MAU22200 2021-2022 Practice Problem Set 9

1. Suppose that X and Y are sets, $f\colon X \to Y$ is a function, $\mathcal{B}$ is a Boolean algebra on X and $\mu\colon \mathcal{B} \to [0, +\infty]$ is a content on $(X, \mathcal{B})$. It was shown in the notes that $f^{**}(\mathcal{B})$ is a Boolean algebra on Y. Define $\nu\colon f^{**}(\mathcal{B}) \to [0, +\infty]$ by
$$\nu(E) = \mu(f^*(E)).$$
Show that ν is a content on $(Y, f^{**}(\mathcal{B}))$.

2. (a) Suppose X is an uncountable set. Show that if $E \in \wp(X)$ then at most one of E or $X \setminus E$ is countable.

   (b) Define $\mathcal{B}$ to be the set of those $E \in \wp(X)$ such that E or $X \setminus E$ is countable. Show that $\mathcal{B}$ is a σ-algebra.

   (c) Define $\mu\colon \mathcal{B} \to [0, +\infty]$ by $\mu(E) = 0$ if E is countable and $\mu(E) = +\infty$ if $X \setminus E$ is countable. Show that μ is a content on $(X, \mathcal{B})$.

3. Show that every interval in $\mathbf{R}$ is a Borel set.
Hint: This is one of those rare instances where case by case analysis of the ten types of intervals is not a terrible idea.