ by

$$R_{f\circ j}(v) = \sum_{x\in X} f(j(x))v(x)$$

and

$$R_f(w) = \sum_{y\in Y} f(y)w(y).$$

Then

$$R_f \circ \varphi = R_{f\circ j}.$$

Define $\alpha_X\colon \mathbf{P}_X \to \wp(U_X)$ by saying $v \in \alpha_X(\mathcal{Q})$ if and only if $(X, \mathcal{B}_X, \mu_X)$, $\mathcal{Q}$ and v are compatible. Similarly, define $\alpha_Y\colon \mathbf{P}_Y \to \wp(U_Y)$ by saying $w \in \alpha_Y(\mathcal{R})$ if and only if $(Y, \mathcal{B}_Y, \mu_Y)$, $\mathcal{R}$ and w are compatible. If $\mathcal{Q} \in \mathbf{P}_X$ then

$$\alpha_X(\mathcal{Q}) \subseteq \varphi_j^*(\alpha_Y((j^*)_*(\mathbf{Q}))).$$

Let $\mathcal{E}_X$ be the upward closure of $\alpha_{X*}(\mathbf{P}_X)$. and let $\mathcal{E}_Y$ be the upward closure of $\alpha_{Y*}(\mathbf{P}_Y)$. Suppose $W \in \mathcal{E}_X$, i.e. that there is a $\mathcal{Q} \in \mathbf{P}_X$ such that $\alpha_X(\mathcal{Q}) \subseteq W$. Then $(j^*)_*(\mathcal{Q}) \in \mathbf{P}_Y$ and $\varphi_j^*(\alpha_Y((j^*)_*(\mathbf{Q}))) \subseteq W$. Therefore $\mathcal{E}_Y \subseteq \mathcal{E}_X$ so $\mathcal{E}_X$ converges if $\mathcal{E}_Y$ does and the limits are the same. $\square$

Proposition 9.5.2. *Suppose $(Y, \mathcal{B}_Y)$ is a measurable space $X \in \mathcal{B}_Y$ and*

$$\mathcal{B}_X = \{E \in \mathcal{B}_Y : E \subseteq X\}$$

Suppose μ_X is a measure on $(X, \mathcal{B}_X)$. Define $\mu_Y\colon \mathcal{B}_Y \to [0, +\infty]$ by

$$\mu_Y(E) = \mu_X(X \cap E).$$

Then $(Y, \mathcal{B}_Y, \mu_Y)$ is a measure space, the inclusion function $j\colon X \to Y$ is a morphism from $(X, \mathcal{B}_X, \mu_X)$ to $(Y, \mathcal{B}_Y, \mu_Y)$, and

$$\int_{s \in X} f(s)\, d\mu_X(s) = \int_{s \in Y} f(s)\, d\mu_Y(s)$$

for every integrable function f on $(Y, \mathcal{B}_Y, \mu_Y)$.

The measure μ_Y is said to be obtained from μ_X by extending it to be zero outside of X. This terminology is motivated by the observation that $\mu_Y(Y \setminus X) = 0$.

Proof.

$$\mu_Y(\varnothing) = \mu_X(X \cap \varnothing) = \mu_X(\varnothing) = 0.$$

If E_0, E_1, … are disjoint elements of $\mathcal{B}_Y$ then $X \cap E_0$, $X \cap E_1$, … are disjoint elements of $\mathcal{B}_X$ and

$$\mu_Y\left(\bigcup E_i\right) = \mu_X\left(X \cap \bigcup E_i\right) = \mu_X\left(\bigcup (X \cap E_i)\right) = \sum \mu_X(X \cap E_i) = \sum \mu_Y(E_i).$$

So μ_Y is a measure on $(Y, \mathcal{B}_Y)$.

If $E \in \mathcal{B}_Y$ then

$$j^*(E) = E \cap X \in \mathcal{B}_X$$

so $E \in j^{**}(\mathcal{B}_X)$. Therefore

$$\mathcal{B}_Y \subseteq B_X.$$

Also,

$$\mu_Y(E) = \mu_X(X \cap E) = \mu_X(j^*(E)).$$

So j is a morphism. The equality of the integrals therefore follows from Proposition 9.5.1. □

Proposition 9.5.3. *Suppose $(Y, \mathcal{B}_Y, \mu_Y)$ is a measure space and $X \in \mathcal{B}_Y$. Define*

$$\mathcal{B}_X = \{E \in \mathcal{B}_Y \colon E \subseteq X\}$$

and define $\mu_X\colon \mathcal{B}_X \to [0, +\infty]$ by

$$\mu_X(E) = \mu_Y(X \cap E).$$

Then $(X, \mathcal{B}_X, \mu_X)$ is a measure space and

$$\int_{t \in X} g(t)\, d\mu_X(t) = \int_{t \in Y} \chi_X(t) g(t)\, d\mu_Y(t)$$

for every integrable function g on $(Y, \mathcal{B}_Y, \mu_Y)$.

The measure μ_X is said to be the restriction of μ_Y from Y to X.

Proof. $\varnothing \in \mathcal{B}_Y$ and $\varnothing \subseteq X$ so $\varnothing \in \mathcal{B}_X$. If $E \in \mathcal{B}_X$ then $E \in \mathcal{B}_Y$. $X \in \mathcal{B}_Y$ so $X \setminus E \in \mathcal{B}_Y$. We also have $X \setminus E \subseteq X$ so $X \setminus E \in \mathcal{B}_X$. If E_0, E_1, … are elements of $\mathcal{B}_X$ then they're all elements of $\mathcal{B}_Y$ as well so $\bigcup E_i \in \mathcal{B}_Y$. Now $E_i \subseteq X$ for each i so $\bigcup E_i \subseteq X$. Therefore $\bigcup E_i \in \mathcal{B}_X$. So $\mathcal{B}_X$ is a σ-algebra.

$$\mu_X(\varnothing) = \mu_Y(X \cap \varnothing) = \mu_Y(\varnothing) = 0.$$

If E_0, E_1, … are disjoint elements of $\mathcal{B}_X$ then $X \cap E_0$, $X \cap E_1$, … are disjoint elements of $\mathcal{B}_Y$ so

$$\mu_X\left(\bigcup E_i\right) \mu_Y\left(X \cap \bigcup E_i\right) = \mu_Y\left(\bigcup (X \cap E_i)\right) = \sum \mu_Y\left(X \cap E_i\right) = \sum \mu_X\left(E_i\right),$$

so μ_X is a measure.

If g is an integrable function on Y then for every $\epsilon > 0$ there are semisimple functions f and h such that $f(t) \leq g(t) \leq h(t)$ for almost all $t \in X$ and

$$\int_{t \in Y} h(t)\, d\mu_Y(t) \leq \int_{t \in Y} f(t)\, d\mu_Y(t) + \epsilon$$

Then

$$\chi_X(t) f(t) \leq \chi_X(t) g(t) \leq \chi_X(t) h(t)$$

for almost all $t \in S$. The functions $\chi_X f$ and $\chi_X h$ are semisimple functions. So there are partitions $\mathcal{P}$ and $\mathcal{Q}$ such that f is constant on elements of $\mathcal{P}$ and h is constant on elements of $\mathcal{Q}$. Let $\mathcal{R}$ be the common refinement of $\mathcal{P}$, $\mathcal{Q}$ and $\{X, Y \setminus X\}$. The both f and h are constant on elements of $\mathcal{R}$, so there are $\varphi, \psi\colon \mathcal{R} \to Z$ such that $f(t) = \varphi(E)$ and $h(t) = \psi(E)$ if $x \in E$. Define

$$\mathcal{S} = \{E \in \mathcal{R} \colon E \subseteq X\}.$$

Then $\mathcal{S}$ is a countable partition of X. Now

$$\int_{t \in Y} f(t)\, d\mu_Y(t) = \sum_{E \in \mathcal{R}} \varphi(E) \mu_Y(E),$$

$$\int_{t \in Y} h(t)\, d\mu_Y(t) = \sum_{E \in \mathcal{R}} \psi(E) \mu_Y(E),$$

$$\int_{t\in X} f(t)\, d\mu_Y(t) = \sum_{E\in\mathcal{S}} \varphi(E)\mu_X(E),$$

and

$$\int_{t\in X} h(t)\, d\mu_Y(t) = \sum_{E\in\mathcal{S}} \psi(E)\mu_X(E).$$

We can rewrite

$$\int_{t\in Y} h(t)\, d\mu_Y(t) \le \int_{t\in Y} f(t)\, d\mu_Y(t) + \epsilon$$

as

$$\sum_{E\in\mathcal{R}} \psi(E)\mu_Y(E) \le \sum_{E\in\mathcal{R}} \varphi(E)\mu_Y(E) + \epsilon.$$

$f(t) \le h(t)$ for almost all $t \in Y$ so

$$\varphi(E)\mu_Y(E) \le \psi(E)\mu_Y(E)$$

for all $E \in \mathcal{R}$. If $E \in \mathcal{S}$ then $\mu_Y(E) = \mu_X(E)$. It follows that

$$\sum_{E\in\mathcal{S}} \psi(E)\mu_Y(E) \le \sum_{E\in\mathcal{S}} \varphi(E)\mu_Y(E) + \epsilon.$$

So for every $\epsilon > 0$ there are semisimple functions f and h such that

$$f(t) \le g(t) \le h(t)$$

for almost all $t \in X$ and

$$\int_{t\in X} h(t)\, d\mu_X(t) \le \int_{t\in X} f(t)\, d\mu_X(t) + \epsilon.$$

So f is integrable with respect to $(X, \mathcal{B}_X, \mu_X)$. If we set

$$\varphi'(E) = \begin{cases} \varphi(E) & \text{if } E \in \mathcal{S}, \\ 0 & \text{if } E \in \mathcal{R} \setminus \mathcal{S}, \end{cases}$$

and

$$\psi'(E) = \begin{cases} \psi(E) & \text{if } E \in \mathcal{S}, \\ 0 & \text{if } E \in \mathcal{R} \setminus \mathcal{S}, \end{cases}$$

Then

$$f'(t) \le \chi_X(t) g(t) \le h'(t)$$

for almost all $t \in Y$, where $f'(t) = \varphi'(E)$ and $h'(t) = \psi'(E)$ if $t \in E$. Therefore

$$\begin{aligned} \int_{t\in Y} f'(t)\, d\mu_Y(t) &\le \int_{t\in Y} \chi_X(t) g(t)\, d\mu_Y(t) \\ &\le \int_{t\in Y} h'(t)\, d\mu_Y(t). \end{aligned}$$

The integrals of f' and h' are equal to

$$\sum_{E\in\mathcal{S}} \varphi(E)\mu_X(E)$$

and

$$\sum_{E\in\mathcal{S}} \psi(E)\mu_X(E)$$

respectively, which are within a distance ϵ of

$$\int_{t\in X} g(t)\, d\mu_X(t)$$

so

$$\left| \int_{t\in Y} \chi_X(t) g(t)\, d\mu_Y(t) - \int_{t\in X} g(t)\, d\mu_X(t) \right| \le \epsilon$$

for all $\epsilon > 0$. The two integrals are therefore equal. □

Note that extending a measure by zero to a superset and then restricting back to the original subset gives us back the measure we started with. Restricting to a subset and then extending by zero back to the original set does not. More precisely, if we start with a measure μ_Y on Y and restrict it to a measure μ_X on $X \subseteq Y$ and then extend this by zero to Y we get a measure ν on Y, where $\nu(E) = \mu_Y(X \cap E)$.

We'll always use the restriction of Lebesgue measure as our measure on subsets of $\mathbf{R}$ unless some other measure is specified.

10 The Fundamental Theorem of Calculus

10.1 Riemann integration

The following theorem is half of the Fundamental Theorem of Calculus for the Riemann integral.

Theorem 10.1.1. *Suppose $f\colon [a,b] \to \mathbf{R}$ is continuous. Define $F\colon [a,b] \to \mathbf{R}$ by*

$$F(y) = \int_a^y f(x)\,dx.$$

Then F is differentiable and $F'(x) = f(x)$ for all $x \in [a,b]$.

The other half is

Theorem 10.1.2. *Suppose $F\colon [a,b] \to \mathbf{R}$ is differentiable and F' is Riemann integrable. Then*

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

These are often called the First Fundamental Theorem of Calculus and the Second Fundamental Theorem of Calculus, respectively. The goal of this chapter is to prove analogous results for the Lebesgue integral.

An easy consequence of Theorem 10.1.2 is the following.

Corollary 10.1.3. *Suppose $f\colon [a,b] \to \mathbf{R}$ is continuous. Then*

$$\lim_{h \searrow 0} \frac{1}{h} \int_y^{y+h} f(x)\,dx = f(y)$$

for all $y \in [a,b)$,

$$\lim_{h \searrow 0} \frac{1}{h} \int_{y-h}^{y} f(x)\,dx = f(y)$$

for all $y \in (a,b]$, and

$$\lim_{h \searrow 0} \frac{1}{2h} \int_{y-h}^{y+h} f(x)\,dx = f(y)$$

for all $y \in (a,b)$.

Proof. Let $F(y) = \int_a^y f(x)\,dx$. Then

$$F'(y) = \lim_{h \to 0} \frac{1}{h} \int_y^{y+h} f(x)\,dx$$

by the definition of the derivative. By Theorem 10.1.2 this limit exists and is equal to $f(y)$. The one sided limits must then exist, subject to a restriction discuss below, and are also equal to $f(y)$, so

$$\lim_{h \searrow 0} \frac{1}{h} \int_y^{y+h} f(x)\,dx = f(y)$$

for all $y \in [a,b)$,

$$\lim_{h \searrow 0} \frac{1}{h} \int_{y-h}^{y} f(x)\,dx = f(y)$$

for all $y \in (a,b]$. We need to exclude $y = b$ in the first limit and $y = a$ in the second limit because there are no points of the form $y + h$ in $[a,b]$ if $y = b$ and there are no points of the form $y - h$ in $[a,b]$ if $y = a$. Now

$$\begin{aligned} &\frac{1}{2h} \int_{y-h}^{y+h} f(x)\,dx \\ &\quad = \frac{1}{2} \left(\frac{1}{h} \int_y^{y+h} f(x)\,dx + \frac{1}{h} \int_{y-h}^{y} f(x)\,dx \right). \end{aligned}$$

Taking limits gives

$$\lim_{h \searrow 0} \frac{1}{2h} \int_{y-h}^{y+h} f(x)\,dx = \frac{1}{2}(f(y) + f(y)) = f(y).$$

For this we require $y \in (a,b)$ so that both limits on the right exist. ☐

10.2 The First Fundamental Theorem

Proposition 10.2.1. *Suppose $(X, \mathcal{B}, \mu)$ is a measure space, $g\colon X \to [0,+\infty]$ is integrable and $\lambda > 0$. Then*

$$\mu(E_\lambda) \le \frac{1}{\lambda} \int_{x \in X} g(x)\,d\mu(x)$$

where

$$E_\lambda = \{x \in X \colon g(x) \ge \lambda\}.$$

The inequality above is known as *Markov's Inequality*.

Proof. Let $f = \lambda \chi_E$. In other words,

$$f(x) = \begin{cases} \lambda & \text{if } x \in E_\lambda, \\ 0 & \text{if } x \notin E_\lambda. \end{cases}$$

Then f is a simple function and

$$\int_{x\in X} f(x)\, d\mu(x) = \lambda\mu(E_\lambda).$$

On the other hand $f(x) \leq g(x)$ for all $x \in X$ so

$$\int_{x\in X} f(x)\, d\mu(x) \leq \int_{x\in X} g(x)\, d\mu(x).$$

Therefore

$$\lambda\mu(E_\lambda) \leq \int_{x\in X} g(x)\, d\mu(x).$$

Dividing by λ gives

$$\mu(E_\lambda) \leq \frac{1}{\lambda}\int_{x\in X} g(x)\, d\mu(x)$$

□

Proposition 10.2.2. *Suppose W is an open subset of $\mathbf{R}$. Show that there is a countable partition $\mathcal{P}$ of U such that if $V \in \mathcal{P}$ then V is an open interval and the endpoints of V, if any, are not in W.*

Proof. For each $x \in W$ let $\mathcal{A}_x$ be set of open intervals U such that $x \in U$ and $U \subseteq W$. Let

$$V_x = \bigcup_{U\in\mathcal{A}_x} U.$$

Then $x \in V_x$, V_x is open and $V_x \subseteq W$. Suppose $p, q, r \in V_x$ and $p \leq q \leq r$. Then there are $U_p \in \mathcal{A}_x$ and $U_r \in \mathcal{A}_x$ such that $p \in U_p$ and $r \in U_r$. If $x \leq q$ then $q \in U_r$ because $x \in U_r$, $r \in U_r$, $x \leq q \leq r$ and U_r is an interval. If $x \geq U_r$ then $q \in U_p$ because $p \in U_p$, $x \in U_p$, $p \leq q \leq x$ and U_p is an interval. In either case, $q \in V_x$. So if $p \leq q \leq r$, $p \in V_x$ and $r \in V_x$ then $q \in V_x$. In other words, V_x is an interval. The endpoints of an open interval do not belong to the interval. Suppose $V_x \cap V_y \neq \varnothing$. Then there is a $z \in V_x \cap V_y$. V_x is an open interval, $z \in V_x$ and $V_x \subseteq W$ so $V_x \in \mathcal{A}_z$ and therefore $V_x \subseteq V_z$. But then $x \in V_z$, V_z is an open interval and $V_z \subseteq W$ so $V_z \in \mathcal{A}_x$ and $V_z \subseteq V_x$. Since we already have the reverse inequality we find that $V_x = V_z$. The same argument with y in place of x gives $V_y = V_z$ and therefore $V_x = V_y$. So for any $x, y \in W$ we either have $V_x = V_y$ or $V_x \cap V_y = \varnothing$. In other words, the set $\mathcal{P}$ of sets of the form V_x for some $x \in W$ is a partition of W. Every element of $\mathcal{P}$ is a non-empty open set and so contains a rational number and there are only countably many rationals so $\mathcal{P}$ is countable. □

Proposition 10.2.3. *Suppose $F\colon [a,b] \to \mathbf{R}$ is continuous. Then there are countable sets $\mathcal{P}^+$, $\mathcal{P}_+$, $\mathcal{P}^-$ and $\mathcal{P}_-$ of open subintervals of $[a,b]$ such that*

(a) If $I, J \in \mathcal{P}^+$ then $I = J$ or $I \cap J = \varnothing$,

(b) If $(\alpha,\beta) \in \mathcal{P}^+$ then $F(\alpha) \leq F(\beta)$ and if $F(\alpha) < F(\beta)$ then $\alpha = a$.

(c) If $a \leq x \leq y \leq b$ and $x \notin \bigcup_{E\in\mathcal{P}^+} E$ then $F(x) \geq F(y)$.

(d) If $I, J \in \mathcal{P}_+$ then $I = J$ or $I \cap J = \varnothing$,

(e) If $(\alpha,\beta) \in \mathcal{P}_+$ then $F(\alpha) \geq F(\beta)$ and if $F(\alpha) > F(\beta)$ then $\alpha = a$.

(f) If $a \leq x \leq y \leq b$ and $x \notin \bigcup_{E\in\mathcal{P}_+} E$ then $F(x) \leq F(y)$.

(g) If $I, J \in \mathcal{P}^-$ then $I = J$ or $I \cap J = \varnothing$,

(h) If $(\alpha,\beta) \in \mathcal{P}^-$ then $F(\alpha) \geq F(\beta)$ and if $F(\alpha) > F(\beta)$ then $\beta = b$.

(i) If $a \leq x \leq y \leq b$ and $y \notin \bigcup_{E\in\mathcal{P}^-} E$ then $F(x) \leq F(y)$.

(j) If $I, J \in \mathcal{P}_-$ then $I = J$ or $I \cap J = \varnothing$,

(k) If $(\alpha,\beta) \in \mathcal{P}_-$ then $F(\alpha) \leq F(\beta)$ and if $F(\alpha) < F(\beta)$ then $\beta = b$.

(l) If $a \leq x \leq y \leq b$ and $y \notin \bigcup_{E\in\mathcal{P}_-} E$ then $F(x) \geq F(y)$.

Proof. It suffices to prove the existence of one of $\mathcal{P}^+$ $\mathcal{P}_+$, $\mathcal{P}^-$ or $\mathcal{P}_-$. We can then get the other three either by changing the directions of some inequalities in the proof or simply by working with $-F(x)$, $F(a+b-x)$ or $-F(a+b-x)$ in place of $F(x)$.

Let W be the set of $x \in (a,b)$ such that there is a $z \in (x,b)$ such that $F(x) < F(z)$. Suppose $x \in W$, i.e. that the is a z as above. Then

$$x \in (a,z) \cap F^*((-\infty, F(z))).$$

If

$$y \in (a, z) \cap F^*((-\infty, F(z)))$$

then $y < z$ and $F(y) < F(z)$ so $y \in W$. F is continuous so $F^*((-\infty, F(z)))$ is open and therefore so is $(a, z) \cap F^*((-\infty, F(z)))$. This is therefore an open neighbourhood of x in W. Since each $x \in W$ has such a neighbourhood it follows that W is open.

We now apply Proposition 10.2.2 to get a partition $\mathcal{P}$ of W by countably many disjoint open intervals. $W \subseteq (a, b)$ so if $E \in \mathcal{P}$ then $E = (\alpha, \beta)$ for some $\alpha, \beta \in [a, b]$.

If $x \notin W$ then there is, by the definition of W, no $z \in (x, b)$ and $F(x) < F(z)$. In other words, if $x \notin W$ then $F(z) \geq F(x)$ for all $z \in (x, b)$. By continuity it then follows that $F(z) \geq F(x)$ for all $z \in [x, b]$. Replacing z by y, this means that if $a \leq x \leq y \leq b$ and $x \notin \bigcup_{E \in \mathcal{P}^+} E$ then $F(x) \leq F(y)$.

Suppose $(\alpha, \beta) \in \mathcal{P}$ and $t \in (\alpha, \beta)$ is such that $F(\beta) < F(t)$. Then

$$A = [t, \beta] \cap F^*([F(t), +\infty))$$

is closed, $t \in A$ and $\beta \notin A$. Let $u = \sup A$. Then

$$u \in [t, b) \subseteq (\alpha, \beta) \subseteq W.$$

In other words, there is a $v \in (u, b)$ such that $F(u) < F(v)$. Then

$$F(\beta) < F(t) \leq F(u) < F(v).$$

$\beta \notin W$ so if $\beta \neq b$ then $\beta \notin (a, b)$ and $F(w) \leq F(\beta)$ for all $w \in [\beta, b]$. This also holds, trivially, if $\beta = b$. So

$$F(w) < F(v)$$

for all $w \in [\beta, b]$. Therefore $v \notin [\beta, b]$. But then $v \in (\alpha, \beta)$ and $F(v) > F(t)$ so $v \in A$. But $v > u = \sup(A)$ so we have a contradiction. There is therefore no $t \in (\alpha, \beta)$ is such that $F(\beta) < F(t)$. In other words, $F(t) \leq F(\beta)$ for all $t \in (\alpha, \beta)$. By continuity then

$$F(\alpha) \leq F(\beta).$$

Suppose $\alpha \neq a$. Then $\alpha \in (a, b)$ but $\alpha \notin W$ so $F(z) \leq F(\alpha)$ for all $z \in [\alpha, b]$. In particular, $F(\beta) \leq F(\alpha)$. □

Proposition 10.2.4. *Suppose $f: \mathbf{R} \to \mathbf{R}$ is integrable and*

$$F(y) = \int_{x \in (-\infty, y)} f(x)\, d\mu(x).$$

Then F is continuous.

Proof. Suppose $x \in \mathbf{R}$ and $\epsilon > 0$. f is integrable so $|f|$ is integrable. Let

$$g_n = \chi_{(x, x+1/n)} |f|.$$

Then

$$\lim_{n \to \infty} g_n(s) = 0.$$

Also

$$|g_n(s)| = g_n(s) \leq |f(s)|$$

for all $s \in \mathbf{R}$ and

$$\int_{s \in \mathbf{R}} |f(s)| dm(s) < +\infty$$

so by the Dominated Convergence Theorem, Theorem **??**, we have

$$\lim_{n \to \infty} \int_{s \in \mathbf{R}} g_n(s)\, dm(s) = \int_{s \in \mathbf{R}} \lim_{n \to \infty} g_n(s)\, dm(s) = 0.$$

In other words, there is a k_+ such that if $n \geq k_+$ then

$$\int_{s \in \mathbf{R}} g_n(s)\, dm(s) = \left| \int_{s \in \mathbf{R}} g_n(s)\, dm(s) - 0 \right| < \epsilon.$$

This holds in particular for $n = k_+$. Choose $\delta_+ = 1/k_+$. If $x < y < x + \delta_+$ then

$$-g_k(s) \leq \chi_{(x,y)}(s) f(s) \leq g_{k_+}(s)$$

so

$$\begin{aligned} -\epsilon &< \int_{s \in (x,y)} f(s)\, dm(s) \\ &= \int_{s \in \mathbf{R}} \chi_{(x,y)}(s) f(s)\, dm(s) < \epsilon. \end{aligned}$$

Now

$$F(y) - F(x) = \int_{s \in (x,y)} f(s)\, dm(s)$$

so

$$|F(y) - F(x)| < \epsilon$$

whenever $y \in (x, x + \delta_+)$. Similarly, if we let

$$h_n = \chi_{(x-1/n,x)}|f|.$$

then

$$\lim_{n\to\infty} h_n(s) = 0.$$

Also

$$|h_n(s)| = h_n(s) \leq |f(s)|$$

so

$$\begin{aligned}\lim_{n\to\infty} \int_{s\in\mathbf{R}} h_n(s)\, dm(s) &= \int_{s\in\mathbf{R}} \lim_{n\to\infty} h_n(s)\, dm(s) \\ &= 0.\end{aligned}$$

by the Dominated Convergence Theorem and there is a k_- such that if $n \geq k_-$ then

$$\int_{s\in\mathbf{R}} h_n(s)\, dm(s) = \left|\int_{s\in\mathbf{R}} h_n(s)\, dm(s) - 0\right| < \epsilon.$$

This holds in particular for $n = k_-$. Choose $\delta_- = 1/k_-$. If $x - \delta_- < y < x$ then

$$-h_k(s) \leq \chi_{(y,x)}(s) f(s) \leq h_{k_+}(s)$$

so

$$\begin{aligned}-\epsilon &< \int_{s\in(y,x)} f(s)\, dm(s) \\ &= \int_{s\in\mathbf{R}} \chi_{(y,x)}(s) f(s)\, dm(s) < \epsilon.\end{aligned}$$

Now

$$F(x) - F(y) = \int_{s\in(y,x)} f(s)\, dm(s)$$

so

$$|F(y) - F(x)| < \epsilon$$

whenever $y \in (x - \delta_-, x)$. If $y = 0$ then

$$|F(y) - F(x)| = 0 < \epsilon,$$

so

$$|F(y) - F(x)| < \epsilon$$

whenever $y \in (x-\delta_-, x+\delta_+)$, where $\delta = \min(\delta_-, \delta_+)$. For any $x \in \mathbf{R}$ and $\epsilon > 0$ there is therefore a δ such that if $|y - x| < \delta$ then $|F(y) - F(x)| < \epsilon$. In other words, F is continuous. □

Proposition 10.2.5. *Suppose $f: \mathbf{R} \to \mathbf{R}$ is integrable. For every $\lambda > 0$ we have*

$$\mu(E_\lambda) \leq \frac{1}{\lambda} \int_{x\in\mathbf{R}} |f(x)|\, dm(x),$$

where

$$E_\lambda = \left\{ y \in \mathbf{R} \colon \sup_{h>0} \frac{1}{h} \int_{x\in[y,y+h]} |f(x)|\, dm(x) \right\}.$$

Proof. Consider the function

$$F(y) = \int_{x\in[a,y]} |f(x)|\, d\mu(x) - \lambda(y - a).$$

F is continuous by Proposition 10.2.4 so we can apply Proposition 10.2.3. There is therefore a countable set $\mathcal{P}^+$ of disjoint open intervals with the properties listed there. In particular, if $a \leq x \leq y \leq b$ and $x \notin \bigcup_{E\in\mathcal{P}} E$ then $F(x) \geq F(y)$. We can rewrite this inequality as

$$\int_{s\in[x,y]} |f(s)|\, dm(s) \leq \lambda(y - x).$$

Equivalently, if

$$\frac{1}{y - x} \int_{s\in[x,y]} |f(s)|\, dm(s) > \lambda$$

for some $y \in (x, b]$ then

$$x \in \bigcup_{E\in\mathcal{P}} E.$$

Another way of saying this is that if

$$\sup \frac{1}{h} \int_{s\in[x,x+h]} |f(s)|\, dm(s) > \lambda$$

then

$$x \in \bigcup_{E\in\mathcal{P}} E.$$

In terms of E_λ we have

$$E_\lambda \subseteq \bigcup_{E\in\mathcal{P}} E.$$

By countable additivity then

$$m(E_\lambda) \le \sum_{E \in \mathcal{P}} m(E).$$

If $(\alpha, \beta) \in \mathcal{P}$ then $F(\alpha) \le F(\beta)$, i.e.

$$\begin{aligned} \int_{s \in E} |f(s)|\, dm(s) &= \int_{s \in (\alpha,\beta)} |f(s)|\, dm(s) \\ &\ge \lambda(\beta - \alpha) = \lambda m(E). \end{aligned}$$

So

$$\begin{aligned} \sum_{E \in \mathcal{P}} m(E) &\le \frac{1}{\lambda} \sum_{E \in \mathcal{P}} \int_{s \in E} |f(s)|\, dm(s) \\ &= \frac{1}{\lambda}_{s \in \bigcup_{E \in \mathcal{P}} E} |f(s)|\, dm(s) \\ &\le \frac{1}{\lambda} \int_{s \in \mathbf{R}} |f(s)\, dm(s). \end{aligned}$$

□

Proposition 10.2.6. *Suppose $(X, \mathcal{T})$ is a locally compact Hausdorff space and μ is a Radon measure on X. If $f\colon X \to \mathbf{R}$ is integrable and $\epsilon > 0$ then there is a compactly supported continuous function $g\colon X \to \mathbf{R}$ such that*

$$\int_{x \in \mathbf{R}} |f(x) - g(x)|\, d\mu(x) < \epsilon.$$

Proof. f is integrable so

$$\overline{\int}_{x \in X} f(x)\, d\mu(x) \le \underline{\int}_{x \in X} f(x)\, d\mu(x)$$

and hence

$$\overline{\int}_{x \in X} f(x)\, d\mu(x) < \underline{\int}_{x \in X} f(x)\, d\mu(x) + \frac{\epsilon}{2}.$$

From the definition of the upper integral as an infimum there is therefore a semisimple function $h\colon X \to \mathbf{R}$ such that $f(x) \le v(x)$ for all $x \in X$ and

$$\begin{aligned} \int_{x \in X} h(x)\, d\mu(x) &< \underline{\int}_{x \in X} f(x)\, d\mu(x) + \frac{\epsilon}{2}. \\ &= \int_{x \in X} f(x)\, d\mu(x) + \frac{\epsilon}{2}. \end{aligned}$$

Then

$$\begin{aligned} \int_{x \in X} |h(x) - f(x)|\, d\mu(x) &= \int_{x \in X} (h(x) - f(x))\, d\mu(x) \\ &< \frac{\epsilon}{2}. \end{aligned}$$

h is semisimple so there is a countable partition $\mathcal{Q}$ of X and a function $\varphi\colon \mathcal{Q} \to \mathbf{R}$ such that $h(x) = \varphi(E)$ if $x \in E$. Let $E_0,\ E_1,\ \ldots$ be an enumeration of $\mathcal{Q}$. Let

$$\delta_j = \frac{\epsilon}{2^{j+2}(|\varphi(E)| + 1)}$$

μ is a Radon measure so there are compact K_j and open U_j such that

$$K_j \subseteq E_j \subseteq U_j,$$

$$\mu(E_j) < \mu(K_j) + \delta_j$$

and

$$\mu(U_j) < \mu(E_j) + \delta_j$$

By Proposition 9.3.1 there is a compactly supported continuous function $g_j\colon X \to [0,1]$ such that $g_j(x) = 1$ if $x \in K$ and $g_j(x) = 0$ if $x \in X \setminus U$. Then

$$\chi_{K_j}(x) \le g_j(x) \le \chi_{U_j}(x)$$

for all $x \in X$ and hence

$$\begin{aligned} \int_{x \in X} g_j(x) &\le \int_{U_j} (x)\, d\mu(x) = m(U_j) \\ &< \mu(E_j) + \delta_j = \int_{x \in E_j} \chi_{E_j}(x) + \delta_j \end{aligned}$$

and

$$\begin{aligned} \int_{x \in X} g_j(x) &\ge \int_{K_j} (x)\, d\mu(x) = m(K_j) \\ &> \mu(E_j) - \delta_j = \int_{x \in E_j} \chi_{E_j}(x) - \delta_j. \end{aligned}$$

Then

$$\int_{x \in K} |\varphi(E_j) g_j(x) - \varphi(E_j)\chi_{E_j}(x)|\, d\mu(x) < |\varphi(E_j)|\delta_j.$$

Now

$$\sum_{j=0}^{\infty} |\varphi(E_j)|\delta_j = \sum_{j=0}^{\infty} \frac{|\varphi(E_j)|}{|\varphi(E_j)| + 1} \frac{\epsilon}{2^{j+2}} < \frac{\epsilon}{2}.$$

So

$$\sum_{j=0}^{\infty} \int_{x\in K} |\varphi(E_j)g_j(x) - \varphi(E_j)\chi_{E_j}(x)|\, d\mu(x) < \frac{\epsilon}{2}.$$

It follows from the Monotone Convergence Theorem that

$$\sum_{j=0}^{\infty} |\varphi(E_j)g_j(x) - \varphi(E_j)\chi_{E_j}(x)|$$

is integrable. It then follows from the Dominated Convergence Theorem that

$$\sum_{j=0}^{\infty} (\varphi(E_j)g_j(x) - \varphi(E_j)\chi_{E_j}(x))$$

is integrable and that its integral is equal to

$$\sum_{j=0}^{\infty} \int_{x\in X} (\varphi(E_j)g_j(x) - \varphi(E_j)\chi_{E_j}(x))\, d\mu(x),$$

which is of absolute value less than $\epsilon/2$. Defining g by

$$g(x) = \sum_{j=0}^{\infty} \varphi(E_j)g_j(x)$$

we then have

$$\int_{x\in X} |g(x) - h(x)|\, d\mu(x) < \frac{\epsilon}{2}.$$

From this and the inequality

$$\int_{x\in X} |h(x) - f(x)|\, d\mu(x) < \frac{\epsilon}{2}.$$

obtained earlier we get

$$\int_{x\in X} |g(x) - f(x)|\, d\mu(x) < \frac{\epsilon}{2}.$$

□

Theorem 10.2.7. *Suppose $f\colon [a,b] \to \mathbf{R}$ is integrable. Then*

$$\lim_{h\searrow 0} \frac{1}{h} \int_{x\in[y,y+h]} f(x)\, dm(x) = f(y),$$

$$\lim_{h\searrow 0} \frac{1}{h} \int_{x\in[y-h,h]} f(x)\, dm(x) = f(y),$$

and

$$\lim_{h\searrow 0} \frac{1}{2h} \int_{x\in[y-h,y+h]} f(x)\, dm(x) = f(y)$$

for almost all $x \in [a,b]$.

Proof. By Proposition 10.2.6 there is a compactly supported continuous function $g\colon \mathbf{R} \to \mathbf{R}$ such that

$$\int_{x\in\mathbf{R}} |f(x) - g(x)|\, dm(x) < \epsilon.$$

Let

$$A_{\lambda,\epsilon} = \{x \in \mathbf{R} \colon |f(x) - g(x)| \geq \lambda\},$$
$$B_{\lambda,\epsilon} = \{x \in \mathbf{R} \colon k(s) \geq \lambda\},$$

where

$$k(x) = \sup_{h>0} \frac{1}{h} \int_{s\in[x,x+h]} |f(s) - g(s)|\, dm(s)$$

and

$$C_{\lambda,\epsilon} = \{x \in \mathbf{R} \colon |f(x) - g(x)| < \lambda, k(x) < \lambda\}.$$

By Proposition 10.2.1 we have

$$m(A_{\lambda,\epsilon}) < \frac{\epsilon}{\lambda}.$$

By Proposition 10.2.5 we have

$$m(B_{\lambda,\epsilon}) < \frac{\epsilon}{\lambda}.$$

Now

$$A_{\lambda,\epsilon} \cup B_{\lambda,\epsilon} \cup C_{\lambda,\epsilon} = \mathbf{R}$$

so

$$\mathbf{R} \setminus C_{\lambda,\epsilon} \subseteq A_{\lambda,\epsilon} \cup B_{\lambda,\epsilon}$$

and

$$\begin{aligned} m(\mathbf{R} \setminus C_{\lambda,\epsilon}) &\leq m(A_{\lambda,\epsilon} \cup B_{\lambda,\epsilon}) \\ &\leq m(A_{\lambda,\epsilon}) + m(B_{\lambda,\epsilon}) < 2\frac{\epsilon}{\lambda}. \end{aligned}$$

From Corollary 10.1.3 we get

$$\left| \frac{1}{h} \int_{s\in[x,x+h]} g(s)\, d\mu(s) - g(x) \right| < \lambda$$

for sufficiently small positive h. If $x \in C_{\lambda,\epsilon}$ then

$$|f(x) - g(x)| < \lambda$$

and

$$\frac{1}{h} \int_{s \in [x,x+h]} |f(s) - g(s)| \, dm(s) < \lambda$$

for all positive h. It then follows that

$$\left| \frac{1}{h} \int_{s \in [x,x+h]} f(s) \, d\mu(s) - f(x) \right| < 3\lambda$$

for sufficiently small positive h. Then

$$\limsup_{h>0} \left| \frac{1}{h} \int_{s \in [x,x+h]} f(s) \, d\mu(s) - f(x) \right| < 3\lambda.$$

This holds for $x \in C_{\lambda,\epsilon}$. Let

$$D_\lambda = \bigcup_{n=0}^{\infty} C_{\lambda,1/2^n}.$$

If $x \in D_\lambda$ then there is an $\epsilon > 0$ such that $x \in C_{\lambda,\epsilon}$ so

$$\limsup_{h>0} \left| \frac{1}{h} \int_{s \in [x,x+h]} f(s) \, d\mu(s) - f(x) \right| < 3\lambda$$

for all $x \in D_\lambda$. Now

$$\mathbf{R} \setminus D_\lambda = \bigcap_{n=0}^{\infty} \left(\mathbf{R} \setminus C_{\lambda,1/2^n} \right)$$

and

$$m\left(\mathbf{R} \setminus C_{\lambda,1/2^n}\right) < \frac{\lambda}{2^{n-1}} \to 0$$

as $n \to \infty$. It follows from Proposition 7.6.8 that that their intersection is null, i.e. that

$$m(\mathbf{R} \setminus D_\lambda) = 0.$$

Let

$$E = \bigcap_{n=0}^{\infty} D_{1/2^n}.$$

Then

$$\mathbf{R} \setminus E = \bigcup_{n=0}^{\infty} \left(\mathbf{R} \setminus D_{1/2^n} \right)$$

so

$$\mu(\mathbf{R} \setminus E) = 0.$$

If $x \in E$ then $x \in D_{1/2^n}$ for all n so

$$\limsup_{h>0} \left| \frac{1}{h} \int_{s \in [x,x+h]} f(s) \, d\mu(s) - f(x) \right| < \frac{3}{2^n}$$

for all n. This is possible only if

$$\limsup_{h>0} \left| \frac{1}{h} \int_{s \in [x,x+h]} f(s) \, d\mu(s) - f(x) \right| = 0$$

and therefore

$$\lim_{h \searrow 0} \left| \frac{1}{h} \int_{s \in [x,x+h]} f(s) \, d\mu(s) - f(x) \right| = 0,$$

or, equivalently,

$$\lim_{h \searrow 0} \frac{1}{h} \int_{s \in [x,x+h]} f(s) \, d\mu(s) = f(x)$$

for $x \in E$.

There is a similar argument for $[x-h, x]$ or, alternatively, we can apply the same argument to $\tilde{f}(x) = f(-x)$.

$$\lim_{h \searrow 0} \frac{1}{h} \int_{s \in [x-h,x]} f(s) \, d\mu(s) = f(x)$$

for x in some set E such that $m(\mathbf{R} \setminus E) = 0$. There is then a E such that $m(\mathbf{R} \setminus E) = 0$ for which both

$$\lim_{h \searrow 0} \frac{1}{h} \int_{s \in [x,x+h]} f(s) \, d\mu(s) = f(x)$$

and

$$\lim_{h \searrow 0} \frac{1}{h} \int_{s \in [x-h,x]} f(s) \, d\mu(s) = f(x)$$

for $x \in E$ and hence

$$\lim_{h \searrow 0} \frac{1}{2h} \int_{s \in [x-h,x+h]} f(s) \, d\mu(s) = f(x).$$

□

Corollary 10.2.8. *Suppose E is a measurable subset of $\mathbf{R}$. For almost all $x \in E$ we have*

$$\lim_{h \searrow 0} \frac{1}{2h} m(E \cap [x-h, x+h]) = \chi_E(x).$$

Proof. We first consider the special case where $m(E) < +\infty$. Applying Proposition 10.2.7 to the function χ_E shows that for almost all $x \in \mathbf{R}$ we have

$$\lim_{h \searrow 0} \frac{1}{2h} m(E \cap [x-h, x+h]) = \chi_E(x).$$

for almost all $x \in \mathbf{R}$. It $m(E) = +\infty$ then we apply the above argument to

$$E_n = (-n, n) \cap E.$$

This gives

$$\lim_{h \searrow 0} \frac{1}{2h} m((-n,n) \cap E \cap [x-h, x+h]) = \chi_{(-n,n)\cap E}(x).$$

for almost all $x \in E$. The exceptional set could be different for each n but the union of countably many null sets is still null so for almost all $x \in \mathbf{R}$ we have the equation above for all n. In particular it holds for n large enough that $x \in (-n, n)$. For such n we have

$$\chi_{(-n,n)\cap E}(x) = \chi_E(x).$$

Also,

$$(-n, n) \cap E \cap [x-h, x+h] = E \cap [x-h, x+h]$$

for sufficiently small h so we can replace the set on the left with the set on the right in the limit. In other words,

$$\lim_{h \searrow 0} \frac{1}{2h} m(E \cap [x-h, x+h]) = \chi_E(x).$$

□

The following is our analogue of Theorem 10.1.2 for Lebesgue integration.

Theorem 10.2.9. *Suppose $f \colon \mathbf{R} \to \mathbf{R}$ is integrable and*

$$F(y) = \int_{x \in (-\infty, y)} f(x)\, d\mu(x).$$

Then F is continuous. For almost all y in $\mathbf{R}$ F is differentiable at y and $F'(y) = f(y)$.

Proof.

$$\frac{F(z) - F(y)}{z - y} = \begin{cases} \frac{1}{h} \int_{x \in [y, y+h]} f(x)\, dm(x) & \text{if } z > y, \\ \frac{1}{h} \int_{x \in [y-h, y]} f(x)\, dm(x) & \text{if } z < y, \end{cases}$$

where $h = |z - y|$. The preceding theorem therefore shows that

$$\lim \frac{F(z) - F(y)}{z - y} = f(y).$$

□

10.3 The Second Fundamental Theorem

Proposition 10.3.1. *Suppose I is a non-empty interval and $F \colon I \to \mathbf{R}$ is Lipschitz continuous. Then there are Lipschitz continuous functions G and H such that G is monotone increasing, H is monotone decreasing and $F = G + H$.*

Proof. It's useful to introduce the quantities

$$V_+(a, b) = \sup \sum_{j=1}^{n} \max(0, F(x_j) - F(x_{j-1}))$$

$$V_-(a, b) = \inf \sum_{j=1}^{n} \min(0, F(x_j) - F(x_{j-1}))$$

$$V(a, b) = \sup \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})|$$

for $a, b \in I$ such that $a \leq b$, where the suprema and infima are over $x_0, x_1, \ldots, x_n$ are such that

$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b.$$

First of all, $V_+(a, b) \geq 0$, $V_-(a, b) \leq 0$ and $V(a, b) \geq 0$. This is clear because the sums in the definitions of V_+ and V are sums of non-negative terms while the sum in the definition of V_- is a sum of non-positive terms.

Next, if $a \leq b \leq c$ then

$$V_+(a, c) = V_+(a, b) + V_+(b, c),$$

$$V_-(a, c) = V_-(a, b) + V_-(b, c),$$

and
$$V(a,c) = V(a,b) + V(b,c).$$
We can see this as follows. All three equations hold trivially if $a = b$ so we'll assume from now on that $a < b$. If
$$a = x_0 \le x_1 \le \cdots \le x_n = b$$
and
$$b = y_0 \le y_1 \le \cdots \le y_p = c$$
then set $z_j = x_j$ if $j \le n$ and $z_j = y_{j-n}$ if $j > n$. Then
$$a = z_0 \le z_1 \le \cdots \le z_q = c$$
where $q = n + p$. Also,
$$\sum_{j=1}^{n} \max(0, F(x_j) - F(x_{j-1})) + \sum_{j=1}^{p} \max(0, F(y_j) - F(y_{j-1})) = \sum_{j=1}^{q} \max(0, F(z_j) - F(z_{j-1})).$$
Now
$$\sum_{j=1}^{q} \max(0, F(z_j) - F(z_{j-1})) \le V_+(a,c)$$
so
$$\sum_{j=1}^{n} \max(0, F(x_j) - F(x_{j-1})) + \sum_{j=1}^{p} \max(0, F(y_j) - F(y_{j-1})) \le V_+(a,c).$$
Taking the supremum over all possible choices of the x's gives
$$V_+(a,b) + \sum_{j=1}^{p} \max(0, F(y_j) - F(y_{j-1})) \le V_+(a,c).$$
Taking the supremum over all possible choices of the y's then gives
$$V_+(a,b) + V_+(b,c) \le V_+(a,c).$$
Suppose now that $z_0, z_1, \ldots, z_q$ are such that
$$a = z_0 \le z_1 \le \cdots \le z_q = c.$$
Let n be the smallest integer such that $z_n \ge b$. Define $x_j = z_j$ if $j < n$ and $x_n = b$. Define $y_0 = b$ and $y_j = z_{n+j-1}$ if $j > 0$. Then
$$a = x_0 \le x_1 \le \cdots \le x_n = b$$
and
$$b = y_0 \le y_1 \le \cdots \le y_p = c,$$
where $p = q - n + 1$. For any real numbers u, v and w such that $u + v = w$ we have
$$\max(0,u) + \max(0,v) \ge \max(0,w).$$
We apply this to
$$u = F(x_n) - F(x_{n-1})$$
$$v = F(y_1) - F(y_0)$$
$$w = F(z_n) - F(z_{n-1}).$$
Note that $n - 1 \ge 0$ since $a < b$. The equation $u + v = w$ holds because $x_n = b = y_0$, $x_{n-1} = z_{n-1}$ and $y_1 = z_n$. So
$$\max(0, F(z_n) - F(z_{n-1})) \le \max(0, F(x_n) - F(x_{n-1})) + \max(0, F(y_1) - F(y_0)).$$
To this inequality we add the equations
$$\sum_{j=1}^{n-1} \max(0, F(x_j) - F(x_{j-1}) = \sum_{j=1}^{n-1} \max(0, F(z_j) - F(z_{j-1})$$
and
$$\sum_{j=2}^{q} \max(0, F(y_j) - F(y_{j-1})) = \sum_{j=n+1}^{q} \max(0, F(z_j) - F(z_{j-1}))$$

to get

$$\sum_{j=1}^{q} \max(0, F(z_j) - F(z_{j-1}))$$
$$\leq \sum_{j=1}^{n} \max(0, F(x_j) - F(x_{j-1}))$$
$$+ \sum_{j=1}^{p} \max(0, F(y_j) - F(y_{j-1})).$$

Now

$$\sum_{j=1}^{n} \max(0, F(x_j) - F(x_{j-1})) \leq V_+(a, b)$$

and

$$\sum_{j=1}^{p} \max(0, F(y_j) - F(y_{j-1})) \leq V_+(b, c)$$

so

$$\sum_{j=1}^{q} \max(0, F(z_j) - F(z_{j-1})) \leq V_+(a, b) + V_+(b, c).$$

Taking the supremum over all possible choices of the z's gives

$$V_+(a, c) \leq V_+(a, b) + V_+(b, c).$$

Since we already have the reverse inequality we get

$$V_+(a, c) = V_+(a, b) + V_+(b, c).$$

The proofs of the corresponding results for V_- and V are similar, with minor changes.

From $V_+(a, c) = V_+(a, b) + V_+(b, c)$ and $V_+(a, b) \geq 0$ and $V_+(b, c) \geq$ we get that $V_+(a, c) \geq V_+(a, b)$ and $V_+(a, c) \geq V_+(b, c)$. So $V_+(a, c)$ is monotone increasing as a function of c and monotone decreasing as a function of a. The same is true for V, while $V_-(a, c)$ is monotone decreasing as a function of c and monotone increasing as a function of a.

Next we note that

$$|F(x_j) - F(x_{j-1})|. = \max(0, F(x_j) - F(x_{j-1}) - \min(0, F(x_j) - F(x_{j-1}).$$

Summing over j and taking suprema over all possible choices of the x's gives

$$V_+(a, b) - V_-(a, b) = V(a, b).$$

Note that the supremum of

$$-\sum_{j=1}^{n} \min(0, F(x_j) - F(x_{j-1}))$$

is minus the infimum of the same sum, i.e. $-V_-(a, b)$.

Next we show that

$$V_+(a, b) + V_-(a, b) = F(b) - F(a).$$

For each $\epsilon > 0$ there is, by the definition of the supremum, a choice of $x_0, \ldots, x_n$ such that

$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b$$

such that

$$V_+(a, b) - \epsilon < \sum_{j=1}^{n} \max(0, F(x_j) - F(x_{j-1})).$$

Similarly, there is, by the definition of the infimum, a choice of

$$a = y_0 \leq y_1 \leq \cdots \leq y_p = b$$

Let $z_0, z_1, \ldots, z_q$ be the x's and y's sorted into increasing order. Then

$$a = z_0 \leq z_1 \leq \cdots \leq z_q = b.$$

For any j there are k and l such that $x_{j-1} = z_k$ and $x_j = z_l$. Then

$$F(x_j) - F(x_{j-1}) = \sum_{i=k+1}^{l} \left(F(z_i) - F(z_{i-1})\right).$$

The inequality

$$\max(0, u) + \max(0, v) \geq \max(0, w)$$

used previously can be extended by induction to finite sums. From this and the equation above we get

$$\max(0, F(x_j) - F(x_{j-1}))$$
$$\leq \sum_{i=k+1}^{l} \max\left(0, F(z_i) - F(z_{i-1})\right).$$

Summing over i gives

$$\sum_{j=1}^{n} \max(0, F(x_j) - F(x_{j-1})) \\ \leq \sum_{j=1}^{q} \max(0, F(z_j) - F(z_{j-1})).$$

The sum on the right hand side is bounded by $V_+(a,b)$ by the definition of the supremum. Combining what we have so far we obtain

$$V_+(a,b) - \epsilon < \sum_{j=1}^{q} \max(0, F(z_j) - F(z_{j-1})) \leq V_+(a,b).$$

A similar argument gives

$$V_-(a,b) \leq \sum_{j=1}^{q} \min(0, F(z_j) - F(z_{j-1})) < V_-(a,b) + \epsilon.$$

Now

$$F(z_j) - F(z_{j-1}) = \max(0, F(z_j) - F(z_{j-1})) \\ + \min(0, F(z_j) - F(z_{j-1})).$$

Summing over j and using the inequalities obtained earlier we find

$$V_-(a,b) + V_+(a,b) - \epsilon < F(b) - F(a) \\ < V_-(a,b) + V_+(a,b) + \epsilon.$$

This holds for all $\epsilon > 0$ so

$$V_-(a,b) + V_+(a,b) = F(b) - F(a).$$

F is Lipschitz continuous so there is a $K \geq 0$ such that

$$|F(x) - F(y)| \leq K|x - y|$$

for all x and y in I. In particular, if $x_{j-1} \leq x_j$ then

$$|F(x_j) - F(x_{j-1})| \leq K|x_j - x_{j-1}| = K(x_j - x_{j-1}).$$

If

$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b$$

then

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \leq K \sum_{j=1}^{n} (x_j - x_{j-1}) = K(b-a)$$

Taking suprema we find that

$$V(a,b) \leq K(x_j - x_{j-1}).$$

From

$$V_+(a,b) \geq 0, \\ V_-(a,b) \leq 0$$

and

$$V(a,b) = V_+(a,b) - V_-(a,b)$$

it follows that

$$0 \leq V_+(a,b) \leq K(b-a)$$

and

$$-K(b-a) \leq V_-(a,b) \leq 0.$$

I is non-empty so there is a $w \in I$. Define

$$G(x) = \begin{cases} \frac{1}{2}F(w) + V_+(w,x) & \text{if } x \geq w, \\ \frac{1}{2}F(w) - V_+(x,w) & \text{if } x < w, \end{cases}$$

and

$$H(x) = \begin{cases} \frac{1}{2}F(w) + V_-(w,x) & \text{if } x \geq w, \\ \frac{1}{2}F(w) - V_-(x,w) & \text{if } x < w, \end{cases}$$

Then

$$G(y) - G(x) = \begin{cases} V_+(x,y) & \text{if } x \leq y, \\ -V_+(y,x) & \text{if } x > y, \end{cases}$$

and

$$H(y) - H(x) = \begin{cases} V_-(x,y) & \text{if } x \leq y, \\ -V_-(y,x) & \text{if } x > y. \end{cases}$$

There are a number of cases to consider depending on the ordering of w, x and y but all of them are immediate consequences of the definitions and the fact that

$$V_+(a,c) = V_+(a,b) + V_+(b,c)$$

and

$$V_-(a,c) = V_-(a,b) + V_-(b,c).$$

so

$$G(y) + H(y) - G(x) - H(x) = F(y) - F(x)$$

We apply this with $x = w$ and use the identity

$$F(w) = G(w) + H(w),$$

which follows immediately from the definitions, to get

$$F(y) = G(y) + H(y)$$

for all $y \in I$.

From

$$0 \le V_+(a, b) \le K(b - a)$$

and

$$-K(b - a) \le V_-(a, b) \le 0$$

we get that

$$0 \le G(b) - G(a) \le K(b - a)$$

and

$$-K(b - a) \le H(b) - H(a) \le 0$$

for all $a, b \in I$ such that $a \le b$. From these inequalities we conclude that G is monotone increasing, H is monotone decreasing, and both are Lipschitz continuous with Lipschitz constant K. □

It's convenient to introduce substitutes for the derivative for functions which may not have them.

Definition 10.3.2. Suppose F is a function from a neighbourhood of $x \in \mathbf{R}$ to $\mathbf{R}$. The *Dini derivatives* of F at x are

$$D^+F(x) = \limsup_{h \searrow 0} \frac{F(x+h) - F(x)}{h},$$

$$D_+F(x) = \liminf_{h \searrow 0} \frac{F(x+h) - F(x)}{h},$$

$$D^-F(x) = \limsup_{h \nearrow 0} \frac{F(x+h) - F(x)}{h}$$

and

$$D_-F(x) = \liminf_{h \nearrow 0} \frac{F(x+h) - F(x)}{h}.$$

If F is defined in a neighbourhood of x then all four Dini derivatives exist as elements of $[-\infty, +\infty]$, even if f is not differentiable, or even continuous.

Proposition 10.3.3. *Suppose F is a function from an open interval in $\mathbf{R}$ to $\mathbf{R}$. Then $D_+F(x) \le D^+F(x)$ and $D_-F(x) \le D^-F(x)$. If F is monotone increasing then $0 \le D_+F(x) \le D^+F(x)$ and $0 \le D_-F(x) \le D^-F(x)$.*

Proof. The lim inf is always less than the lim sup so $D_+F(x) \le D^+F(x)$ and $D_-F(x) \le D^-F(x)$. If F is monotone increasing then

$$\frac{F(x+h) - F(x)}{h} \ge 0$$

for all $h > 0$ so

$$D_+F(x) = \liminf_{h \searrow 0} \frac{F(x+h) - F(x)}{h} \ge 0$$

but also

$$\frac{F(x+h) - F(x)}{h} \ge 0$$

for all $h < 0$ so

$$D_-F(x) = \liminf_{h \nearrow 0} \frac{F(x+h) - F(x)}{h} \ge 0.$$

□

Proposition 10.3.4. *F is differentiable at F if and only if*

$$D^+F(x) = D_+F(x) = D^-F(x) = D_-F(x) \in \mathbf{R}.$$

Proof.

$$\lim_{h \searrow 0} \frac{F(x+h) - F(x)}{h}$$

exists if and only if the $\liminf_{h \searrow 0}$ and $\limsup h \searrow 0$ are finite and equal, i.e. if and only if $D_+F(x) = D^+F(x) \in \mathbf{R}$. It is then equal to their common value. Similarly,

$$\lim_{h \nearrow 0} \frac{F(x+h) - F(x)}{h}$$

exists if and only if the $\liminf_{h \nearrow 0}$ and $\limsup h \nearrow 0$ are finite and equal, i.e. if and only if $D_-F(x) = D^-F(x) \in \mathbf{R}$. It is then equal to their common value. Finally,

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

exists if and only if $\lim_{h \searrow 0}$ and $\lim_{h \nearrow 0}$ exist and are equal. □

Proposition 10.3.5. *If I is an interval and $F\colon I \to \mathbf{R}$ is continuous then the Dini derivatives of F are all measurable.*

In fact the continuity assumption is not needed, but it makes the proof considerably simpler.

Proof. It suffices to prove this for one of the four derivatives. The other three can then be obtained by replacing $F(x)$ by $F(-x)$, $-F(x)$ or $-F(-x)$. We'll prove it for DF^+.

$$\begin{aligned} DF^+(x) &= \limsup_{h\to 0} \frac{F(x+h)-F(x)}{h} \\ &= \inf_{k\in(0,+\infty)} \sup_{h\in(0,k)} \frac{F(x+h)-F(x)}{h}. \end{aligned}$$

Let

$$y = \sup_{h\in(0,k)\cap\mathbf{Q}} \frac{F(x+h)-F(x)}{h}.$$

Then

$$y \le \sup_{h\in(0,k)} \frac{F(x+h)-F(x)}{h}$$

because $(0,k)\cap\mathbf{Q}$ is a subset of $(0,k)$. Suppose

$$y < \sup_{h\in(0,k)} \frac{F(x+h)-F(x)}{h}.$$

Then there is an $h\in(0,k)$ such that

$$y < \frac{F(x+h)-F(x)}{h}.$$

Another way to say this is that the set

$$\{h\in(0,k)\colon \frac{F(x+h)-F(x)}{h} > y\}$$

is non-empty. It's open by the continuity of F and every non-empty open subset of the reals contains a rational, so there is an $h\in(0,k)\cap\mathcal{Q}$ such that

$$y < \frac{F(x+h)-F(x)}{h}$$

But this contradicts the definition of y, so the assumption that

$$y < \sup_{h\in(0,k)} \frac{F(x+h)-F(x)}{h}$$

was false and therefore

$$y = \sup_{h\in(0,k)} \frac{F(x+h)-F(x)}{h},$$

i.e.

$$\sup_{h\in(0,k)\cap\mathbf{Q}} \frac{F(x+h)-F(x)}{h}, = \sup_{h\in(0,k)} \frac{F(x+h)-F(x)}{h}.$$

So

$$\begin{aligned} DF^+(x) &= \inf_{k\in(0,+\infty)} \sup_{h\in(0,k)} \frac{F(x+h)-F(x)}{h}. \\ &= \inf_{k\in(0,+\infty)} \sup_{h\in(0,k)\cap\mathbf{Q}} \frac{F(x+h)-F(x)}{h}. \end{aligned}$$

Let

$$z = \inf_{k\in(0,+\infty)\cap\mathbf{Q}} \sup_{h\in(0,k)\cap\mathbf{Q}} \frac{F(x+h)-F(x)}{h}.$$

$(0,+\infty)\cap\mathbf{Q}$ is a subset of $(0,+\infty)$ so

$$z \ge \inf_{k\in(0,+\infty)} \sup_{h\in(0,k)\cap\mathbf{Q}} \frac{F(x+h)-F(x)}{h}.$$

If

$$z > \inf_{k\in(0,+\infty)} \sup_{h\in(0,k)\cap\mathbf{Q}} \frac{F(x+h)-F(x)}{h}$$

then there is a $k\in(0+\infty)$ such that

$$z > \sup_{h\in(0,k)\cap\mathbf{Q}} \frac{F(x+h)-F(x)}{h}$$

Every non-empty open subset of the reals contains a rational number so there is a $j\in(0,k)\cap\mathbf{Q}$. Then

$$(0,j)\cap\mathbf{Q} \subseteq (0,k)\cap\mathbf{Q}$$

so

$$\sup_{h\in(0,j)\cap\mathbf{Q}} \frac{F(x+h)-F(x)}{h} \le \sup_{h\in(0,k)\cap\mathbf{Q}} \frac{F(x+h)-F(x)}{h}$$

and therefore

$$z > \sup_{h\in(0,j)\cap\mathbf{Q}} \frac{F(x+h)-F(x)}{h}.$$

But this contradicts the definition of z, so

$$z = \inf_{k\in(0,+\infty)} \sup_{h\in(0,k)\cap\mathbf{Q}} \frac{F(x+h)-F(x)}{h}.$$

In other words,

$$D^+F(x) = \inf_{k\in(0,+\infty)\cap\mathbf{Q}} \sup_{h\in(0,k)\cap\mathbf{Q}} \frac{F(x+h)-F(x)}{h}.$$

Now $\frac{F(x+h)-F(x)}{h}$ is a measurable function of x for each h and Proposition 8.5.8 tells us that infima and suprema of countable sets of measurable functions are measurable so D^+F is measurable. □

Now that we know that the Dini derivatives are measurable it makes sense to try to estimate the measure of the set on which they are larger than a given value. This will require a preliminary lemma.

Lemma 10.3.6. *Suppose that $\mathcal{A}$ is a finite set of intervals in $\mathbf{R}$. Then there is a $\mathcal{B}\subseteq\mathcal{A}$ such that*

(a) $\bigcup_{I\in\mathcal{A}} = \bigcup_{J\in\mathcal{B}}$, *and*

(b) for all $x\in\mathbf{R}$ the set $\{J\in\mathcal{B}\colon x\in J\}$ has at most two elements.

Proof. Let n be the number of elements in $\mathcal{A}$. Define a finite sequence $\mathcal{C}_k$ as follows. $\mathcal{C}_0=\mathcal{A}$. For $k\geq 0$ we terminated the sequence with $\mathcal{C}_k$ if no element of C_k is contained in the union of two other elements. If there is an $I\in\mathcal{C}_k$ such that I is contained in the union of two other elements of $\mathcal{C}_k$ then we set $\mathcal{C}_{k+1}=\mathcal{C}_k\setminus\{I\}$. The intervals we remove belong to the union of the ones which remain so

$$\bigcup_{I\in\mathcal{C}_k} I = \bigcup_{I\in\mathcal{C}_{k+1}} I$$

for each k and hence, by induction,

$$\bigcup_{I\in\mathcal{C}_k} I = \bigcup_{I\in\mathcal{A}} I.$$

An even easier induction shows that $\mathcal{C}_k\subseteq\mathcal{A}$ for all k and that the number of elements in $\mathcal{C}_k$ is $n-k$. From this last fact it follows that the sequence ends with some $\mathcal{C}_m$ with $m\leq n$. Call this final element $\mathcal{B}$. No element of $\mathcal{B}$ is contained in the union of two other elements.

Suppose $x\in I$, $x\in J$ and $x\in K$ for distinct $I,J,K\in\mathcal{C}_k$. Let

$$L = I\cup J\cup K,$$

$$a=\inf L$$

and

$$b=\sup L.$$

a could be finite or $-\infty$ and b could be finite or $+\infty$. In any case there is an $M\in\{I,J,K\}$ such that $\inf M=a$ and an $N\in\{I,J,K\}$ such that $\sup N=b$. If $y\in(a,x]$ then $y\in M$. If $y\in[x,b)$ then $y\in N$. If $x\in L$ then $x\in(a,b)$ so $x\in M$ or $x\in N$, i.e. $x\in M\cup N$. So

$$I\cup J\cup K = L\subseteq M\cup N.$$

I, J and K were distinct so there is an $H\in\{I,J,K\}$ such that $H\neq M$ and $H\neq N$. Then

$$H\subseteq I\cup J\cup K\subseteq M\cup N.$$

so one element of $\mathcal{C}_k$ is contained in to the union of two others. Thus $k\neq m$. Equivalently, if $\| = m$ then we can't have $x\in I$, $x\in J$ and $x\in K$ for distinct $I,J,K\in\mathcal{C}_k=\mathcal{B}$. So each x is an element of at most two elements of $\mathcal{B}$. □

Proposition 10.3.7. *Suppose $F\colon[a,b]\to\mathbf{R}$ is monotone increasing and $\lambda>0$. Then there is a $C>0$ such that*

$$m(E^+)\leq C\frac{F(b)-F(a)}{\lambda},$$

$$m(E_+)\leq C\frac{F(b)-F(a)}{\lambda},$$

$$m(E^-)\leq C\frac{F(b)-F(a)}{\lambda},$$

and

$$m(E_-)\leq C\frac{F(b)-F(a)}{\lambda}$$

where

$$E^+=\left\{x\in[a,b]\colon D^+F(x)\geq\lambda\right\},$$

$$E_+ = \{x \in [a,b] \colon D_+F(x) \ge \lambda\},$$
$$E^- = \left\{x \in [a,b] \colon D^-F(x) \ge \lambda\right\},$$

and

$$E_- = \{x \in [a,b] \colon D_-F(x) \ge \lambda\}.$$

If F is continuous then the inequalities above hold with $C = 1$.

Proof. It's sufficient to prove and one of these since the other three can then be obtained from that one by applying it to $F(-x)$, $-F(x)$ and $-F(-x)$. We'll prove the one for E^+.

Suppose K is a compact subset of E^+ and $\kappa \in (0,1)$. For each $x \in K$ we have

$$D^+F(x) = \limsup_{h \searrow 0} \frac{F(x+h) - F(x)}{h} \ge \lambda.$$

Since $\kappa\lambda < \lambda$ there is an $h > 0$ such that

$$\frac{F(x+h) - F(x)}{h} > \kappa\lambda.$$

Let

$$I_{x,h} = \left(\frac{2\kappa x + \kappa h - h}{2\kappa}, \frac{2\kappa x + \kappa h + h}{2\kappa}\right).$$

Then $x \in I_{x,h}$ and

$$\ell(I_{x,h}) = \frac{h}{\kappa}.$$

We can therefore rewrite the inequality above as

$$\ell(I_{x,h}) \le \frac{F(x+h) - F(x)}{\kappa^2\lambda}.$$

The sets $I_{x,h}$ as above form an open cover of K so they have a finite subcover. In other words, there are $x_1, \ldots, x_n$ and $h_1, \ldots, h_n$ such that

$$K \subseteq \bigcup_{j=1}^{n} I_{x_j,h_j}$$

and

$$\ell(I_{x_j,h_j}) \le \frac{F(x_j+h_j) - F(x_j)}{\kappa^2\lambda}.$$

By Lemma 10.3.6 we can assume, without loss of generality, that each $x \in K$ belongs to at most two of the intervals I_{x_j,h_j}. By countable subadditivity we have

$$m(K) \le \sum_{j=1}^{n} m(I_{x_j,h_j}) = \sum_{j=1}^{n} \ell(I_{x_j,h_j}).$$

On the other hand

$$\sum_{j=1}^{n} (F(x_j+h_j) - F(x_j)) \le 2(F(b) - F(a))$$

so

$$m(K) \le 2\frac{F(b) - F(a)}{\kappa^2\lambda}.$$

This holds for all $\kappa \in (0,1)$ so

$$m(K) \le 2\frac{F(b) - F(a)}{\lambda}.$$

Lebesgue measure is a Radon measure so

$$\mu(E^+) = \sup m(K)$$

where the supremum is over all compact subsets K of E^+, all of which satisfy the inequality above, so

$$m(E^+) \le 2\frac{F(b) - F(a)}{\lambda},$$

which is what we were looking for, with $C = 2$.

To get $C = 1$ in the continuous case we need a different argument. Define G by

$$G(x) = F(x) - \lambda x.$$

If F is continuous then so is G. We can therefore apply Proposition 10.2.3 to get a set $\mathcal{P}^+$ of disjoint open intervals contained in $[a,b]$ such that

$$G(\alpha) \le G(\beta)$$

for each $(\alpha, \beta) \in \mathcal{P}^+$ and

$$G(x) \ge G(y)$$

if $a \le x \le y \le b$ and $x \notin \bigcup_{E \in \mathcal{P}^+}$. In terms of F these inequalities can be written as

$$\frac{F(\beta) - F(\alpha)}{\beta - \alpha} \ge \lambda$$

and
$$\frac{F(y)-F(x)}{y-x} \le \lambda.$$
If $x \in (a,b)$ and $x \notin \bigcup_{E\in\mathcal{P}^+}$ then
$$\frac{F(y)-F(x)}{y-x} \le \lambda$$
for all $y \in (x,b]$ and so
$$D^+F(x) = \limsup_{h\searrow 0} \frac{F(y)-F(x)}{y-x} \le \lambda.$$
Equivalently, if $D^+F(x) > \lambda$ then $x \in \bigcup_{E\in\mathcal{P}^+}$, i.e. $x \in (\alpha,\beta)$ where $(\alpha,\beta) \in \mathcal{P}^+$. Therefore
$$\{x \in (a,b) \colon D^+F(x) > \lambda\} \subseteq \bigcup_{E\in\mathcal{P}^+} E$$
so
$$m\left(\{x \in (a,b) \colon D^+F(x) > \lambda\}\right) \le \sum_{E\in\mathcal{P}^+} m(E).$$
If $(\alpha,\beta) \in \mathcal{P}^+$ then
$$\frac{F(\beta)-F(\alpha)}{\beta-\alpha} \ge \lambda$$
then
$$m((\alpha,\beta)) = \beta-\alpha \le \frac{F(\beta)-F(\alpha)}{\lambda}$$
so
$$m\left(\{x \in (a,b) \colon D^+F(x) > \lambda\}\right) \le \sum_{(\alpha,\beta)\in\mathcal{P}^+} \frac{F(\beta)-F(\alpha)}{\lambda}$$
Any finite subset of $\mathcal{P}^+$ can be ordered as (α_1,β_1), …, (α_n,β_n) such that
$$a \le \alpha_1 \le \beta_1 \le \cdots \le \alpha_n, \beta_n \le b.$$
F was assumed to be monotone increasing so
$$F(\alpha_1) - F(a) \ge 0$$
$$\sum_{j=1}^{n-1} (F(\alpha_{j+1} - F(\beta_j)) \ge 0$$
and
$$F(b) - F(\beta_n) \ge 0$$
Adding the left hands of these inequalities to $\sum_{j=1}^{n}(F(\beta_j) - F(\alpha_j)$ gives $F(b)-F(a)$ since all the remaining terms cancel in pairs. From this we see that
$$\sum_{j=1}^{n}(F(\beta_j) - F(\alpha_j) \le F(b) - F(a).$$
Taking the supremum over all finite subsets of $\mathcal{P}^+$ gives
$$\sum_{(\alpha,\beta)\in\mathcal{P}^+} \frac{F(\beta)-F(\alpha)}{\lambda} \le \frac{F(b)-F(a)}{\lambda}$$
and therefore
$$m\left(\{x \in (a,b) \colon D^+F(x) > \lambda\}\right) \le \sum_{(\alpha,\beta)\in\mathcal{P}^+} \frac{F(\beta)-F(\alpha)}{\lambda}.$$
Now $\{a,b\}$ is of measure zero so we can immediately improve this to
$$m\left(\{x \in [a,b] \colon D^+F(x) > \lambda\}\right) \le \sum_{(\alpha,\beta)\in\mathcal{P}^+} \frac{F(\beta)-F(\alpha)}{\lambda}.$$
This is still not quite the statement of the proposition though because we have "$> \lambda$" here rather than "$\ge \lambda$". To get the version as it appears there we take a $\kappa \in (0,\lambda)$ and apply the argument above with κ in place of λ, obtaining
$$m\left(\{x \in [a,b] \colon D^+F(x) > \kappa\}\right) \le \sum_{(\alpha,\beta)\in\mathcal{P}^+} \frac{F(\beta)-F(\alpha)}{\kappa}.$$
Now
$$\{x \in [a,b] \colon D^+F(x) \ge \lambda\} \subseteq \{x \in [a,b] \colon D^+F(x) > \kappa\}$$
so the measure of the left hand side is less than or equal to that of the right hand side. It then follows

that

$$m\left(\{x \in [a,b] \colon D^+F(x) \geq \lambda\}\right) \leq \sum_{(\alpha,\beta)\in\mathcal{P}^+} \frac{F(\beta)-F(\alpha)}{\kappa}.$$

This holds for all $\kappa \in (0,\lambda)$ so

$$m\left(\{x \in [a,b] \colon D^+F(x) \geq \lambda\}\right) \leq \sum_{(\alpha,\beta)\in\mathcal{P}^+} \frac{F(\beta)-F(\alpha)}{\lambda}.$$

□

Proposition 10.3.8. *Suppose I is a non-empty interval and $F \colon I \to \mathbf{R}$ is continuous and monotone. Then F is differentiable at x for almost all $x \in X$.*

Proof. If $D_-F(x) < D^+F(x)$ then there are rational numbers p and q such that

$$D_-F(x) < p < q < D^+F(x).$$

In other words,

$$\{x \in I \colon D_-F(x) < D^+F(x)\} \subseteq \bigcup_{\substack{(p,q)\in\mathbf{Q}^2\\ p<q}} A_{p,q},$$

where

$$A_{p,q} = \{y \in \mathbf{R} \colon D_-F(y) < p < q < D^+F(y)\}$$

Suppose $[a,b] \subseteq I$. We apply Proposition 10.2.3 to the function

$$G(x) = F(x) - px.$$

This gives a countable set $\mathcal{P}^-$ of disjoint open intervals such that if $(\alpha,\beta) \in \mathcal{P}^-$ then

$$G(\alpha) \geq G(\beta)$$

and if $a \leq x \leq y \leq b$ and $y \notin \bigcup_{E\in\mathcal{P}^-}$ then

$$G(x) \leq G(y)$$

In terms of F these inequalities are

$$\frac{F(\beta)-F(\alpha)}{\beta-\alpha} \leq p$$

and

$$\frac{F(y)-F(x)}{y-x} \geq p$$

If $y \in (a,b)$ and $y \notin \bigcup_{E\in\mathcal{P}^-} E$ then

$$\frac{F(y)-F(x)}{y-x} \geq p$$

for all $x \in [a,y)$ then

$$D_-F(y) \geq p$$

so

$$y \notin A_{p,q}.$$

Equivalently, if $y \in A_{p,q}$ then $y \in \bigcup_{E\in\mathcal{P}^-} E$, i.e. there is an interval $(\alpha,\beta) \in \mathcal{P}^-$ such that $y \in (\alpha,\beta)$. Then

$$\frac{F(\beta)-F(\alpha)}{\beta-\alpha} \leq p$$

$$m(A_{p,q} \cap (\alpha,\beta)) \leq \frac{F(\beta)-F(\alpha)}{q}$$

by Proposition 10.3.7. so

$$m(A_{p,q} \cap (\alpha,\beta)) \leq \frac{p}{q}(\beta-\alpha) = \frac{p}{q}m(\alpha,\beta).$$

Now

$$A_{p,q} = \bigcup_{(\alpha,\beta)\in\mathcal{P}^-} A_{p,q} \cap (\alpha,\beta)$$

and this is a disjoint union, so

$$m(A_{p,q}) \cap (a,b) = \sum_{(\alpha,\beta)\in\mathcal{P}^-} m(A_{p,q} \cap (\alpha,\beta)).$$

Also,

$$\bigcup_{(\alpha,\beta)\in\mathcal{P}^-} (\alpha,\beta) \subseteq (a,b)$$

and this union is also disjoint so

$$\sum_{(\alpha,\beta)\in\mathcal{P}^-} m((\alpha,\beta)) \leq m((a,b)).$$

Combining all of these, we have

$$m(A_{p,q} \cap (a,b)) \leq \frac{p}{q}m((a,b)).$$

In particular, if $x \in I^\circ$ then

$$\frac{1}{2h} m(A_{p,q} \cap (x-h, x+h)) \leq \frac{p}{q}$$

for sufficiently small positive h. We can write this as

$$\frac{1}{2h} \int_{x \in (x-h,x+h)} \chi_{A_{p,q}}(s)\, ds.$$

The limit as $h \searrow 0$ is $\chi_{A_{p,q}}(x)$ for almost all x by Theorem 10.2.7. On the other hand we must have

$$\lim_{h \searrow 0} \frac{1}{2h} m(A_{p,q} \cap (x-h, x+h)) \leq \frac{p}{q} < 1$$

so

$$\chi_{A_{p,q}}(x) < 1$$

for almost all x. In other words, $x \notin A_{p,q}$ for almost all x, or $m(A_{p,q}) = 0$. As we saw earlier,

$$\{x \in I \colon D_- F(x) < D^+ F(x)\} \subseteq \bigcup_{\substack{(p,q) \in \mathbf{Q}^2 \\ p<q}} A_{p,q}.$$

This is a countable union, so

$$m\left(\{x \in I \colon D_- F(x) < D^+ F(x)\}\right) = 0.$$

A similar argument with

$$B_{p,q} = \{y \in \mathbf{R} \colon D_+ F(y) < p < q < D^- F(y)\}$$

and $\mathcal{P}^+$ gives

$$m\left(\{x \in I \colon D_+ F(x) < D^- F(x)\}\right) = 0.$$

So for almost all $x \in I$ we have

$$D_- F(x) \geq D^+ F(x)$$

and

$$D_+ F(x) \geq D^- F(x).$$

On the other hand, Proposition 10.3.3 gives

$$D_- F(x) \leq D^- F(x)$$

and

$$D_+ F(x) \geq D^+ F(x)$$

so

$$D^+ F(x) = D_+ F(x) = D^- F(x) = D_- F(x)$$

for almost all $x \in I$. It follows from Proposition 10.3.7 that they are finite for almost all x as well. By Proposition 10.3.4 it then follows that F is differentiable for almost all $x \in I$. □

Theorem 10.3.9. *Suppose I is a non-empty interval and $F \colon I \to \mathbf{R}$ is Lipschitz continuous. Let E be the set of x such that $F'(x)$ is differentiable at x. Then $m(I \setminus E) = 0$, F' is integrable on $E \cap [a,b]$ for all $a, b \in I$ and*

$$\int_{x \in E \cap [a,b]} F'(x)\, dm(x) = F(b) - F(a).$$

Proof. From Propositions 10.3.1 and Proposition 10.3.8 we know that F is differentiable almost everywhere. Choose a sequence h_0, h_1, ... of positive numbers with $\lim_{j \to \infty = 0}$ and define

$$f_j(x) = \frac{F(x+h_j) - F(x)}{h_j}.$$

This will require us to evaluate F at points outside of $[a,b]$ so we extend F by defining $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$. It's easy to see that this extension is also Lipschitz continuous. Then

$$\lim_{j \to \infty} f_j(x) = F'(x)$$

for all x at which F is differentiable, i.e. almost all x. Also, if $K \geq 0$ is a Lipschitz constant then

$$|f_j(x)| \leq K$$

for all x. We also has

$$\int_{x \in [a,b]} K\, dm(x) = K(b-a) < +\infty.$$

We can therefore apply the Dominated Convergence Theorem to get

$$\lim_{j \to \infty} \int_{x \in [a,b]} f_j(x)\, dm(x) = \int_{x \in [a,b]} F'(x)\, dm(x).$$

Now

$$\begin{aligned}
\int_{x\in[a,b]} f_j(x)\,dm(x) &= \int_{x\in[a,b]} \frac{F(x+h_j)-F(x)}{h_j}\,dm(x) \\
&= \frac{1}{h_j}\left(\int_{x\in[a,b]} F(x+h_j)\,dm(x)\right. \\
&\qquad \left. - \int_{x\in[a,b]} F(x)\,dm(x)\right) \\
&= \frac{1}{h_j}\left(\int_{x\in[a+h_j,b+h_j]} F(x)\,dm(x)\right. \\
&\qquad \left. - \int_{x\in[a,b]} F(x)\,dm(x)\right) \\
&= \frac{1}{h_j}\int_{x\in[b,b+h_j]} F(x)\,dm(x) \\
&\qquad - \frac{1}{h_j}\int_{x\in[a,a+h_j]} F(x)\,dm(x).
\end{aligned}$$

From Corollary 10.1.3 we have

$$\lim_{j\to\infty} \frac{1}{h_j}\int_{x\in[a,a+h_j]} F(x)\,dm(x) = F(a)$$

and

$$\lim_{j\to\infty} \frac{1}{h_j}\int_{x\in[b,b+h_j]} F(x)\,dm(x) = F(b),$$

since F is Lipschitz continuous and hence continuous. Corollary 10.1.3 refers to Riemann integrals but these are equal to the Lebesgue integrals. From the equations above it follows that

$$\int_{x\in[a,b]} F'(x)\,dm(x) = F(b) - F(a),$$

as claimed. □

11 Affine spaces and convex sets

11.1 Affine spaces

Definition 11.1.1. $A \in \wp(\mathbf{R}^n)$ is called *affine* if $(1-t)\mathbf{x} + t\mathbf{y} \in A$ whenever $t \in \mathbf{R}$ and $\mathbf{x}, \mathbf{y} \in A$. The *affine span* of a subset of V is the intersection of its affine supersets.

Proposition 11.1.2. *(a) The intersection of any collection of affine spaces is affine.*

(b) The affine span of a set is the smallest affine space containing it.

(c) If A is an affine space, $t_0, \ldots, t_m \in \mathbf{R}$ are such that

$$\sum_{i=0}^{m} t_i = 1$$

and $\mathbf{x}_0, \ldots \mathbf{x}_m \in A$ then

$$\sum_{i=0}^{m} t_i \mathbf{x}_i \in A.$$

Proof. Suppose $\mathcal{A}$ is a set of affine spaces and $B = \bigcap_{A\in\mathcal{A}} A$. If $\mathbf{x}, \mathbf{y} \in B$ then $\mathbf{x}, \mathbf{y} \in A$ for all $A \in \mathcal{A}$. Each A is an affine space so if $t \in \mathbf{R}$ then $(1-t)\mathbf{x} + t\mathbf{y} \in A$. This holds for all $A \in \mathcal{A}$ so $(1-t)\mathbf{x}+t\mathbf{y} \in B$. This establishes 11.1.2a.

Let $\mathcal{A}$ be the set of affine spaces containing S and let B be its affine span, i.e. $B = \bigcap_{A\in\mathcal{A}} A$. This is an affine space by 11.1.2a and it contains S so $A \in \mathcal{A}$. Any other element of $\mathcal{A}$ contains the intersection of all elements and so contains B. This establishes 11.1.2b.

Let $N_m = \{0, 1, \ldots, m\}$. We define $r\colon N_m \to N_m$ as follows. $r(0)$ is the i with the maximum value of t_i. $r(1)$ is the i with the maximum value, other than $i = r(0)$. $r(2)$ is the i with the maximum value, other than $i = r(0)$ or $i = r(1)$. If there is more than one choice at any stage the we make a choice arbitrarily and continue. After $m+1$ steps we have a function $r\colon N_m \to N_m$, which is injective by construction, since at each stage we chose a number other than those already chosen. It's also bijective since any injective function from a finite set to itself is also surjective and hence bijective. If $s_j \le 0$ then some summand t_i is less than or equal to zero. If $k > j$ then $k > i$ and $t_k \le t_i \le 0$, because otherwise $r(k)$ would have been chosen earlier than $r(i)$. It follows that

$$s_m = s_j + \sum_{k=j+1}^{m} t_{r(j)} \le 0.$$

But

$$s_m = 1$$

since this is just a reordering of the finite sum

$$\sum_{i=0}^{m} t_i.$$

But then $1 \leq 0$, which is impossible so there can be no j for which $s_j \leq 0$. In other words,

$$s_j = \sum_{i=0}^{j} t_{r(j)} > 0$$

for each $j \in N_m$. We now define

$$u_j = \frac{t_{r(j)}}{s_j}.$$

and

$$\mathbf{y}_j = \sum_{i=0}^{j} \frac{t_{r(i)}}{s_j} \mathbf{x}_{r(i)}.$$

Then

$$\mathbf{y}_0 = \mathbf{x}_{r(0)}$$

and

$$\mathbf{y}_m = \sum_{i=0}^{m} t_{r(i)} \mathbf{x}_{r(i)} = \sum_{i=0}^{m} t_i \mathbf{x}_i.$$

Also,

$$\begin{aligned}
\mathbf{y}_{j+1} &= \sum_{i=0}^{j+1} \frac{t_{r(i)}}{s_{j+1}} \mathbf{x}_{r(i)} \\
&= \sum_{i=0}^{j} \frac{t_{r(i)}}{s_{j+1}} \mathbf{x}_{r(i)} + \frac{t_{r(j+1)}}{s_{j+1}} \mathbf{x}_{r(j+1)} \\
&= \frac{s_j}{s_{j+1}} \mathbf{y}_j + u_{j+1} \mathbf{x}_{r(j+1)} \\
&= (1 - u_j) \mathbf{y}_j + u_j \mathbf{x}_{r(j+1)}.
\end{aligned}$$

By induction we see that $\mathbf{y}_j \in A$ for each $j \in N_m$ and hence for $j = m$. In other words,

$$\sum_{i=0}^{m} t_i \mathbf{x}_i \in A.$$

This establishes 11.1.2c. □

Proposition 11.1.3. *If V is a linear subspace of $\mathbf{R}^n$ then V is an affine set and $\mathbf{0} \in V$. Conversely, if V is an affine subset of $\mathbf{R}^n$ and $\mathbf{0} \in V$ then V is a linear subspace.*

Definition 11.1.4. If $\mathbf{x} \in \mathbf{R}^n$ and $A \in \wp(\mathbf{R}^n)$ then the *translate* of A by $\mathbf{x}$ is the set

$$\{\mathbf{y} \in \mathbf{R}^n : \mathbf{y} - \mathbf{x} \in A\}.$$

Proposition 11.1.5. *If A is a non-empty affine set then there is an $\mathbf{x} \in \mathbf{R}^n$ and a linear subspace V of $\mathbf{R}^n$ such that $A = \mathbf{x} + V$. If $\mathbf{x} \in \mathbf{R}^n$ and V' are such that $A = \mathbf{x}' + V'$ then $V' = V$ and $\mathbf{x}' - \mathbf{x} \in V$.*

Definition 11.1.6. If A is a non-empty affine subset of $\mathbf{R}^n$ then we define the *dimension* of A to be the dimension of V, with V as in the proposition above. If A is empty then we say that the dimension of A is -1. An affine subset of dimension $n - 1$ is called a *hyperplane*.

Proposition 11.1.7. *$A \in \wp(\mathbf{R}^n)$ is a hyperplane if and only if there are $a_1, \ldots, a_n \in \mathbf{R}$, not all of which are zero, and a $b \in \mathbf{R}$ such that*

$$A = \left\{ (x_1, \ldots, x_n) \in \mathbf{R}^n : \sum_{i=1}^{n} a_i x_i + b = 0 \right\}.$$

11.2 Convex sets

Definition 11.2.1. $C \in \wp(\mathbf{R}^n)$ is called *convex* if $(1 - t)\mathbf{x} + t\mathbf{y} \in C$ whenever $t \in [0, 1]$ and $\mathbf{x}, \mathbf{y} \in C$. The *convex hull* of a subset of V is the intersection of its convex supersets.

Definition 11.2.2. If $C \in \wp(\mathbf{R}^n)$ is convex then the *relative interior* of C, denoted $C^\diamond$, is the interior of C regarded as a subset of its affine span, with the subspace topology. $U \in \wp(\mathbf{R}^n)$ is called *relatively open* if $U = U^\diamond$.

As a subset of the affine span U is open if and only if it is equal to its interior so U is relatively open if and only if it is open as a subset of the affine span of A, with its subspace topology.

You might wonder why we don't define a set to be relatively closed to mean that it's closed as a subset of its affine span with respect to the subspace topology. A relatively closed set would be closed and vice versa so this would simply be an unnecessary synonym for closed. By contrast, not every relatively open set is open. In fact $\{\mathbf{x}\}$ is relatively open for each $\mathbf{x} \in \mathbf{R}^n$. $\{\mathbf{x}\}$ is an affine space and a superset of $\{\mathbf{x}\}$ and so is equal to the affine span of $\{\mathbf{x}\}$. $\{\mathbf{x}\}$ is open as a subset of $\{\mathbf{x}\}$ in the subspace topology so it's relatively open. $\{\mathbf{x}\}$ is not, of course, open if $n > 0$.

The following properties are consequences of the definitions above.

Proposition 11.2.3. *(a) Every affine set is convex.*

(b) The intersection of any collection of convex sets is convex.

(c) The intersection of any collection of relatively open sets is relatively open.

(d) The convex hull of a set is the smallest convex set containing it.

(e) If C is a convex set, $t_0, \ldots, t_m \in [0, +\infty)$ are such that

$$\sum_{i=0}^{m} t_i = 1$$

and $\mathbf{x}_0, \ldots \mathbf{x}_m \in C$ then

$$\sum_{i=0}^{m} t_i \mathbf{x}_i \in C.$$

Proof. If $t \in [0, 1]$ then $t \in \mathbf{R}$, so if $(1-t)\mathbf{x} + t\mathbf{y} \in S$ for all $t \in \mathbf{R}$ then $(1-t)\mathbf{x} + t\mathbf{y} \in S$ for all $t \in [0, 1]$. This establishes 11.2.3a.

Suppose $\mathcal{C}$ is a set of convex sets and $B = \bigcap_{C \in \mathcal{C}} C$. If $\mathbf{x}, \mathbf{y} \in B$ then $\mathbf{x}, \mathbf{y} \in C$ for all $C \in \mathcal{C}$. Each C is a convex set so if $t \in [0, 1]$ then $(1-t)\mathbf{x} + t\mathbf{y} \in C$. This holds for all $C \in \mathcal{C}$ so $(1-t)\mathbf{x} + t\mathbf{y} \in B$. This establishes 11.2.3b.

Suppose $U_1, \ldots, U_m$ are relatively open. In other words, U_i is open as a subset of A_i for each i, where A_i is the affine span of U_i. Let

$$V = \bigcap_{i=1}^{m} U_i$$

and let B be the affine span of V. $V \subseteq B$ so

$$V = V \cap B = V \cap \bigcap_{i=1}^{m} U_i = \bigcap_{i=1}^{m} (B \cap U_i).$$

U_i is relatively open, so open in the subspace topology on A_i. In other words, there is an open W_i such that

$$U_i = A_i \cap W_i.$$

Then

$$V = \bigcap_{i=1}^{m} (B \cap A_i \cap W_i).$$

But $B \subseteq A_i$ so $B \cap A_i = B$. Therefore

$$V = \bigcap_{i=1}^{m} (B \cap W_i) = B \cap \left(\bigcap_{i=1}^{m} W_i \right).$$

$\bigcap_{i=1}^{m} W_i$ is an open set, so V is open in the subspace topology on B. In other words, it's relatively open. This establishes 11.2.3c.

Let $\mathcal{C}$ be the set of convex sets containing S and let C be its affine span, i.e. $B = \bigcup_{C \in \mathcal{C}} C$. This is a convex set by 11.2.3b and it contains S so $C \in \mathcal{C}$. Any other element of $\mathcal{C}$ contains the intersection of all elements and so contains B. This establishes 11.2.3d.

The proof of 11.2.3e is the same as that of 11.1.2c except that we note that $t_i \geq 0$ for each i implies $s_j \leq s_{j+1}$ for each j and hence $u_j \in [0, 1]$. ☐

Definition 11.2.4. An *open halfspace* in $\mathbf{R}^n$ is a set of the form

$$\left\{ (x_1, \ldots, x_n) \in \mathbf{R}^n \colon \sum_{i=1}^{n} a_i x_i + b > 0 \right\}$$

for some $a_1, \ldots, a_n$ in $\mathbf{R}$, not all of which are zero and some $b \in \mathbf{R}$. A *closed halfspace* is the same, but with a non-strict inequality, i.e.

$$\left\{ (x_1, \ldots, x_n) \in \mathbf{R}^n \colon \sum_{i=1}^{n} a_i x_i + b \geq 0 \right\}$$

A set which is either an open halfspace or a closed halfspace is called a *halfspace*.

Note that the open and closed halfspaces are, as the terminology suggests, open and closed subsets, respectively, of $\mathbf{R}^n$, since linear functions are continuous.

Proposition 11.2.5. *Halfspaces are convex.*

Definition 11.2.6. $C \in \wp(\mathbf{R}^n)$ is called a *convex polytope* if there is a finite set S such that C is the convex hull of S. It is called a *simplex* if there is such an S with $m+1$ elements, where m is the dimension of C.

Proposition 11.2.7. *C is simplex of dimension m if and only if there are $\mathbf{x}_0, \ldots, \mathbf{x}_m \in C$ such that for every $\mathbf{y} \in C$ there are unique $t_0, \ldots, t_m \in [0,1]$ such that*

$$\sum_{i=0}^{m} t_i = 1$$

and

$$\sum_{i=0}^{m} t_i \mathbf{x}_i = \mathbf{y}.$$

The set $\{\mathbf{x}_0, \ldots, \mathbf{x}_m\}$ is uniquely determined by the simplex.

Definition 11.2.8. The points $\mathbf{x}_0, \ldots, \mathbf{x}_m$ as above are called the *vertices* of C. If C is a simplex then the *barycentre* of C is the point

$$\sum_{i=0}^{m} \frac{1}{m+1} \mathbf{x}_i.$$

The vertices and barycentre is well defined because the simplex determines the set $\{\mathbf{x}_0, \ldots, \mathbf{x}_m\}$ uniquely. Note that it's only the set of vertices which is determined, not their order. This is sufficient to make the barycentre well defined though since the coefficients of the vertices are all equal.

Definition 11.2.9. The *dimension* of a convex set is the dimension of its affine span.

Proposition 11.2.10. *Suppose C is a convex subset of $\mathbf{R}^n$, $\mathbf{x} \in C^\diamond$ and $\mathbf{y} \in C$. Then $(1-t)\mathbf{x} + t\mathbf{y} \in C^\diamond$ for all $t \in [0,1)$.*

Proposition 11.2.11. *Suppose $C \in \wp(\mathbf{R}^n)$ is convex.*

(a) $C^\diamond$ is convex.

(b) $\overline{C}$ is convex.

(c) $C^\diamond$ and $\overline{C}$ have the same affine span.

(d) $C^\diamond$ and $\overline{C}$ have the same dimension.

(e) If $C \neq \varnothing$ then $C^\diamond \neq \varnothing$.

Proposition 11.2.12. *(a) If $C \in \wp(\mathbf{R})$ is convex then*

$$\overline{(C^\diamond)} = \overline{C}$$

and

$$\left(\overline{C}\right)^\diamond = C^\diamond.$$

(b) Suppose $C_1, C_2 \in \wp(\mathbf{R}^n)$ are convex. Then $\overline{C_1} = \overline{C_2}$ if and only if $C_1^\diamond = C_2^\diamond$.

(c) If $C \in \wp(\mathbf{R}^n)$ is convex, $U \in \wp(\mathbf{R}^n)$ is open and $C^\diamond \cap U = \varnothing$ then $\overline{C} \cap U = \varnothing$.

(d) If $C_1, C_2 \in \wp(\mathbf{R}^n)$ are convex, $C_2 \neq \varnothing$, and $C_1 \subseteq C_2^\diamond$, $C_1 \cap C_2 = \varnothing$ then the dimension of C_1 is less than the dimension of C_2.

11.3 Faces

Definition 11.3.1. Suppose $C \in \wp(\mathbf{R}^n)$ is convex. $F \in \wp(C)$ is said to be a *face* of C if whenever $\mathbf{x}, \mathbf{y} \in C$, $t \in (0,1)$ and $(1-t)\mathbf{x} + t\mathbf{y} \in F$ we have $\mathbf{x} \in F$ and $\mathbf{y} \in F$.

Note that by this definition $\varnothing$ is a face of every convex set because the condition above holds vacuously.

Proposition 11.3.2. *Suppose F is a face of a convex subset C in $\mathbf{R}^n$.*

(a) F is convex.

(b) If $D \in \wp(C)$ is convex and $D^\diamond \cap F \neq \varnothing$ then $D \subseteq F$.

(c) If E is a face of F then E is a face of C.

(d) If $f\colon \mathbf{R}^n$ is a linear function $\mathbf{y} \in C$ and $f(\mathbf{x}) \leq f(\mathbf{y})$ for all $\mathbf{x} \in C$ then the set

$$\{\mathbf{x} \in C \colon f(\mathbf{x}) = f(\mathbf{y})\}$$

is a face of C.

(e) *If D is a convex set such that $F \subseteq D \subseteq C$ then F is a face of D.*

(f) *If D is a convex set such that $D^\diamond \cap F \neq \varnothing$ then F is a face of D.*

(g) *$F = C \cap \overline{F}$.*

(h) *If C is closed then F is closed.*

(i) *If E is a face of C and $E^\diamond \cap F^\diamond \neq \varnothing$ then $E = F$.*

(j) *If $F \neq C$ then $C^\diamond \cap F = \varnothing$*

(k) *If $F \neq C$ then the dimension of F is less than the dimension of C.*

Proposition 11.3.3. *Suppose C is a convex set. Let $\mathcal{Q}$ be the set of relative interiors of faces of C and let $\mathcal{P} = \mathcal{Q} \setminus \{\varnothing\}$. Then $\mathcal{P}$ is a partition of C. If D is a relatively open convex subset of C then $D \subseteq E$ for some $E \in \mathcal{P}$. If $\overline{C}$ is a convex polytope then $\mathcal{P}$ is finite.*

Definition 11.3.4. Suppose (X, d) is a metric space and $A \in \wp(A)$. The *diameter* of A is

$$\sup_{x,y\in A} d(x, y).$$

This is an element of $[0, +\infty]$. It is an element of $[0, +\infty)$ if and only if A is bounded.

11.4 Complexes

Definition 11.4.1. A *complex* is a finite $\mathcal{C} \in \wp(\wp(\mathbf{R}^n))$ satisfying the following properties:

(a) If $E \in \mathcal{C}$ then E is a compact convex polytope.

(b) If $E \in \mathcal{C}$ and F is a face of E then $F \in \mathcal{C}$.

(c) If $E_1, E_2 \in \mathcal{C}$ then $E_1 \cap E_2$ is a face of both E_1 and E_2.

The *dimension* of $\mathcal{C}$ is the maximum of the dimensions of E for all $E \in \mathcal{C}$. The *underlying space* of $\mathcal{C}$ is the set $\bigcup_{E\in\mathcal{C}} E$. The *mesh* of a complex is the maximum of the diameters of its elements.

Note that 11.4.1c includes the case $E_1 \cap E_2 = \varnothing$.

Proposition 11.4.2. *A finite set $\mathcal{C}$ of compact convex polytopes is a complex if and only if the following two conditions are satisfied.*

(a) *If $E \in \mathcal{C}$ and F is a face of E then $F \in \mathcal{C}$.*

(b) *If $\mathbf{x}$ belongs to the underlying set of $\mathcal{C}$ then there is a unique $E \in \mathcal{C}$ such that $\mathbf{x} \in E^\diamond$.*

Proposition 11.4.3. *Every compact convex polytope is the underlying set of a complex.*

It's certainly possible for the same compact convex polytope to be the underlying set for more than one complex. In fact this always happens for complexes of positive dimension, and is quite useful.

Definition 11.4.4. Suppose $\mathcal{A}$ and $\mathcal{C}$ are complexes and $\mathcal{A} \subseteq \mathcal{C}$. Then $\mathcal{A}$ is said to be *subcomplex* of $\mathcal{C}$.

Not every subset of a complex is a complex and therefore not every subset is a subcomplex, but we do have the following proposition.

Proposition 11.4.5. *Suppose $\mathcal{C}$ is a complex and $\mathcal{A} \in \wp(\mathcal{C})$. Then $\mathcal{A}$ is a subcomplex if and only if $F \in \mathcal{A}$ whenever if $E \in \mathcal{A}$ and F is a face of E.*

Definition 11.4.6. A complex $\mathcal{C}'$ is said to be a *refinement* of a complex $\mathcal{C}$ if they have the same underlying set and for every $E' \in \mathcal{C}'$ there is an $E \in \mathcal{C}$ such that $E' \subseteq E$.

Proposition 11.4.7. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are complexes with the same underlying set then there is a complex $\mathcal{D}$ which is a refinement of both $\mathcal{C}_1$ and $\mathcal{C}_2$*

11.5 Simplicial complexes

Definition 11.5.1. A complex $\mathcal{C} \in \wp(\wp(\mathbf{R}^n))$ is called *simplicial* if every $E \in \mathcal{C}$ is a simplex.

Proposition 11.5.2. *Suppose $\mathcal{C}$ is a complex. Let $\mathcal{C}^- = \setminus\{\varnothing\}$. Suppose $\varphi\colon \mathcal{C}^- \to \mathbf{R}^n$ is a function such that $\varphi(E) \in E^\diamond$ for each $E \in \mathcal{C}$. Let S be the set of strictly increasing sequences of elements of $\mathcal{C}^-$, i.e sequences $E_0, E_1, \ldots, E_k \in \mathcal{C}^-$ such that*

$$E_0 \subset E_1 \subset \cdots \subset E_k.$$

Define $\psi\colon S \to \wp(\mathbf{R}^n)$ by saying that ψ takes the sequence $E_0, E_1, \ldots, E_k$ to the convex hull of $\{\varphi(E_0), \varphi(E_1), \ldots, \varphi(E_k)\}$. This is a k-dimensional simplex. Let $\mathcal{R}$ be image of ψ. Then $\mathcal{R}$ is a simplicial complex and a refinement of $\mathcal{C}$. If $\mathcal{C}$ is a simplicial complex and $\varphi(E)$ is the barycentre of E for each $E \in \mathcal{C}$ then the mesh of $\mathcal{R}$ is at most $\frac{k}{k+1}$ times the mesh of $\mathcal{C}$, where k is the dimension of $\mathcal{C}$.

Theorem 11.5.3. *Suppose $\mathcal{C}_1, \mathcal{C}_2, \ldots \mathcal{C}_m$ are complexes with the same underlying set and $\delta > 0$. Then there is a simplicial complex $\mathcal{D}$ which is a refinement of each of the $\mathcal{C}$'s and has mesh less than δ.*

12 Higher dimensions

$\mathbf{R}$ has a natural order structure while $\mathbf{R}^n$ for $n > 1$ does not. There are several places where we used this order structure in developing the theory of Riemann integration in one dimension and these will need modification to get the corresponding theory for higher dimensions. To get from the Riemann integral to the Lebesgue integral, by contrast, we used the Riesz Representation Theorem, which works equally well for any locally compact σ-compact Hausdorff topological space, including $\mathbf{R}^n$, so no changes are required there.

12.1 Semilinear sets

In $\mathbf{R}$ we obtained the Jordan algebra $\mathcal{J}$ by completing the Boolean algebra of finite unions of intervals, which we called $\mathcal{I}$. We need an analogue of $\mathcal{I}$ for higher dimensions. There are a number of distinct choices one could make which lead to the same $\mathcal{J}$ in the end. The particular one I'm making here is based on the observation that every element of $\mathcal{I}$ is describable by linking a finite number of linear inequalities with Boolean operators and that any set which can be so described is an element of $\mathcal{I}$.

Definition 12.1.1. The *semilinear algebra* is the Boolean algebra generated by the open halfspaces. A *semilinear set* is an element of the semilinear algebra.

Proposition 12.1.2. *The following subsets of $\mathbf{R}^n$ are semilinear:*

(a) $\varnothing$

(b) $\mathbf{R}^n$

(c) any closed halfspace

(d) any hyperplane

(e) any affine subspace

(f) any intersection of finitely many hyperplanes and affine subspaces

(g) any union of finitely many intersections of finitely many hyperplanes and affine subspaces

We'll see soon that every semilinear set can be written in the last of these ways.

Proof. We use repeatedly the fact that complements, finite unions and finite intersections of elements of a Boolean algebra belong to that Boolean algebra.

(a) $\varnothing$ is the union of an empty collection of open hyperplanes.

(b) $\mathbf{R}^n$ is the complement of $\varnothing$.

(c)
$$\left\{(x_1, \ldots, x_n) \in \mathbf{R}^n \colon \sum_{i=1}^n a_i x_i + b \geq 0\right\}$$
is the complement of
$$\left\{(x_1, \ldots, x_n) \in \mathbf{R}^n \colon \sum_{i=1}^n (-a_i) x_i - b > 0\right\}.$$

(d) Every hyperplane is of the form
$$\left\{(x_1, \ldots, x_n) \in \mathbf{R}^n \colon \sum_{i=1}^n a_i x_i + b = 0\right\}.$$
This is the intersection of
$$\left\{(x_1, \ldots, x_n) \in \mathbf{R}^n \colon \sum_{i=1}^n a_i x_i + b \geq 0\right\}$$

and

$$\left\{(x_1,\ldots,x_n)\in\mathbf{R}^n\colon \sum_{i=1}^{n}(-a_i)x_i-b\geq 0\right\},$$

both of which are semilinear by the preceding part, and the intersection of finitely many semilinear sets is semilinear.

(e) Every affine subspace is the intersection of finitely many hyperplanes, and the intersection of finitely many semilinear sets is semilinear.

(f) The intersection of finitely many semilinear sets is semilinear.

(g) The union of finitely many semilinear sets is semilinear.

□

Proposition 12.1.3. *The semilinear algebra is a subset of the Borel algebra.*

Proof. The open halfspaces are open sets so the σ-algebra generated by the open halfspaces is a subset of the σ-algebra generated by the open sets, i.e. of the Borel algebra. The σ-algebra generated by the open halfspaces is a σ-algebra and hence is a Boolean algebra. The semilinear algebra is the Boolean algebra generated by the open halfspaces and so is the smallest Boolean algebra containing them. It is therefore a subset of the σ-algebra generated by them, which, as we just saw, is itself a subset of the Borel algebra. □

The proof above only shows that the semilinear algebra is a subset of the Borel algebra, not that it is a proper subset, but that's easy to see for $n>0$. $\mathbf{Q}^n$ is a Borel set which is not a semilinear set.

Proposition 12.1.4. *Suppose $\mathcal{C}$ is a complex and $\mathcal{A}\in\wp(\mathcal{C})$. Then $\bigcup_{E\in\mathcal{A}}E^\diamond$ is a bounded semilinear set.*

Proof. Every simplex is bounded and every subset of a bounded set is bounded so $E^\diamond$ is bounded for each $E\in\mathcal{A}$. The union of finitely many bounded sets is bounded, so $\bigcup_{E\in\mathcal{C}}E^\diamond$ is bounded. Each $E^\diamond$ is a finite intersection of open half spaces and so is a semilinear set. The union of finitely many of them is therefore also a semilinear set. □

The converse is true as well. For every bounded semilinear set A there is a simplicial complex $\mathcal{C}$ and an $\mathcal{A}\in\wp(\mathcal{C})$ such that $A=\bigcup_{E\in\mathcal{C}}E^\diamond$. Proving this will require the following proposition, which gives a more explicit characterisation of the Boolean algebra generated by a set.

Proposition 12.1.5. *Suppose X is a set $\mathcal{A}\in\wp(\wp(X))$ and $\mathcal{B}$ is the Boolean algebra generated by $\mathcal{A}$. Let $\mathcal{S}_1$ be the set of E such that $E\in\mathcal{A}$ or $X\setminus E\in\mathcal{A}$. Let $\mathcal{S}_2$ be the set of intersections of finitely many elements of $\mathcal{S}_1$. Let $\mathcal{S}_3$ be the set of unions of finitely many elements of $\mathcal{S}_2$. Then $\mathcal{B}=\mathcal{S}_3$.*

Proof. By the definition of the Boolean algebra generated by a set we have $\mathcal{A}\subseteq\mathcal{B}$ and $\mathcal{B}$ is a Boolean algebra on X. From the former we see that if $E\in\mathcal{A}$ then $E\in\mathcal{B}$. Since $\mathcal{B}$ is a Boolean algebra on X we then have $X\setminus E\in\mathcal{B}$. So $\mathcal{S}_1\subseteq\mathcal{B}$. The intersection of finitely many elements of a Boolean algebra belongs to the Boolean algebra, so $\mathcal{S}_2\subseteq\mathcal{B}$. The union of finitely many elements of a Boolean algebra belongs to the Boolean algebra, so $\mathcal{S}_3\subseteq\mathcal{B}$.

We next show that $\mathcal{S}_3$ is a Boolean algebra. Suppose $E\in\mathcal{S}_3$. Then there is a finite $\mathcal{C}\subseteq\mathcal{S}_2$ such that

$$E=\bigcup_{F\in\mathcal{C}}F.$$

Each $F\in\mathcal{S}_2$ so there is a set $\mathcal{D}(F)\subseteq\mathcal{S}_1$ such that

$$F=\bigcap_{G\in\mathcal{D}(F)}G.$$

So

$$E=\bigcup_{F\in\mathcal{C}}\bigcap_{G\in\mathcal{D}(F)}G.$$

Then

$$X\setminus E=\bigcap_{F\in\mathcal{C}}\bigcup_{G\in\mathcal{D}(F)}(X\setminus G).$$

$\mathcal{D}$ is a function from $\mathcal{C}$ to $\mathcal{S}_1$. Let P be its product, i.e. the set of all functions $s\colon\mathcal{C}\to\mathcal{S}_1$ such that

$$s(F)\in\mathcal{D}(F)$$

for all $F\in\mathcal{C}$. If $x\in X\setminus E$ then $x\in\bigcup_{G\in\mathcal{D}(F)}(X\setminus G)$ for each $F\in\mathcal{C}$ so for each $F\in\mathcal{C}$ there is $G\in\mathcal{D}(F)$

such that $x \in (X \setminus G)$. If we choose such a G for each $F \in \mathcal{C}$ and call it $s(F)$ then s is a function from $\mathcal{C}$ to $\mathcal{S}_1$ such that $s(F) \in \mathcal{D}(F)$, i.e. an element of P. So

$$x \in \bigcup_{s \in P} \bigcap_{F \in \mathcal{C}} (X \setminus s(F)).$$

Conversely, if the statement above holds, i.e. if there is an $s \in P$ such that $x \in (X \setminus s(F))$ for all $F \in \mathcal{C}$ then $s(F) \in \mathcal{D}(F)$ and there is, for each $\mathcal{F} \in \mathcal{C}$, a $G \in \mathcal{D}(C)$ such that $x \in X \setminus G$, namely $G = s(F)$. So $x \in \bigcup_{G \in \mathcal{D}(F)} (X \setminus G)$ for each $F \in \mathcal{C}$. Since this holds for all $F \in \mathcal{C}$ we have

$$x \in \bigcap_{F \in \mathcal{C}} \bigcup_{G \in \mathcal{D}(F)} (X \setminus G),$$

i.e. $x \in X \setminus E$. In other words,

$$X \setminus E = \bigcup_{s \in P} \bigcap_{F \in \mathcal{C}} (X \setminus s(F)).$$

For each $s \in P$ and $F \in \mathcal{C}$ we have $s(F) \in \mathcal{S}_1$ so

$$X \setminus s(F) \in \mathcal{S}_1.$$

Therefore

$$\bigcap_{F \in \mathcal{C}} (X \setminus s(F)) \in \mathcal{S}_2$$

since $\mathcal{C}$ is finite, and

$$X \setminus E \in \mathcal{S}_3$$

since P is finite.

If $E_1, E_2 \in \mathcal{S}_3$ then there are finite $\mathcal{C}_1 \subseteq S_2$ and $\mathcal{C}_2 \subseteq \mathcal{S}_2$ such that

$$E_1 = \bigcup_{F \in \mathcal{C}_1} F$$

and

$$E_2 = \bigcup_{F \in \mathcal{C}_2} F.$$

Then

$$E_1 \cup E_2 = \bigcup_{F \in \mathcal{C}_1 \cup \mathcal{C}_2} F.$$

$\mathcal{C}_1 \cup \mathcal{C}_2$ is a finite subset of $\mathcal{S}_2$ so

$$E_1 \cup E_2 \in \mathcal{S}_3.$$

Therefore $\mathcal{S}_3$ is a Boolean algebra.

$\mathcal{S}_3$ is a Boolean algebra and contains $\mathcal{A}$. $\mathcal{B}$ is the smallest Boolean algebra containing $\mathcal{A}$ so $\mathcal{B} \subseteq \mathcal{S}_3$. But we've already seen that $\mathcal{S}_3 \subseteq \mathcal{B}$, so $\mathcal{B} = \mathcal{S}_3$. $\square$

The proof above used the fact that the product of finitely many finite sets is finite. It is not the case that a product of countably many countable sets is countable so the argument above cannot be adapted to prove the corresponding statement for σ-algebras, and indeed that statement isn't true, since there are Borel sets which cannot be written as countable unions of countable intersections of open or closed sets.

Proposition 12.1.6. *Suppose E is a bounded semi-linear set. Then there is a complex $\mathcal{C}$ and a $\mathcal{A} \in \wp(\mathcal{C})$ such that*

$$E = \bigcup_{F \in \mathcal{A}} F^{\diamond}.$$

Proof. By Proposition 12.1.5 we can write E as

$$E = \bigcup_{i=1}^{k} \bigcap_{j=1}^{l_i} H_{i,j}$$

where $H_{i,j}$ is a halfspace for each i and j. Let $P_{i,j} = \overline{H_{i,j}}$ and $Q_{i,j} = \overline{\mathbf{R}^n \setminus H_{i,j}}$. These are closed halfspaces. Define $\mathbf{T} \in \wp(\wp(\wp(\mathbf{R}^n)))$ by saying that $\mathcal{S} \in \mathbf{T}$ if and only if both the following conditions are satisfied:

- If $H \in \mathcal{S}$ then $H = P_{i,j}$ or $H = Q_{i,j}$ for some i and j.
- For each i and j we have $P_{i,j} \in \mathcal{S}$ or $Q_{i,j} \in \mathcal{S}$, or both.

For each $\mathcal{S} \in \mathbf{T}$ we have a closed convex set

$$I(\mathcal{S}) = \bigcap_{H \in \mathcal{S}} H.$$

This set may, of course, be empty.

Every $\mathbf{y} \in \mathbf{R}^n$ belongs to $I(\mathcal{S})$ for some $\mathcal{S} \in \mathbf{T}$. To see this note that each $H_{i,j}$ is of the form

$$H_{i,j} = \{\mathbf{x} \in \mathbf{R}^n \colon g_{i,j}(\mathbf{x}) > 0\}$$

or

$$H_{i,j} = \{\mathbf{x} \in \mathbf{R}^n : g_{i,j}(\mathbf{x}) \geq 0\},$$

depending on whether $H_{i,j}$ is an open or closed half-space, for some linear function $g_{i,j} : \mathbf{R}^n \to \mathbf{R}$. Then

$$P_{i,j} = \{\mathbf{x} \in \mathbf{R}^n : g_{i,j}(\mathbf{x}) \geq 0\}$$

and

$$Q_{i,j} = \{\mathbf{x} \in \mathbf{R}^n : g_{i,j}(\mathbf{x}) \leq 0\}.$$

Let $\mathcal{S}$ be the set of $P_{i,j}$ such that $g_{i,j}(\mathbf{y}) \geq 0$ and $Q_{i,j}$ such that $g_{i,j}(\mathbf{y}) \leq 0$. Then $\mathbf{y} \in I(\mathcal{S})$. Also, for every i and j we have $g_{i,j}(\mathbf{y}) \geq 0$ or $g_{i,j}(\mathbf{y}) \leq 0$ or both, so $P_{i,j} \in \mathcal{S}$ or $Q_{i,j} \in \mathcal{S}$ or both. In other words, $\mathcal{S} \in \mathbf{T}$.

Suppose D is a non-empty face of $I(\mathcal{S})$ for some S. Then there are $\mathbf{y} \in I(\mathcal{S})^\diamond$ and $\mathbf{z} \in D^\diamond$. Let $\mathcal{R}$ be the set of $H \in \mathcal{S}$ such that $\mathbf{y} \in H^\circ$ and $\mathbf{z} \notin H^\circ$. Each such H is of the form

$$H = \{\mathbf{x} \in \mathbf{R}^n : f_H(\mathbf{x}) \geq 0\}$$

for some linear function $f_H : \mathbf{R}^n \to \mathbf{R}$. $\mathbf{z} \notin H^\circ$ so $f(\mathbf{z}) = 0$. Then $H \in \mathcal{S}$ so $I(\mathcal{S}) \subseteq H$ and therefore $f_H(\mathbf{x}) \geq 0 = f_H(\mathbf{z})$ for all $\mathbf{x} \in I(\mathcal{S})$. It follows from Proposition 11.3.2d that

$$\{\mathbf{x} \in I(\mathcal{S}) : f_H(\mathbf{x}) = 0\}$$

is a face of $I(\mathcal{S})$. Also,

$$\mathbf{z} \in \{\mathbf{x} \in I(\mathcal{S}) : f_H(\mathbf{x}) = 0\}.$$

Let F be the intersection of these sets for all $H \in \mathcal{R}$. Then F is the intersection of set of faces of $I(\mathcal{S})$ and so is a face of $I(\mathcal{S})$. $\mathbf{z} \in F$ and $\mathbf{z} \in D^\diamond$ so F is a face D by Proposition 11.3.2f. But then $D = F$ by Proposition 11.3.2j. In other words,

$$D = I(\mathcal{S}) \cap \bigcap_{H \in \mathcal{R}} \{\mathbf{x} \in \mathbf{R}^n : f_H(\mathbf{x}) = 0\}.$$

For each $H \in \mathcal{R}$ we can write

$$\bigcap_{H \in \mathcal{R}} \{\mathbf{x} \in \mathbf{R}^n : f_H(\mathbf{x}) = 0\} = H \cap H'$$

where $H' = Q_{i,j}$ if $H = P_{i,j}$ and vice versa. So

$$D = I(\mathcal{S}')$$

where

$$\mathcal{S}' = \mathcal{S} \cap \{H' : H \in \mathcal{R}\}.$$

If $\mathcal{S} \in \mathbf{T}$ then $\mathcal{S}' \in \mathbf{T}$. So every non-empty face of $I(\mathcal{S})$ for an $\mathcal{S} \in \mathbf{T}$ is $I(\mathcal{S}')$ for some $\mathcal{S}' \in \mathbf{T}$.

Suppose

$$I(\mathcal{S})^\diamond \cap E \neq \varnothing,$$

i.e. that there is a $\mathbf{y} \in I(\mathcal{S})^\diamond \cap E$. Then

$$\mathbf{y} \in \bigcap_{j=1}^{l_i} H_{i,j}$$

for some i. Here we use the representation of E as a union of intersections of halfspaces described earlier. Now

$$I(\mathcal{S}) = \bigcup_{K \in \mathcal{S}} K.$$

Each of these K is of the form

$$K = \{\mathbf{x} \in \mathbf{R}^n : f_K(\mathbf{x}) \geq 0\}$$

for some linear function f_K. If $\mathbf{z} \in I(\mathcal{S})^\diamond$ then $f_K(\mathbf{z}) = 0$ for all $K \in \mathcal{S}$ such that $f_K(\mathbf{y}) = 0$ and $f_K(\mathbf{z}) > 0$ for all those $K \in \mathcal{S}$ such that $f_K(\mathbf{y}) > 0$. It follows that $\mathbf{z} \in H_{i,j}$ whenever $\mathbf{y} \in H_{i,j}$. So

$$I(\mathcal{S})^\diamond \subseteq E.$$

In other words, every set $I(\mathcal{S})^\diamond$ is either contained entirely within E or entirely within its complement.

Let $\mathcal{A}$ be the set of sets of the form $I(\mathcal{S})$ such that $I(\mathcal{S})^\diamond \subseteq E$. Then

$$E = \bigcup_{F \in \mathcal{A}} F^\circ.$$

Let $\mathcal{C}$ be the set of faces of elements of $\mathcal{A}$. Every element of $\mathcal{A}$ is a face of itself, so

$$\mathcal{A} \subseteq \mathcal{C}.$$

If $F \in \mathcal{A}$ then $F = I(\mathcal{S})$ for some $\mathcal{S} \in \mathbf{T}$. If D is a face of F then $D = \mathcal{I}(\mathcal{S}')$ for some $\mathcal{S}' \in \mathbf{T}$. If $D \in \mathcal{C}$ then $D \subseteq \overline{E}$. E is bounded so $\overline{E}$ is bounded and hence D is bounded. So D is a bounded intersection of finitely many closed halfplanes and hence is a convex polytope. Suppose $D_1, D_2 \in \mathcal{C}$, i.e. that there are

$F_1, F_2 \in \mathcal{A}$ such that D_1 is a face of F_1 and D_2 is a face of F_2. Then there are $\mathcal{S}'_1, \mathcal{S}'_2 \in \mathbf{T}$ such that $D_1 = I(\mathcal{S}'_1)$ and $D_2 = I(\mathcal{S}'_2)$. Then

$$D_1 \cap D_2 = I(\mathcal{S}'_1 \cup \mathcal{S}'_2).$$

This is a face of both D_1 and D_2 so it's a face of F_1 and F_2 and therefore an element of $\mathcal{C}$. So $\mathcal{C}$ is a set of convex polytopes with the property that every face of an element of $\mathcal{C}$ is an element of $\mathcal{C}$ and the property that the intersection of any two elements of $\mathcal{C}$ is an element of $\mathcal{C}$. In other words, $\mathcal{C}$ is a complex. □

Theorem 12.1.7. *Suppose E is a bounded semilinear set. Then there is a simplicial complex $\mathcal{C}$ and a $\mathcal{A} \in \wp(\mathcal{C})$ such that*

$$E = \bigcup_{F \in \mathcal{A}} F^\diamond.$$

Proof. We take the complex $\mathcal{C}$ from the preceding proposition and refine it to a simplicial complex by means of Proposition 11.5.2. □

12.2 Jordan Content

Lemma 12.2.1. *Let M be the $n+1$ by $n+2$ matrix*

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ v_{1,0} & v_{1,1} & \cdots & v_{1,n} & v_{1,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-1,0} & v_{n-1,1} & \cdots & v_{n-1,n} & v_{n-1,n+1} \\ v_{n,0} & v_{n,1} & \cdots & v_{n,n} & v_{n,n+1} \end{bmatrix}.$$

and let N_j be the $(-1)^j$ times the matrix M with its $j+1$'s column removed. Then

$$\sum_{j=0}^{n+1} \det(N_j) = 0.$$

Proof. Consider the $n+2$ by $n+2$ matrix

$$P = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ v_{1,0} & v_{1,1} & \cdots & v_{1,n} & v_{1,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-1,0} & v_{n-1,1} & \cdots & v_{n-1,n} & v_{n-1,n+1} \\ v_{n,0} & v_{n,1} & \cdots & v_{n,n} & v_{n,n+1} \end{bmatrix}.$$

Expanding this matrix on its first row gives

$$\det(P) = \sum_{j=0}^{n+1} \det(N_i).$$

But the first two rows of P are equal so P is singular and

$$\det(P) = 0.$$

□

Lemma 12.2.2. *Suppose $\mathbf{v}_0$, $\mathbf{v}_1$, …, $\mathbf{v}_n$ are affinely independent. Define $t_j\colon \mathbf{R}^n \to \mathbf{R}$ to be the linear function*

$$t_j(\mathbf{x}) = -\det(N_j(\mathbf{x}))/\det(N_{n+1})$$

where N_j is $(-1)^j$ times the matrix obtained by deleting the $j+1$'st column from the $n+1$ by $n+2$ matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ v_{1,0} & v_{1,1} & \cdots & v_{1,n} & x_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-1,0} & v_{n-1,1} & \cdots & v_{n-1,n} & x_{n-1} \\ v_{n,0} & v_{n,1} & \cdots & v_{n,n} & x_n \end{bmatrix}.$$

Then

$$\mathbf{x} = \sum_{j=0}^{n} t_j(\mathbf{x})\mathbf{v}_j$$

for all $\mathbf{x} \in \mathbf{R}^n$ and

$$\sum_{j=0}^{n} t_j(\mathbf{x}) = 1.$$

Proof. If $j \neq k$ then $N_j(\mathbf{v}_k)$ has a repeated column so

$$t_j(\mathbf{v}_k) = 0.$$

On the other hand, $N_j(\mathbf{v}_j)$ differs from N_{n+1} only by a permutation of the columns and multiplication by the sign of that permutation so $N_j(\mathbf{v}_j) = -N_{n+1}$ and hence

$$t_j(\mathbf{v}_j) = 1.$$

The affine span of $\mathbf{v}_0$, $\mathbf{v}_1$, …, $\mathbf{v}_n$ is all of $\mathbf{R}^n$ so any $\mathbf{x} \in \mathbf{R}^n$ can be written as

$$\mathbf{x} = \sum_{k=0}^{n} s_k \mathbf{v}_k.$$

Then

$$t_j(\mathbf{x}) = \sum_{k=0}^{n} s_k t_j(\mathbf{v}_k) = s_j.$$

So

$$\mathbf{x} = \sum_{j=0}^{n} t_j(\mathbf{x})\mathbf{v}_j$$

The equation

$$\sum_{j=0}^{n} t_j(\mathbf{x}) = 1$$

follows from Lemma 12.2.1. □

Corollary 12.2.3. *Suppose $\mathbf{v}_0$, $\mathbf{v}_1$, …, $\mathbf{v}_n$ are affinely independent. Let E be the simplex of which they are vertices. Let t_j be the functions from Lemma 12.2.2. Then*

$$E = \{\mathbf{x} \in \mathbf{R}^n : t_0(\mathbf{x}) \geq 0, t_1(\mathbf{x}) \geq 0, \ldots, t_n(\mathbf{x}) \geq 0\}.$$

Lemma 12.2.4. *Let S' be the simplex with vertices $\mathbf{v}_0$, $\mathbf{v}_1$, …, $\mathbf{v}_{n-1}$ and $\mathbf{v}'_n$ let S'' be the simplex with vertices $\mathbf{v}_0$, $\mathbf{v}_1$, …, $\mathbf{v}_{n-1}$ and $\mathbf{v}''_n$. Let F be the simplex with vertices $\mathbf{v}_0$, $\mathbf{v}_1$, …, $\mathbf{v}_{n-1}$. If*

$$\det\left(\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ v_{1,0} & v_{1,1} & \cdots & v_{1,n-1} & v'_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-1,0} & v_{n-1,1} & \cdots & v_{n-1,n-1} & v'_{n-1,n} \\ v_{n,0} & v_{n,1} & \cdots & v_{n,n-1} & v'_{n,n} \end{bmatrix}\right)$$

and

$$\det\left(\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ v_{1,0} & v_{1,1} & \cdots & v_{1,n-1} & v''_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-1,0} & v_{n-1,1} & \cdots & v_{n-1,n-1} & v''_{n-1,n} \\ v_{n,0} & v_{n,1} & \cdots & v_{n,n-1} & v''_{n,n} \end{bmatrix}\right)$$

have opposite signs then $S' \cap S'' = F$. If they have the same sign then $S' \cap S''$ is of dimension n.

Implicit in the statement that S' and S'' are simplices is the assumption that $\mathbf{v}_0$, $\mathbf{v}_1$, …, $\mathbf{v}_{n-1}$ and $\mathbf{v}'_n$ are affinely independent, as are $\mathbf{v}_0$, $\mathbf{v}_1$, …, $\mathbf{v}_{n-1}$ and $\mathbf{v}''_n$. Then $\mathbf{v}_0$, $\mathbf{v}_1$, …, and $\mathbf{v}_{n-1}$are also affinely independent. This affine independence assumption ensures that the determinants are non-zero.

Proof. Define t'_j and t''_j as in Lemma 12.2.2 with $\mathbf{v}_n$ replaced by $\mathbf{v}'_n$ and $\mathbf{v}''_n$ respectively. Now

$$t'_n(\mathbf{x}) = -\det(N'_n(\mathbf{x}))/\det(N'_{n+1})$$

and

$$t''_n(\mathbf{x}) = -\det(N''_n(\mathbf{x}))/\det(N''_{n+1}),$$

with notation which is an obvious modification of that in the lemma. The numerators in these fractions are functions of $\mathbf{x}$ but the denominators are not, since these involve dropping the last column, the only one which depends on $\mathbf{x}$. The determinants appearing in the statement of corollary are just the determinants of $(-1)^{n+1}N'_{n+1}$ and $(-1)^{n+1}N''_{n+1}$. The matrices $N'_n(\mathbf{x})$ and $N''_n(\mathbf{x})$ are both equal to

$$(-1)^n \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ v_{1,0} & v_{1,1} & \cdots & v_{1,n-1} & x_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-1,0} & v_{n-1,1} & \cdots & v_{n-1,n-1} & x_{n-1} \\ v_{n,0} & v_{n,1} & \cdots & v_{n,n-1} & x''_n \end{bmatrix}.$$

If this matrix is non-zero and the determinants of $(-1)^{n+1}N'_{n+1}$ and $(-1)^{n+1}N''_{n+1}$ are of opposite sign then $t'_n(\mathbf{x})$ and $t''_n(\mathbf{x})$ are of opposite sign. If $\mathbf{x} \in S' \cap S''$ then $t'_n(\mathbf{x}) \geq 0$ and $t''_n(\mathbf{x}) \geq 0$. So if the determinants are of opposite sign then $t'_n(\mathbf{x}) = 0$ for all $\mathbf{x} \in S' \cap S''$. But then we can rewrite

$$\mathbf{x} = \sum_{j=0}^{n-1} t'_j(\mathbf{x})\mathbf{v}_j + t'_n(\mathbf{x})\mathbf{v}'$$

and

$$\sum_{j=0}^{n-1} t'_j(\mathbf{x}) + t'_n(\mathbf{x}) = 1$$

as

$$\mathbf{x} = \sum_{j=0}^{n-1} t'_j(\mathbf{x})\mathbf{v}_j$$

and

$$\sum_{j=0}^{n-1} t'_j(\mathbf{x}) = 1.$$

In other words, $\mathbf{x} \in F$. Therefore $S' \cap S'' \subseteq F$ if the determinants are of opposite sign. The reverse inclusion holds in any case so we conclude that

$$S' \cap S'' = F.$$

Let $\mathbf{u}$ be the barycentre of F. Then

$$t'_j(\mathbf{u}) = \frac{1}{n} = t''_j(\mathbf{u})$$

for $j < n$ and

$$t'_n(\mathbf{u}) = 0 = t''_n(\mathbf{u}).$$

Define

$$\mathbf{w}(s) = (1-s)\mathbf{u} + s\mathbf{v}'.$$

Then $\mathbf{w}(s) \in S'$ for $s \in [0,1]$.

$$t''_n(\mathbf{w}(s)) = (1-s)t''_n(\mathbf{u}) + st''_n(\mathbf{v}') = st''_n(\mathbf{v}').$$

But

$$t''_n(\mathbf{v}') = -\det(N''_n(\mathbf{v}'))/\det(N''_{n+1}).$$

We've already seen that N''_n and N'_n are the same function so we can write this as

$$t''_n(\mathbf{v}') = -\det(N'_n(\mathbf{v}'))/\det(N''_{n+1}).$$

or

$$t''_n(\mathbf{v}') = \frac{\det(N'_{n+1})}{\det(N''_{n+1})}t'_n(\mathbf{v}') = \frac{\det(N'_{n+1})}{\det(N''_{n+1})}.$$

The numerator and denominator are, as we've already seen, the two determinants appearing in the hypotheses of the theorem. If they are of the same sign then

$$t''_n(\mathbf{w}(s)) = st''_n(\mathbf{v}') > 0$$

for all $s > 0$. We also have

$$t''_j(\mathbf{w}(0)) = t''_j(\mathbf{u}) = 1/n > 0$$

for $j < n$, so for sufficiently small positive s we have, by continuity,

$$t''_j(\mathbf{w}(s)) > 0.$$

Choose some such s with $s < 1$. Then $t_j(\mathbf{s}) > 0$ for all j so $\mathbf{w}(s) \in S''$. We already have $\mathbf{w}(s) \in S'$ for all $s \in (0,1)$ so

$$\mathbf{w}(s) \in S' \cap S''.$$

Let C be the convex hull of $\mathbf{v}_0, \ldots, \mathbf{v}_{n-1}, \mathbf{w}(s)$. C is convex and its vertices lie in $S' \cap S''$ and so $C \subseteq S' \cap S''$ and the dimension of C is less than or equal to that of $S' \cap S''$. $t'_n(\mathbf{x}) = 0$ for all $\mathbf{x}$ in the affine span $\mathbf{v}_0, \ldots, \mathbf{v}_{n-1}$ while $t_n(\mathbf{w}(s)) > 0$ so $\mathbf{w}(s)$ is not in the affine span of $\mathbf{v}_0, \ldots, \mathbf{v}_{n-1}$. So $\mathbf{v}_0, \ldots, \mathbf{v}_{n-1}$, $\mathbf{w}(s)$ are affinely independent and C is a simplex of dimension n. $S' \cap S''$ is therefore also of dimension n. □

Corollary 12.2.5. *If $\mathcal{C}$ is a simplicial complex in $\mathbf{R}^n$ and $F \in \mathcal{C}$ is of dimension $n-1$ then there are at most two simplices of dimension n in $\mathcal{C}$ of which F is a face.*

Proof. Let $\mathbf{v}_0$, $\mathbf{v}_1$, $\ldots \mathbf{v}_{n-1}$ be the vertices of a simplex F. Suppose E_1, E_2 and E_3 were distinct simplices in $\mathcal{C}$ such that F is a face of each of the three. Let $\mathbf{w}_k$ be the vertex of E_k which is not in F and

$$P_k = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ v_{1,0} & v_{1,1} & \cdots & v_{1,n-1} & w_{1,k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-1,0} & v_{n-1,1} & \cdots & v_{n-1,n-1} & w_{n-1,k} \\ v_{n,0} & v_{n,1} & \cdots & v_{n,n-1} & w_{n,k} \end{bmatrix}.$$

$E_1, E_2 \in \mathcal{C}$ so $E_1 \cap E_2 \in \mathcal{C}$ by the definition of a complex. If $E_1 \cap E_2$ were of dimension n then its relative interior would lie in the relative interiors of both E_1 and E_2. Every point in the underlying set of $\mathcal{C}$ lies in the relative interior of only one element of $\mathcal{C}$ though, so the dimension of $E_1 \cap E_2$ is at most $n-1$. It follows from Lemma 12.2.4 that $\det(P_1)$ and $\det(P_2)$ are of opposite sign. But the same argument, with obvious modifications, shows that $\det(P_1)$ and $\det(P_3)$ are of opposite sign and that $\det(P_2)$ and $\det(P_3)$ are of opposite sign. But this is impossible. □

Theorem 12.2.6. *Suppose $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ are simplicial complexes, with $\mathcal{C}$ being a refinement of both $\mathcal{C}'$ and $\mathcal{C}''$. For any n dimensional simplex E define*

$$V(E) = \frac{1}{n!}|\det(A_E)|$$

where A_E is the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ w_{E,1,0} & w_{E,1,1} & \cdots & w_{E,1,n-1} & w_{E,1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{E,n-1,0} & w_{E,n-1,1} & \cdots & w_{E,n-1,n-1} & w_{E,n-1,n} \\ w_{E,n,0} & w_{E,n,1} & \cdots & w_{E,n,n-1} & w_{E,n,n} \end{bmatrix},$$

$w_{E,i,j}$ is i'th coordinate of $\mathbf{w}_{E,j}$, and the vertices of E are $\mathbf{w}_{E,0}$, $\mathbf{w}_{E,1}$, …, $\mathbf{w}_{E,n}$. Then

$$\sum_{E\in\mathcal{C}'_n} V(E) = \sum_{E\in\mathcal{C}_n} V(E) = \sum_{E\in\mathcal{C}''_n} V(E)$$

where $\mathcal{C}_n$, $\mathcal{C}'_n$ and $\mathcal{C}''_n$ are the sets of simplices of dimension n in $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$, respectively.

Proof. Suppose $S \in \mathcal{C}'_n$. Let $\mathcal{S}$ be the simplicial complex in $\mathbf{R}^n$ consisting of the n dimensional simplex S and its faces. Let $\mathcal{R}$ be the simplicial complex consisting of those elements of $\mathcal{C}$ which are subsets of S. Then $\mathcal{R}$ is a refinement of $\mathcal{S}$. Let $\mathcal{R}_k$ and $\mathcal{S}_k$ be the sets of k dimensional elements of $\mathcal{R}$ and $\mathcal{S}$ respectively. In particular $\mathcal{R}_0$ is the set of vertices of elements of $\mathcal{R}$, and contains $\mathcal{S}_0$, the set of vertices of S. We can number the elements of $\mathcal{R}_0$, starting at 0, in such a way the the first $n+1$ are the elements of $\mathcal{R}_0$. Let $\mathbf{v}_0$, $\mathbf{v}_1$, …, $\mathbf{v}_m$ be the elements of $\mathcal{R}_0$, with this numbering. For each $E \in \mathcal{R}_k$ let $\mathbf{w}_{E,0}$, $\mathbf{w}_{E,1}$, …, $\mathbf{w}_{E,k}$ be the vertices of E, numbered in increasing order with respect to the ordering chosen for the elements of $\mathbf{R}_0$. In particular,

$$\mathbf{w}_{S,k} = \mathbf{v}_k.$$

Let M_E be the $n+1$ by $n+2$ matrix

$$\begin{bmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ v_{1,0} & \cdots & v_{1,n-k} & w_{E,1,0} & \cdots & w_{E,1,k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{n-1,0} & \cdots & v_{n-1,n-k} & w_{E,n-1,0} & \cdots & w_{E,n-1,k} \\ v_{n,0} & \cdots & v_{n,n-k} & w_{E,n,0} & \cdots & w_{E,n,k} \end{bmatrix}$$

and let $N_{E,k}$ be $(-1)^k$ times the $n+1$ by $n+2$ matrix obtained by removing the $k+1$'st column from M_E. By Lemma 12.2.1 we have

$$\sum_{k=0}^{n+1} \det(N_{E,k}) = 0.$$

Now

$$N_{E,n+h-k+1} = (-1)^h N_{F_h,n-k+1}$$

so

$$\det(N_{E,n+h-k+1}) = (-1)^h \det(N_{F_h,n-k+1})$$

where F_h is that face of E which does not have $\mathbf{w}_{E,h}$ as a vertex. Let $\mathcal{E}_k$ be the subset of $\mathcal{R}_k$ consisting of those simplices which are subsets of the k dimensional simplex in $\mathcal{S}_k$ whose vertices are $\mathbf{v}_{n-k}$, …, $\mathbf{v}_n$. Let $\mathcal{F}_k$ be the subset of $\mathcal{R}_k$ consisting of those simplices which are subsets of the $k+1$ dimensional simplex in $\mathcal{S}_k$ whose vertices are $\mathbf{v}_{n-k-1}$, …, $\mathbf{v}_n$. Then $\mathcal{E}_k \subseteq \mathcal{F}_k$. If $E \in \mathcal{E}_k$ and $j < n-k$ then the $n+1$ columns of $N_{E,j}$ are linear combinations of the n vectors $\mathbf{v}_0$, …, $\mathbf{v}_{j-1}$, $\mathbf{v}_{j+1}$, …, $\mathbf{v}_n$ and so

$$\det(N_{E,j}) = 0$$

for such j. We can therefore rewrite the equation

$$\sum_{j=0}^{n+1} \det(N_{E,j}) = 0$$

as

$$\det(N_{E,n-k}) = -\sum_{h=0}^{k} \det(N_{E,n+h-k+1})$$

or

$$\det(N_{E,n-k}) = \sum_{h=0}^{k} (-1)^{h+1} \det(N_{F_h,n-k+1})$$

It follows that

$$\sum_{E\in\mathcal{E}_k} \rho_E \det(N_{E,n-k}) = \sum_{E\in\mathcal{E}_k} \sum_{F\in\mathcal{F}_{k-1}} \rho_E \sigma_{E,F} \det(N_{F_h,n-k})$$

where ρ_E is the sign of $\det(N_{E,n-k})$ and $\sigma_{E,F}$ is $(-1)^{h+1}$ if F is that face of E for which $\mathbf{w}_{E,h}$ is not a vertex and 0 if F is not a face of E. For any $F \in \mathcal{F}_{k-1}$ either

- $F \in \mathcal{E}_{k-1}$ and there is one $E \in \mathcal{E}_k$ such that F is a face of E, with $\rho_F = \rho_E \sigma_{E,F}$, or
- $F \notin \mathcal{E}_{k-1}$ and there are two E's such that F is a face of E, with opposite values of $\rho_E \sigma_{E,F}$.

This follows from Lemma 12.2.4. Therefore

$$\sum_{E\in\mathcal{E}_k} \rho_E \det(N_{E,n-k}) = \sum_{F\in\mathcal{E}_{k-1}} \rho_F \det(N_{F,n-k+1}).$$

We therefore have

$$\sum_{E \in \mathcal{E}_0} \rho_E \det(N_{E,n}) = \sum_{E \in \mathcal{E}_n} \rho_E \det(N_{E,0}).$$

Now $\mathcal{E}_0$ consists of the single element $\mathbf{v}_n$ while $\mathcal{E}_n = \mathcal{R}_n$. This equation therefore tells us that the absolute value of the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ v_{1,0} & v_{1,1} & \cdots & v_{1,n-1} & v_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-1,0} & v_{n-1,1} & \cdots & v_{n-1,n-1} & v_{n-1,n} \\ v_{n,0} & v_{n,1} & \cdots & v_{n,n-1} & v_{n,n} \end{bmatrix}$$

is equal to the sum of the absolute values of the determinants of

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ w_{E,1,0} & w_{E,1,1} & \cdots & w_{E,1,n-1} & w_{E,1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{E,n-1,0} & w_{E,n-1,1} & \cdots & w_{E,n-1,n-1} & w_{E,n-1,n} \\ w_{E,n,0} & w_{E,n,1} & \cdots & w_{E,n,n-1} & w_{E,n,n} \end{bmatrix}$$

over all $E \in \mathbf{R}_n$.

Summing over all $S \in \mathcal{C}'_n$ and dividing by $n!$ gives

$$\sum_{E \in \mathcal{C}'_n} V(E) = \sum_{E \in \mathcal{C}_n} V(E).$$

The same argument with $\mathcal{C}''$ in place of $\mathcal{C}'$ gives

$$\sum_{E \in \mathcal{C}_n} V(E) = \sum_{E \in \mathcal{C}''_n} V(E).$$

□

Corollary 12.2.7. *If $\mathcal{C}'$ and $\mathcal{C}''$ are simplicial complexes with the same underlying space then*

$$\sum_{E \in \mathcal{C}'_n} V(E) = \sum_{E \in \mathcal{C}''_n} V(E).$$

Proof. By Theorem 11.5.3 there is a simplicial complex $\mathcal{C}$ which is a refinement of both $\mathcal{C}'$ and $\mathcal{C}''$, so we can apply the theorem above. □

Corollary 12.2.8. *Suppose E is a bounded semilinear set, $\mathcal{C}'$ and $\mathcal{C}''$ are simplicial complexes whose underlying set is $\overline{E}$, and $\mathcal{A}'$ and $\mathcal{A}''$ are subsets of $\mathcal{C}'$ and $\mathcal{C}''$ respectively such that*

$$\bigcup_{F \in \mathcal{A}'} F^\diamond = E = \bigcup_{F \in \mathcal{A}''} F^\diamond.$$

Then

$$\sum_{E \in \mathcal{C}'_n} V(E) = \sum_{E \in \mathcal{C}''_n} V(E).$$

Definition 12.2.9. Suppose E is a simplicial set If E is bounded then we say the *semilinear content* of E is $\sum_{E \in \mathcal{C}'_n} V(E)$, where $\mathcal{C}$ is a simplicial complex with underlying set $\overline{E}$, $\mathcal{C}'_n$ is the set of n-dimensional simplices in $\mathcal{C}'$ and $V(E)$ is the volume of the simplex E, given by the determinantal formula above. If E is unbounded then we say the semilinear content of E is $+\infty$.

That volume is well defined follows from Theorem 11.5.3, which assures that that there is such a simplicial complex $\mathcal{C}'$, and from the corollary above, which shows that the semilinear content is independent of which such simplicial complex is chosen.

Proposition 12.2.10. *The semilinear content is a content on the semilinear algebra.*

Proof. Clearly the semilinear content of the empty set is 0. If E and F are disjoint bounded semilinear sets then we can choose a simplicial complex $\mathcal{C}$ with underlying set $\overline{E}$ and a simplicial complex $\mathcal{D}$ with underlying set $\overline{F}$. Then $\mathcal{C} \cup \mathcal{D}$ is a simplicial complex with underlying set $\overline{E \cup F}$ and each n-dimensional simplex in $\mathcal{C} \cup \mathcal{D}$ belongs either to $\mathcal{C}_n$ or to $\mathcal{D}_n$ but not both. It follows that the semilinear content of $E \cup F$ is the sum of the semilinear contents of E and of F. If E or F is unbounded then so is $E \cup F$, since it's a superset of both, and so again the semilinear content of $E \cup F$ is the sum of the semilinear contents of E and of F, since the sum of $+\infty$ and any element of $[0, +\infty]$ is $+\infty$. □

Definition 12.2.11. The completion of the semilinear content space on $\mathbf{R}^n$ is called the Jordan content space. Its Boolean algebra is called the *Jordan algebra* on $\mathbf{R}^n$ and the elements of this algebra are called the *Jordan sets* in $\mathbf{R}^n$. The content is called *Jordan content* on $\mathbf{R}^n$.

Proposition 12.2.12. *Every compactly supported continuous function on* $\mathbf{R}^n$ *is integrable with respect to Jordan content.*

Proof. Suppose g is compactly supported and continuous. Then g is supported in $[-M, M]^n$ for some $M \in \mathbf{N}$. The restriction of g to $[-M, M]^n$ is a continuous function on a compact set and so is uniformly continuous. For any $\epsilon > 0$ there is therefore a $\delta > 0$ such that

$$|g(\mathbf{x}) - g(\mathbf{y})| < \frac{\epsilon}{(2M)^n}$$

if $\|\mathbf{x} - \mathbf{y}\| < \delta$. Choose a $k \in \mathbf{N}$ such that

$$k > \frac{\sqrt{n}}{\delta}.$$

Let $\mathcal{P}$ be the set of hypercubes of side length $1/k$ in $[-M, M]^n$ whose vertices are rational numbers with denominators divisible by k. There are $(2Mk)^n$ such hypercubes. Let $\mathcal{Q}$ be $\mathcal{P}$ together with the complement of $[-M, M]^n$. For each $E \in \mathcal{Q}$ let

$$f(\mathbf{x}) = \inf_{\mathbf{y} \in E} g(\mathbf{y})$$

and

$$h(\mathbf{x}) = \sup_{\mathbf{y} \in E} g(\mathbf{y})$$

If $\mathbf{x} \in E$ where $E \in \mathcal{Q}$ then

$$f(\mathbf{x}) \le h(\mathbf{x}) \le f(\mathbf{x}) + \frac{\epsilon}{(2M)^n}$$

because of the inequalities for $|g(\mathbf{x} - \mathbf{y})|$ above. If $\mathbf{x} \in \mathbf{R}^n \setminus [-M, M]^n$ then

$$f(\mathbf{x}) = 0 = h(\mathbf{x}).$$

We therefore have simple functions f and h such that

$$f(\mathbf{x}) \le g(\mathbf{x}) \le h(\mathbf{x})$$

for all $\mathbf{x} \in \mathbf{R}^n$ and

$$\int_{\mathbf{x} \in \mathbf{R}^n} h(\mathbf{x})\, dm(\mathbf{x}) \le \int_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{x})\, dm(\mathbf{x}) + \epsilon.$$

We have such f and h for all $\epsilon > 0$ so g is integrable by our usual criterion for integrability. □

The following is the Fubini Theorem for Riemann integrals.

Theorem 12.2.13. *Suppose* g *is a compactly supported continuous function on* $\mathbf{R}^n$ *where* $n = n' + n''$. *Then*

$$\int_{\mathbf{x}' \in \mathbf{R}^{n'}} \int_{\mathbf{x}'' \in \mathbf{R}^{n''}} g(\mathbf{x}', \mathbf{x}'')\, d\mu_{n''}(\mathbf{x}'')\, d\mu_{n'}(\mathbf{x}'),$$

$$\int_{\mathbf{x} \in \mathbf{R}^n} g(\mathbf{x})\, d\mu_n(\mathbf{x}),$$

and

$$\int_{\mathbf{x}'' \in \mathbf{R}^{n''}} \int_{\mathbf{x}' \in \mathbf{R}^{n'}} g(\mathbf{x}', \mathbf{x}'')\, d\mu_{n'}(\mathbf{x}')\, d\mu_{n''}(\mathbf{x}'')$$

are all equal, where $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$ *and* μ_k *denotes Jordan content on* $\mathbf{R}^k$.

Proof. For each $\epsilon > 0$ we set up f and h as in the proof of the preceding proposition. f is a simple function so we can write it as a sum over elements of $\mathcal{Q}$ or, since the summand for $\mathbf{R}^n \setminus [-M, M]^n$ is zero, a sum over elements of $\mathcal{P}$. It doesn't matter how we order this sum, so we can sum first over those elements where the projection from $\mathbf{R}^n$ to $\mathbf{R}^{n'}$ is a particular hypercube, of which there are $(2Mk)^{n''}$, and then over all $(2Mk)^{n'}$ choices for this hypercube. The inner sum is $k^{-n'}$ times the integral

$$\int_{\mathbf{x}'' \in \mathbf{R}^{n''}} f(\mathbf{x}', \mathbf{x}'')\, d\mu_{n''}(\mathbf{x}'')$$

by our formula for integrals of simple functions on $\mathbf{R}^{n''}$ and the outer sum is

$$\int_{\mathbf{x}' \in \mathbf{R}^{n'}} \int_{\mathbf{x}'' \in \mathbf{R}^{n''}} f(\mathbf{x}', \mathbf{x}'')\, d\mu_{n''}(\mathbf{x}'')\, d\mu_{n'}(\mathbf{x}')$$

by the same formula, but for functions on $\mathbf{R}^{n'}$. This sum is also equal to

$$\int_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{x})\, d\mu_n(\mathbf{x}),$$

again by the formula for integrals of simple functions, but this time for functions on $\mathbf{R}^n$. So

$$\int_{\mathbf{x}' \in \mathbf{R}^{n'}} \int_{\mathbf{x}'' \in \mathbf{R}^{n''}} f(\mathbf{x}', \mathbf{x}'')\, d\mu_{n''}(\mathbf{x}'')\, d\mu_{n'}(\mathbf{x}')$$

and

$$\int_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{x})\,d\mu_n(\mathbf{x})$$

are equal. A similar argument, but grouping the hypercubes by their projection onto $\mathbf{R}^{n''}$, shows that this integral is the same as

$$\int_{\mathbf{x}''\in\mathbf{R}^{n''}} \int_{\mathbf{x}'\in\mathbf{R}^{n'}} f(\mathbf{x}',\mathbf{x}'')\,d\mu_{n'}(\mathbf{x}')\,d\mu_{n''}(\mathbf{x}'').$$

We can do the same with h in place of f. Now

$$g(\mathbf{x}) \le h(\mathbf{x})$$

for all $\mathbf{x}\in\mathbf{R}^n$ so

$$\int_{\mathbf{x}''\in\mathbf{R}^{n''}} g(\mathbf{x}',\mathbf{x}'')\,d\mu_{n''}(\mathbf{x}'')$$

is less than or equal to

$$\int_{\mathbf{x}''\in\mathbf{R}^{n''}} h(\mathbf{x}',\mathbf{x}'')\,d\mu_{n''}(\mathbf{x}'')$$

and hence

$$\int_{\mathbf{x}'\in\mathbf{R}^{n'}} \int_{\mathbf{x}''\in\mathbf{R}^{n''}} g(\mathbf{x}',\mathbf{x}'')\,d\mu_{n''}(\mathbf{x}'')\,d\mu_{n'}(\mathbf{x}')$$

is less than or equal to

$$\int_{\mathbf{x}'\in\mathbf{R}^{n'}} \int_{\mathbf{x}''\in\mathbf{R}^{n''}} h(\mathbf{x}',\mathbf{x}'')\,d\mu_{n''}(\mathbf{x}'')\,d\mu_{n'}(\mathbf{x}'),$$

which, as we just saw, is equal to

$$\int_{\mathbf{x}\in\mathbf{R}^n} h(\mathbf{x})\,d\mu_n(\mathbf{x})$$

This, in turn, is less than or equal to

$$\int_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{x})\,d\mu_n(\mathbf{x}) + \epsilon$$

which is less than or equal to

$$\int_{\mathbf{x}\in\mathbf{R}^n} g(\mathbf{x})\,d\mu_n(\mathbf{x}) + \epsilon.$$

Combining these,

$$\int_{\mathbf{x}'\in\mathbf{R}^{n'}} \int_{\mathbf{x}''\in\mathbf{R}^{n''}} g(\mathbf{x}',\mathbf{x}'')\,d\mu_{n''}(\mathbf{x}'')\,d\mu_{n'}(\mathbf{x}')$$

is less than or equal to

$$\int_{\mathbf{x}\in\mathbf{R}^n} g(\mathbf{x})\,d\mu_n(\mathbf{x}) + \epsilon.$$

We can also run that argument in the reverse direction though.

$$\int_{\mathbf{x}\in\mathbf{R}^n} g(\mathbf{x})\,d\mu_n(\mathbf{x})$$

is less than or equal to

$$\int_{\mathbf{x}\in\mathbf{R}^n} h(\mathbf{x})\,d\mu_n(\mathbf{x})$$

which in turn is less than or equal to

$$\int_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{x})\,d\mu_n(\mathbf{x}) + \epsilon.$$

But

$$\int_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{x})\,d\mu_n(\mathbf{x})$$

is equal to

$$\int_{\mathbf{x}''\in\mathbf{R}^{n''}} \int_{\mathbf{x}'\in\mathbf{R}^{n'}} f(\mathbf{x}',\mathbf{x}'')\,d\mu_{n'}(\mathbf{x}')\,d\mu_{n''}(\mathbf{x}''),$$

which is less than or equal to

$$\int_{\mathbf{x}''\in\mathbf{R}^{n''}} \int_{\mathbf{x}'\in\mathbf{R}^{n'}} g(\mathbf{x}',\mathbf{x}'')\,d\mu_{n'}(\mathbf{x}')\,d\mu_{n''}(\mathbf{x}'').$$

Combining all of these,

$$\int_{\mathbf{x}\in\mathbf{R}^n} g(\mathbf{x})\,d\mu_n(\mathbf{x})$$

is less than or equal to

$$\int_{\mathbf{x}''\in\mathbf{R}^{n''}} \int_{\mathbf{x}'\in\mathbf{R}^{n'}} g(\mathbf{x}',\mathbf{x}'')\,d\mu_{n'}(\mathbf{x}')\,d\mu_{n''}(\mathbf{x}'') + \epsilon.$$

Together with what we showed earlier this means that the difference between

$$\int_{\mathbf{x}\in\mathbf{R}^n} g(\mathbf{x})\,d\mu_n(\mathbf{x})$$

and

$$\int_{\mathbf{x}''\in\mathbf{R}^{n''}} \int_{\mathbf{x}'\in\mathbf{R}^{n'}} g(\mathbf{x}',\mathbf{x}'')\,d\mu_{n'}(\mathbf{x}')\,d\mu_{n''}(\mathbf{x}'')$$

is of absolute value at most ϵ. This holds for all $\epsilon > 0$ so the difference must be zero. The same holds, by an almost identical argument, for the difference between

$$\int_{\mathbf{x}\in\mathbf{R}^n} g(\mathbf{x})\, d\mu_n(\mathbf{x})$$

and

$$\int_{\mathbf{x}'\in\mathbf{R}^{n'}} \int_{\mathbf{x}''\in\mathbf{R}^{n''}} g(\mathbf{x}', \mathbf{x}'')\, d\mu_{n''}(\mathbf{x}'')\, d\mu_{n'}(\mathbf{x}').$$

□

12.3 Lebesgue measure

Proposition 12.3.1. *There is a unique Radon measure μ_B on $\mathbf{R}^n$ such that*

$$\int_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{x})\, d\mu_B(\mathbf{x}) = \int_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{x})\, d\mu_J(\mathbf{x})$$

where μ_J is Jordan content on $\mathbf{R}^n$.

Proof. This follows immediately from the Riesz Representation Theorem applied to

$$I(f) = \int_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{x})\, d\mu_J(\mathbf{x}),$$

which Proposition 12.2.12 assures us is defined for all compactly supported continuous functions f on the locally compact σ-compact Hausdorff topological space $\mathbf{R}^n$. □

We're less interested in the Borel measure μ_B than we are in its completion, which is described by the following theorem.

Theorem 12.3.2. *There is a σ-algebra $\mathcal{B}_L$ on $\mathbf{R}^n$ and a measure m on $(\mathbf{R}^n, \mathcal{B}_L)$ with the following properties.*

(a) The Borel σ-algebra is a subset of $\mathcal{B}_L$.

(b) If $K \in \mathcal{B}_L$ is compact then $m(K) < +\infty$.

(c) If $E \in \mathcal{B}_L$ then

$$m(E) = \sup m(K)$$

where the supremum is over all compact $K \in \mathcal{B}_L$ such that $K \subseteq E$.

(d) If $E \in \mathcal{B}_L$ then

$$m(E) = \inf m(U)$$

where the infimum is over all open $U \in \mathcal{B}_L$ such that $K \subseteq U$.

(e)

$$\int_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{x})\, dm(\mathbf{x}) = \int_{\mathbf{x}\in\mathbf{R}^n} f(\mathbf{x})\, d\mu_J(\mathbf{x})$$

where μ_J is Jordan content on $\mathbf{R}^n$.

(f) If $F \in \mathcal{B}_L$, $m(F) = 0$ and $E \subseteq F$ then $E \in \mathcal{B}_L$ and $m(E) = 0$.

Proof. We apply Theorem 7.6.11 to $(\mathbf{R}^n, \mathcal{B}_B, \mu_B)$, where $\mathcal{B}_B$ is the Borel σ-algebra and μ_B is the measure from the preceding proposition. All of the properties of m are consequences of the corresponding properties of μ_B except for the last, which follows from Proposition 7.6.13. □

Definition 12.3.3. The σ-algebra $\mathcal{B}_L$ from the preceding theorem is called the Lebesgue σ-algebra on $\mathbf{R}^n$ and its elements are called *Lebesgue sets*. The measure m is called *Lebesgue measure* on $\mathbf{R}^n$.

12.4 Fubini-Tonelli

Suppose $n = n' + n''$. For functions g on $\mathbf{R}^n$ define the iterated integrals

$$I_1(g) = \int_{\mathbf{x}'\in\mathbf{R}^{n'}} \int_{\mathbf{x}''\in\mathbf{R}^{n''}} g(\mathbf{x}', \mathbf{x}'')\, dm_{n''}(\mathbf{x}'')\, dm_{n'}(\mathbf{x}')$$

and

$$I_2(g) = \int_{\mathbf{x}''\in\mathbf{R}^{n''}} \int_{\mathbf{x}'\in\mathbf{R}^{n'}} g(\mathbf{x}', \mathbf{x}'')\, dm_{n'}(\mathbf{x}')\, dm_{n''}(\mathbf{x}'')$$

where m_k denotes Lebesgue measure in $\mathbf{R}^k$, provided these integrals exist. By Theorem 12.2.13 we know that

$$I_1(g) = I(g) = I_2(g)$$

for compactly supported continuous functions, where

$$I(g) = \int_{\mathbf{x}\in\mathbf{R}^n} g(\mathbf{x}', \mathbf{x}'')\, dm_n(\mathbf{x}).$$

The goal of this section is to prove the same result for all integrable functions.

We define

$$\mu_1(E) = I_1(\chi_E)$$

and

$$\mu_2(E) = I_2(\chi_E)$$

If E is a Borel set then for each $\mathbf{x}' \in \mathbf{R}^{n'}$ we have

$$\chi_E(\mathbf{x}', \mathbf{x}'') = \chi_{\varphi^*_{\mathbf{x}'}(E)}(\mathbf{x}'')$$

where $\varphi_{\mathbf{x}'}$ is the function from $\mathbf{R}^{n''}$ to $\mathbf{R}$ defined by

$$\varphi_{\mathbf{x}'}(\mathbf{x}'') = (\mathbf{x}', \mathbf{x}'').$$

This function is continuous so $\varphi^*_{\mathbf{x}'}(E)$ is a Borel set by Proposition 7.2.9. So $\chi_{\varphi^*_{\mathbf{x}'}(E)}$ is Borel measurable and we can write

$$\mu_1(E) = \int_{\mathbf{x}' \in \mathbf{R}^{n'}} m_{n''}\left(\varphi^*_{\mathbf{x}'}(E)\right) \, dm_{n'}(\mathbf{x}')$$

with the integrand being defined for all $\mathbf{x}'$. Similarly,

$$\mu_2(E) = \int_{\mathbf{x}'' \in \mathbf{R}^{n''}} m_{n'}\left(\psi^*_{\mathbf{x}''}(E)\right) \, dm_{n''}(\mathbf{x}'')$$

where

$$\psi_{\mathbf{x}''}(\mathbf{x}') = (\mathbf{x}', \mathbf{x}'').$$

For Lebesgue measurable sets the situation is more complicated. Even if E is Lebesgue measurable the sets $\varphi^*_{\mathbf{x}'}(E)$ and $\psi^*_{\mathbf{x}''}(E)$ needn't be Lebesgue measurable for all $\mathbf{x}' \in \mathbf{R}^{n'}$ and $\mathbf{x}'' \in \mathbf{R}^{n''}$. They are Lebesgue measurable for almost all $\mathbf{x}'$ and $\mathbf{x}''$ though, and this suffices to define the integrals.

μ_1 and μ_2 are in fact just m_n, but we don't know this yet. We will show this gradually, by showing that they share various properties of m_n until we have enough properties that they must be equal to m_n.

If $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^n$ then

$$\int_{\mathbf{x}'' \in \mathbf{R}^{n''}} f(\mathbf{x}', \mathbf{x}'') \, dm_{n''}(\mathbf{x}'')$$

is less than or equal

$$\int_{\mathbf{x}'' \in \mathbf{R}^{n''}} g(\mathbf{x}', \mathbf{x}'') \, dm_{n''}(\mathbf{x}'')$$

for all $\mathbf{x}' \in \mathbf{R}^{n'}$ by the monotonicity property of integrals in $\mathbf{R}^{n''}$ and so

$$\int_{\mathbf{x}' \in \mathbf{R}^{n'}} \int_{\mathbf{x}'' \in \mathbf{R}^{n''}} f(\mathbf{x}', \mathbf{x}'') \, dm_{n''}(\mathbf{x}'') \, dm_{n'}(\mathbf{x}')$$

is less than or equal

$$\int_{\mathbf{x}' \in \mathbf{R}^{n'}} \int_{\mathbf{x}'' \in \mathbf{R}^{n''}} g(\mathbf{x}', \mathbf{x}'') \, dm_{n''}(\mathbf{x}'') \, dm_{n'}(\mathbf{x}').$$

In other words,

$$I_1(f) \leq I_1(g).$$

A similar arguments gives

$$I_2(f) \leq I_2(g).$$

If $E \subseteq F$ then

$$\chi_E(\mathbf{x}) \leq \chi_F(\mathbf{x})$$

for all $\mathbf{x} \in \mathbf{R}^n$ so

$$I_j(\chi_E) \leq I_j(\chi_F)$$

and hence

$$\mu_j(E) \leq \mu_j(F),$$

where $j = 1$ or $j = 2$.

If $E_0 \subseteq E_1 \subseteq \cdots$ then

$$\int_{\mathbf{x}'' \in \mathbf{R}^{n''}} \lim_{k \to \infty} \chi_{E_k}(\mathbf{x}', \mathbf{x}'') \, dm_{n''}(\mathbf{x}'')$$

and

$$\lim_{k \to \infty} \int_{\mathbf{x}'' \in \mathbf{R}^{n''}} \chi_{E_k}(\mathbf{x}', \mathbf{x}'') \, dm_{n''}(\mathbf{x}'')$$

are equal by the Monotone Convergence Theorem. If $k \leq l$ then

$$\int_{\mathbf{x}'' \in \mathbf{R}^{n''}} \chi_{E_k}(\mathbf{x}', \mathbf{x}'') \, dm_{n''}(\mathbf{x}'')$$

is less than or equal to

$$\int_{\mathbf{x}'' \in \mathbf{R}^{n''}} \chi_{E_l}(\mathbf{x}', \mathbf{x}'') \, dm_{n''}(\mathbf{x}'')$$

so we can also apply the Monotone Convergence Theorem to the integrals over $\mathcal{R}^{n'}$ to get that

$$\lim_{k\to\infty} \mu_1(E_k) = \mu_1\left(\bigcup_{k=0}^{\infty} E_k\right).$$

Of course we also have

$$\lim_{k\to\infty} \mu_2(E_k) = \mu_2\left(\bigcup_{k=0}^{\infty} E_k\right).$$

If $E_0 \supseteq E_1 \supseteq \cdots$ and $\mu(E_0) < +\infty$ then we can apply a similar argument, but with the Dominated Convergence Theorem in place of the Monotone Convergence Theorem, to get

$$\lim_{k\to\infty} \mu_1(E_k) = \mu_1\left(\bigcap_{k=0}^{\infty} E_k\right).$$

and

$$\lim_{k\to\infty} \mu_2(E_k) = \mu_2\left(\bigcap_{k=0}^{\infty} E_k\right).$$

Proposition 12.4.1. *Suppose K is a compact subset of $\mathbf{R}^n$ and $j = 1$ or $j = 2$. Then*

$$\mu_j(K) \le m_n(K).$$

Proof. $m_n(K) < +\infty$ by one of the defining properties of Radon measures. Choose some $\lambda > m_n(K)$. By one of the other properties of Radon measures we have

$$m_n(K) = \inf m_n(U)$$

where the infimum is over open supersets U of K. There is therefore an open superset U such that

$$m_n(U) < \lambda.$$

By Proposition 9.3.1 there is a compactly supported continuous function $h\colon \mathbf{R}^n \to [0,1]$ such that $h(\mathbf{x}) = 1$ if $\mathbf{x} \in K$ and $h(\mathbf{x}) = 0$ if $\mathbf{x} \notin U$. Then

$$\chi_K(\mathbf{x}) \le h(\mathbf{x}) \le \chi_U(\mathbf{x})$$

for all $\mathbf{x} \in \mathbf{R}^n$ and hence

$$\mu_j(K) = I_j(K) \le I_j(h).$$

Now h is compactly supported and continuous so

$$I_j(h) = I(h).$$

By monotonicity

$$I(h) \le I(\chi_U) = m_n(U) < \lambda,$$

so

$$\mu_j(K) < \lambda.$$

This holds for all $\lambda > m_n(K)$ so

$$\mu_j(K) \le m_n(K).$$

□

Proposition 12.4.2. *Suppose U is an open subset of $\mathbf{R}^n$ and $j = 1$ or $j = 2$. Then*

$$\mu_j(U) \ge m_n(U).$$

Proof. Choose some $\lambda < m_n(U)$. By one of the properties of Radon measures we have

$$m_n(U) = \sup m_n(K)$$

where the supremum is over compact subsets K of U. There is therefore a compact subset K such that

$$m_n(K) > \lambda.$$

By Proposition 9.3.1 there is a compactly supported continuous function $f\colon \mathbf{R}^n \to [0,1]$ such that $f(\mathbf{x}) = 1$ if $\mathbf{x} \in K$ and $f(\mathbf{x}) = 0$ if $\mathbf{x} \notin U$. Then

$$\chi_K(\mathbf{x}) \le f(\mathbf{x}) \le \chi_U(\mathbf{x})$$

for all $\mathbf{x} \in \mathbf{R}^n$ and hence

$$\mu_j(U) = I_j(U) \ge I_j(f).$$

Now f is compactly supported and continuous so

$$I_j(f) = I(f).$$

By monotonicity

$$I(f) \ge I(\chi_K) = m_n(K) > \lambda,$$

so

$$\mu_j(U) > \lambda.$$

This holds for all $\lambda < m_n(U)$ so

$$\mu_j(U) \ge m_n(U).$$

□

Proposition 12.4.3. *Suppose K is a compact subset of $\mathbf{R}^n$ and $j = 1$ or $j = 2$. Then*

$$\mu_j(K) = m_n(K).$$

Proof. We've already proved that

$$\mu_j(K) \le m_n(K)$$

so it suffices to prove

$$\mu_j(K) \ge m_n(K).$$

K is compact so

$$m_n(K) < +\infty$$

There is therefore a λ such that $m_n(K) < \lambda < +\infty$ and we can choose an open superset U of K such that

$$m_n(U) < \lambda < +\infty.$$

Let

$$V_k = U \cap \bigcup_{\mathbf{x} \in K} B(\mathbf{x}, 1/2^k).$$

Then $V_0 \supseteq V_1 \supseteq \cdots$. Also $V_0 \subseteq U$ so

$$m_n(V_0) < +\infty.$$

It follows that

$$\lim_{k \to \infty} \mu(V_k) = \mu_j\left(\bigcap_{k=0}^{\infty} V_k\right).$$

V_k is open so

$$\mu_j(V_k) \ge m_n(V_k)$$

by Proposition 12.4.2. Therefore

$$\lim_{k \to \infty} m_n(V_k) \le \mu\left(\bigcap_{k=0}^{\infty} V_k\right).$$

Now $K \subseteq V_k$ for each k so

$$K \subseteq \bigcap_{k=0}^{\infty} V_k.$$

On the other hand, if $\mathbf{x} \in \bigcap_{k=0}^{\infty} V_k$ then $\mathbf{x} \in V_k$ for all k, so for every k there is a $\mathbf{y}_k \in K$ such that $\mathbf{x} \in B(\mathbf{y}_k, 1/2^k)$ or, equivalently, $\mathbf{y}_k \in B(\mathbf{x}, 1/2^k)$. Then $\mathbf{x} = \lim_{k \to \infty} \mathbf{y}_k$ and so $\mathbf{x} \in \overline{K} = K$. Therefore

$$\bigcap_{k=0}^{\infty} V_k \subseteq K$$

and hence

$$K = \bigcap_{k=0}^{\infty} V_k.$$

So we can rewrite

$$\lim_{k \to \infty} m_n(V_k) \le \mu_j\left(\bigcap_{k=0}^{\infty} V_k\right).$$

as

$$\lim_{k \to \infty} m_n(V_k) \le \mu_j(K).$$

Now $K \subseteq V_k$ for all k so

$$m_n(K) \le m_n(V_k)$$

and hence

$$m_n(K) \le \lim_{k \to \infty} m_n(V_k)$$

and hence

$$m_n(K) \le \mu_j(K).$$

We already obtained the reverse inequality in Proposition 12.4.1 so

$$\mu(K) = m_n(K).$$

□

Proposition 12.4.4. *Suppose U is an open subset of $\mathbf{R}^n$ and $j = 1$ or $j = 2$. Then*

$$\mu_j(U) = m_n(U).$$

Proof. We've already proved that

$$\mu_j(U) \ge m_n(U)$$

so it suffices to prove

$$\mu_j(U) \le m_n(U).$$

Let $\mathcal{A}$ be the set of balls with rational coordinates and radii which are subsets of U. There are only countably many such balls so we can write

$$\mathcal{A} = \{B(\mathbf{y}_0, r_0), B(\mathbf{y}_1, r_1), \ldots\}.$$

Let

$$K_k = \bigcup_{i=0}^{k} \bar{B}(\mathbf{y}_i, r_i/2).$$

Then $K_0 \subseteq K_1 \subseteq \cdots$. It follows that

$$\lim_{k\to\infty} \mu_j(K_k) = \mu_j\left(\bigcup_{k=0}^{\infty} K_k\right).$$

K_k is compact so

$$\mu_j(K_k) \leq m_n(K_k)$$

by Proposition 12.4.1. Therefore

$$\lim_{k\to\infty} m_n(K_k) \geq \mu_j\left(\bigcup_{k=0}^{\infty} K_k\right).$$

Now $K_k \subseteq U$ for each k so

$$\bigcup_{k=0}^{\infty} K_k \subseteq U$$

On the other hand, if $\mathbf{x} \in U$ then $B(\mathbf{x}, s) \subseteq U$ for some s. There is a $\mathbf{y}$ with rational coefficients in $B(\mathbf{x}, s/4)$. Then $B(\mathbf{y}, 3s/4) \subseteq U$. Choose a rational $r \in (s/2, 3s/4)$ Then $\mathbf{x} \in \bar{B}(\mathbf{y}, r/2)$ and $B(\mathbf{y}, r) \subseteq U$. Therefore $B(\dagger, r) \in \mathcal{A}$ and so $\mathbf{x} \in K_k$ for some k. So

$$U \subseteq \bigcup_{k=0}^{\infty} K_k$$

and, since we already have the reverse inclusion,

$$U = \bigcup_{k=0}^{\infty} K_k$$

We can therefore rewrite

$$\lim_{k\to\infty} m_n(K_k) \geq \mu_j\left(\bigcup_{k=0}^{\infty} K_k\right)$$

as

$$\lim_{k\to\infty} m_n(K_k) \geq \mu_j(U).$$

Now $K_k \subseteq U$ for all k so

$$m_n(K_k) \leq m_n(U)$$

and hence

$$\lim_{k\to\infty} m_n(V_k) \leq m_n(U)$$

and therefore

$$\mu_j(U) \leq m_n(U).$$

We already obtained the reverse inequality in Proposition 12.4.2 so

$$\mu_j(U) = m_n(U).$$

□

Proposition 12.4.5. *Suppose E is a Borel subset of $\mathbf{R}^n$ and $j = 1$ or $j = 2$. Then*

$$\mu_j(E) = m_n(E).$$

Proof. Suppose $\lambda < m_n(E)$ There is a compact subset K of E such that

$$m_n(K) > \lambda.$$

By Proposition 12.4.4 then

$$\mu_j(K) > \lambda.$$

K is a subset of E so

$$\mu_j(K) \leq \mu_j(E)$$

and hence

$$\lambda < \mu_j(E)$$

This holds for all $\lambda < m_n(E)$ so

$$m_n(E) \leq \mu_j(E).$$

If $m_n(E) = +\infty$ then we have

$$m_n(E) \geq \mu_j(E)$$

since every number is less than or equal to $+\infty$. If $m_n(E) < +\infty$ then we can find a λ with

$$m_n(E) < \lambda < +\infty$$

and then an open superset U of E such that

$$\mu_j(U) < \lambda.$$

E is a subset of U so

$$\mu_j(E) \leq \mu_j(U)$$

and hence

$$\mu_j(E) < \lambda.$$

This holds for all λ such that $m_n(E) < \lambda < +\infty$ so

$$m_n(E) \geq \mu_j(E)$$

We found the same inequality whether $m_n(E) = +\infty$ or $m_n(E) < +\infty$. We already have the reverse inequality, so

$$\mu_j(E) = m_n(E).$$

□

Proposition 12.4.6. *Suppose E is a Lebesgue subset of $\mathbf{R}^n$ and $j = 1$ or $j = 2$. Then*

$$\mu_j(E) = m_n(E).$$

Proof. By Proposition 7.6.12 there are Borel sets D and H such that

$$E \triangle H \subseteq D$$

and

$$m_n(D) = 0.$$

Then

$$H \setminus D \subseteq E \subseteq D \cup H \subseteq (H \setminus D) \cup D$$

so

$$\begin{aligned}\mu_j(E) &\leq \mu_j(D \cup H) = m_n(D \cup H) \\ &\leq m_n(H \setminus D) + m_n(D) = m_n(H \setminus D) \\ &\leq m_n(E).\end{aligned}$$

Also,

$$\begin{aligned}\mu_j(E) &\geq \mu_j(H \setminus D) = m_n(H \setminus D) \\ &= m_n(H \setminus D) + m_n(D) = m_n(H \cup D) \\ &\geq m_n(E).\end{aligned}$$

So

$$\mu_j(E) = m_n(E).$$

Note that only use the equality of μ_j and m_n on $D \cup H$ and $H \setminus D$, both of which are Borel sets, so we can use Proposition 12.4.5 to avoid what would otherwise be a circular argument. □

Proposition 12.4.7. *Suppose g is a simple function on $\mathbf{R}^n$ and $j = 1$ or $j = 2$. Then*

$$I_j(g) = I(g).$$

Proof. For characteristic functions this follows immediately from the definition of μ_1 and μ_2 and Proposition 12.4.6. Simple functions are linear combination of characteristic functions and both integrals over $\mathbf{R}^n$ and I_1 and I_2 are linear. □

Proposition 12.4.8. *Suppose g is a non-negative semisimple function on $\mathbf{R}^n$ and $j = 1$ or $j = 2$. Then*

$$I_j(g) = I(g).$$

Proof. Non-negative semilinear functions are limits of increasing sequences of simple functions so this follows from Proposition 12.4.7 and the Monotone Convergence Theorem. □

The following is known as *Tonelli's Theorem*

Theorem 12.4.9. *Suppose g is a non-negative measurable function on $\mathbf{R}^n$ and $j = 1$ or $j = 2$. Then*

$$I_j(g) = I(g).$$

Proof. Every non-negative measurable function on $\mathbf{R}^n$ is a limit of increasing sequences of semisimple functions so this follows from Proposition 12.4.8 and the Monotone Convergence Theorem. □

Proposition 12.4.10. *Suppose $|g|$ is an integrable function on $\mathbf{R}^n$ and $j = 1$ or $j = 2$. Then*

$$|I_j(g)| \leq I(|g|).$$

Proof.

$$-|g(\mathbf{x})| \leq g(\mathbf{x}) \leq |g(\mathbf{x})|$$

for all $\mathbf{x}$ so

$$I_j(-|g|) \leq I_j(g) \leq I_j(|g|)$$

by monotonicity. From Theorem 12.4.9 we have

$$I_j(|g|)| = I(|g|).$$

Also, by linearity,

$$I_j(-|g|) = -I_j(|g|)$$

so

$$-I(|g|) \leq I_j(g) \leq I(|g|),$$

which is equivalent to

$$|I_j(g)| \leq I(|g|).$$

□

The following is known as *Fubini's Theorem*

Theorem 12.4.11. *Suppose f is an integrable function on $\mathbf{R}^n$ Then the following three integrals are all equal:*

$$\int_{\mathbf{x}' \in \mathbf{R}^{n'}} \int_{\mathbf{x}'' \in \mathbf{R}^{n''}} f(\mathbf{x}', \mathbf{x}'')\, dm_{n''}(\mathbf{x}'')\, dm_{n'}(\mathbf{x}'),$$

$$\int_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{x})\, dm_n(\mathbf{x}),$$

and

$$\int_{\mathbf{x}'' \in \mathbf{R}^{n''}} \int_{\mathbf{x}' \in \mathbf{R}^{n'}} f(\mathbf{x}', \mathbf{x}'')\, dm_{n'}(\mathbf{x}')\, dm_{n''}(\mathbf{x}'').$$

In the notation we've been using so far this is the statement that

$$I_1(f) = I(f) = I_2(f)$$

for all integrable f.

Proof. By Proposition 10.2.6 there are for any $\epsilon > 0$ a compactly supported continuous function g and an integrable function h such that

$$f = g + h$$

and

$$I(|h|) < \epsilon.$$

□

Then

$$I_j(g) = I(g)$$

by Theorem 12.2.13. By Proposition 12.4.10 we have

$$|I_j(h)| \leq I(|h|) < \epsilon.$$

Now

$$I_j(f) = I_j(g) + I_j(h)$$

and

$$I(f) = I(g) + I(h)$$

so

$$|I_j(f) - I(f)| < 2\epsilon.$$

This holds for all $\epsilon > 0$ so

$$I_j(f) = I(f).$$

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