MAU22200 2021-2022 Practice Problem Set 9

Solutions

1. Suppose that X and Y are sets, $f\colon X \to Y$ is a function, $\mathcal{B}$ is a Boolean algebra on X and $\mu\colon \mathcal{B} \to [0, +\infty]$ is a content on $(X, \mathcal{B})$. It was shown in the notes that $f^{**}(\mathcal{B})$ is a Boolean algebra on Y. Define $\nu\colon f^{**}(\mathcal{B}) \to [0, +\infty]$ by
$$\nu(E) = \mu(f^*(E)).$$
Show that ν is a content on $(Y, f^{**}(\mathcal{B}))$.
Solution: We need to check that $\nu(\varnothing) = 0$ and that if $E, F \in f^{**}(\mathcal{B})$ and $E \cap F = \varnothing$ then
$$\nu(E \cap F) = \nu(E) + \nu(F).$$
The first of these is easy because
$$\nu(\varnothing) = \mu(f^*(\varnothing)) = \mu(\varnothing) = 0.$$
The second requires a bit more work.
$$f^*(E) \cap f^*(F) = f^*(E \cap F) = f^*(\varnothing) = \varnothing$$
so
$$\mu(f^*(E) \cup f^*(F)) = \mu(f^*(E)) + \mu(f^*(F).$$
But
$$\mu(f^*(E)) = \nu(E),$$
$$\mu(f^*(F)) = \nu(F),$$
and
$$\mu(f^*(E) \cup f^*(F)) = \mu(f^*(E \cup F)) = \nu(E \cup F)$$
so
$$\nu(E \cap F) = \nu(E) + \nu(F).$$

2. (a) Suppose X is an uncountable set. Show that if $E \in \wp(X)$ then at most one of E or $X \setminus E$ is countable.
Solution: If E and $X \setminus E$ were both countable then $X = E \cup (X \setminus E$ would be countable by Proposition 2.9.3e from the notes.

   (b) Define $\mathcal{B}$ to be the set of those $E \in \wp(X)$ such that E or $X \setminus E$ is countable. Show that $\mathcal{B}$ is a σ-algebra.
Solution: We need to show that $\varnothing \in \mathcal{B}$, that $X \setminus E \in \mathcal{B}$ if $E \in \mathcal{B}$, and that $\bigcup_{E \in \mathcal{A}} E \in \mathcal{B}$ if $\mathcal{A}$ is a countable subset of $\mathcal{B}$.
$\varnothing$ is countable so $\varnothing \in \mathcal{B}$.
If $E \in \mathcal{B}$ then E is countable or $X \setminus E$ is countable. In the former case $X \setminus (X \setminus E)$ is countable. In the latter case $X \setminus E$ is countable. So if $E \in \mathcal{B}$ then either $X \setminus E$ or its complement is countable, so $X \setminus E \in \mathcal{B}$.

Suppose $\mathcal{A}$ is a countable subset of $\mathcal{B}$. If every $E \in \mathcal{A}$ is countable then $\bigcup_{E\in\mathcal{A}} E$ is countable by Proposition 2.9.3e. In this case $\bigcup_{E\in\mathcal{A}} E \in \mathcal{B}$. If not then there is an $F \in \mathcal{A}$ such that F is uncountable. But $F \in \mathcal{A}$ and $\mathcal{A} \subseteq \mathcal{B}$ so $F \in \mathcal{B}$. F is not countable and $F \in \mathcal{B}$ so $X \setminus F$ is countable. But then

$$F \subseteq \bigcup_{E\in\mathcal{A}} E$$

so

$$X \setminus \bigcup_{E\in\mathcal{A}} E \subseteq X \setminus F.$$

Subsets of countable sets are countable by Proposition 2.9.3.b so $\bigcup_{E\in\mathcal{A}} E$ is countable and therefore $\bigcup_{E\in\mathcal{A}} E \in \mathcal{B}$.

(c) Define $\mu\colon \mathcal{B} \to [0, +\infty]$ by $\mu(E) = 0$ if E is countable and $\mu(E) = +\infty$ if $X \setminus E$ is countable. Show that μ is a content on $(X, \mathcal{B})$.
Solution: We need to show that $\mu(\varnothing) = 0$ and that if $E \cap F = \varnothing$ then $\mu(E \cup F) = \mu(E) + \mu(F)$.
$\varnothing$ is countable, $\mu(\varnothing = 0)$. If E and F are countable then $E \cup F$ is countable so

$$\mu(E \cup F) = 0 = 0 + 0 = \mu(E) + \mu(F).$$

If either E or F is uncountable then it was shown in the answer to the previous part that $X \setminus (E \cup F)$ is countable so $\mu(E \cup F) = +\infty$ and $\mu(E) = +\infty$ or $\mu(F) = +\infty$. In either case

$$\mu(E \cup F) = \mu(E) + \mu(F).$$

3. Show that every interval in $\mathbf{R}$ is a Borel set.
Hint: This is one of those rare instances where case by case analysis of the ten types of intervals is not a terrible idea.
Solution: The open sets generate the Borel σ-algebra so every open set is a Borel set. The complement of any Borel set is a Borel set, so closed sets are also Borel sets. $\mathbf{R} = (-\infty, +\infty)$, $\varnothing$, $(a, +\infty)$, $(-\infty, b)$ and (a, b) for any $a, b \in \mathbf{R}$ are open. $[a, b]$, $[a, +\infty)$ and $(-\infty, b]$ are closed. So all of the sets above are Borel sets. The intersection of countably many Borel sets is a Borel set, so the intersection of any two Borel sets is a Borel set. In particular

$$(a, b] = (a, +\infty) \cap (-\infty, b]$$

and

$$[a, b) = [a, +\infty) \cap (-\infty, b)$$

are Borel sets. By Proposition 7.1.7 every interval is of one of the forms considered above, so all intervals are Borel sets.