Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Science, Technology, Engineering and Mathematics

School of Mathematics

SF Maths
SF TJH

2021-2022

MAU22200 Advanced Analysis

Wednesday 4 May RDS Main Hall 14:00 — 17:00

Prof. John Stalker

Instructions to Candidates:

This is a practice exam!

Answer two questions from Part A and two questions from Part B. In questions with multiple parts you are allowed to use the results of earlier parts in later parts, whether or not you did those parts.

Materials Permitted for this Examination:

Formulae and Tables are available from the invigilators, if required.

Calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

Part A

1. (20 points) The graph of a function $f\colon X \to Y$ is the set

$$G(f) = \{(x, y) \in X \times Y \colon f(x) = y\}.$$

   (a) (5 points) Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ is a continuous function. Prove that $G(f)$ is a closed subset of $\mathbf{R}^2$.

   (b) (5 points) Prove that

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

   is discontinuous, but $G(f)$ is nonetheless closed.

   (c) (5 points) Suppose $(X, \mathcal{T}_X)$ is a topological space, $(Y, \mathcal{T}_Y)$ is a Hausdorff topological space and $f\colon X \to Y$ is continuous. Prove that $G(f)$ is closed.

   Note: The first part of this problem is of course a special case of this one, so if you have a *correct* proof of this part you're allowed to use it for the first part.

   (d) (5 points) Give an example of topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ and a continuous function $f\colon X \to Y$ such that $G(f)$ is not closed.

   Hint: It is possible to find examples with very small X and Y.

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2. (20 points) A pseudometric on a set X is a function $d\colon X \times X \to \mathbf{R}$ such that

   (a) If $x, y \in X$ then $d(x, y) \geq 0$ and $d(x, y) = 0$ if $x = y$.

   (b) If $x, y \in X$ then $d(x, y) = d(y, x)$.

   (c) If $x, y, z \in X$ then $d(x, z) \leq d(x, y) + d(y, z)$.

   (a) (3 points) In what way does this differ from the definition of a metric?

   (b) (4 points) Give an example of pseudometric which is not a metric.

   (c) (7 points) Suppose we define balls and open sets analogously to how we did for metric spaces, i.e.

   $$B(x, r) = \{y \in X\colon d(x, y) < r\}$$

   and U is open if and only if for each $x \in U$ there is and $r > 0$ such that $B(x, r) \subseteq U$. Prove that the set of open sets is a topology on X.

   (d) (6 points) Prove that this topology is Hausdorff if and only if d is a metric.

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3. (20 points) Suppose A and B are subsets of $\mathbf{R}^n$ and C is the subset consisting of all $\mathbf{z} \in \mathbf{R}^n$ such that there are $\mathbf{x} \in A$ and $\mathbf{y} \in B$ with $\mathbf{x} + \mathbf{y} = \mathbf{z}$.

   (a) (5 points) Prove that if A and B are open then C is open.

   (b) (5 points) Prove that if A and B are compact then C is compact.

   (c) (5 points) Prove that if A and B are connected then C is connected.

   (d) (5 points) Prove that if A and B are convex then C is convex.

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Part B

1. (20 points)

   (a) (6 points) Give an example of a content μ on a measurable space $(X, \mathcal{B})$ which is not a measure. In other words, give a set X, a σ algebra $\mathcal{B}$ on X and a function $\mu\colon \mathcal{B} \to [0, +\infty]$ such that $\mu(\varnothing) = 0$ and μ is finitely additive but not countably additive.

   (b) (8 points) Suppose $(X, \mathcal{B})$ is a measurable space and μ is a content on $(X, \mathcal{B})$, but not necessarily a measure. Prove that if E_0, E_1, ... is a sequence of disjoint elements of $\mathcal{B}$ then

$$\mu\left(\bigcup_{i=0}^{\infty} E_i\right) \geq \sum_{i=0}^{\infty} \mu(E_i).$$

   (c) (6 points) Suppose $(X, \mathcal{B})$ is a measurable space and μ is a content on $(X, \mathcal{B})$. Suppose also that μ is countably subadditive, i.e. that for every sequence E_0, E_1, ... of elements of $\mathcal{B}$, not necessarily disjoint, we have

$$\mu\left(\bigcup_{i=0}^{\infty} E_i\right) \leq \sum_{i=0}^{\infty} \mu(E_i).$$

   Prove that μ is a measure.

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2. (20 points) Suppose $h\colon \mathbf{R}^n \to \mathbf{R}$ is measurable. Define f_n for $n \in \mathbf{N}$ by

$$f_n(\mathbf{x}) = \begin{cases} h(\mathbf{x}) & \text{if } \|\mathbf{x}\| \le n, -n \le h(\mathbf{x}) \le n, \\ n & \text{if } \|\mathbf{x}\| \le n, h(\mathbf{x}) > n, \\ -n & \text{if } \|\mathbf{x}\| \le n, h(\mathbf{x}) < -n, \\ 0 & \text{if } \|\mathbf{x}\| > n. \end{cases}$$

(a) (4 points) Prove that f_n is measurable for each n.

(b) (3 points) Prove that f_n is a compactly supported bounded function for each n.

(c) (3 points) Prove that

$$\lim_{n\to\infty} f_n(\mathbf{x}) = h(\mathbf{x})$$

for all $\mathbf{x}$.

(d) (3 points) Prove that

$$|f_n(\mathbf{x})| \le |h(\mathbf{x})|$$

for all n and $\mathbf{x}$.

(e) (7 points) Prove that

$$\lim_{n\to\infty} \int_{\mathbf{x}\in\mathbf{R}^n} f_n(\mathbf{x})\, dm(\mathbf{x}) = \int_{\mathbf{x}\in\mathbf{R}^n} h(\mathbf{x})\, dm(\mathbf{x})$$

if

$$\int_{\mathbf{x}\in\mathbf{R}^n} |h(\mathbf{x})|\, dm(\mathbf{x}) < +\infty$$

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3. (20 points) Suppose $g\colon \mathbf{R} \to [0, +\infty]$ is measurable. Let

$$E = \left\{(x, y) \in \mathbf{R}^2 \colon 0 \leq y \leq f(x)\right\}.$$

(a) (6 points) Prove that E is a measurable subset of $\mathbf{R}^2$.

(b) (4 points) Prove that

$$\int_{y \in \mathbf{R}} \chi_E(x, y)\, dm_1(y) = f(x)$$

for each $x \in \mathbf{R}$. Here m_1 is Lebesgue measure in $\mathbf{R}$ and χ_E is the characteristic function of E.

(c) (10 points) Prove that

$$\int_{\mathbf{x} \in \mathbf{R}} f(x)\, dm_1(x) = m_2(E).$$

Here m_1 is Lebesgue measure in $\mathbf{R}$ and m_2 is Lebesgue measure in $\mathbf{R}^2$.

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