MAU22200 2021-2022 Practice Problem Set 8
Solutions

1. Define $f\colon [-\pi/2, \pi/2] \to [-\infty, +\infty]$ by

$$f(x) = \begin{cases} -\infty & \text{if } x = -\pi/2, \\ \tan(x) & \text{if } -\pi/2 < x < \pi/2, \\ +\infty & \text{if } x = \pi/2. \end{cases}$$

Show that f is continuous and has a continuous inverse.
Hint: You can save yourself some time by using Proposition 3.6.3 from the notes and proving the following lemma:

> Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f\colon X \to Y$ is a function. If $x \in X$, $U \in \mathcal{O}(x)$ and the restriction of f to U is continuous then f is continuous at x.

Solution: First of all, f has an inverse because

$$g(y) = \begin{cases} -\pi/2 & \text{if } y = -\infty, \\ \arctan(y) & \text{if } -\infty < y < +\infty \\ \pi/2 & \text{if } y = +\infty, \end{cases}$$

satisfies $f(g(y)) = y$ for all $y \in [-\infty, +\infty]$ and $g(f(x)) = x$ for all $x \in [-\pi/2, \pi/2]$.

Next we prove the lemma from the hint. If the restriction of f to U is continuous then it is continuous at x so for every $V \in \mathcal{O}(f(x))$ there is a $W \in \mathcal{O}_U(x)$ such that

$$W \subseteq f^*(V).$$

The subscript U on $\mathcal{O}_U(x)$ is there to indicate that this is an open subset of U in the subspace topology, i.e. the intersection of U with an open set in the topology $\mathcal{T}_X$. But the intersection of two open sets is open so W is in fact in $\mathcal{O}_X(x)$. In other words, for each $V \in \mathcal{O}(f(x))$ there is a $W \in \mathcal{O}_X(x)$ such that

$$W \subseteq f^*(V).$$

So f is continuous at x.

For any $x \in (-\pi/2, \pi/2)$ we take $U = (-\pi/2, \pi/2)$ and note the this is an open neighbourhood of x and that the restriction of f to U is the tangent function, which is already known to be continuous. So f is continuous at every point in $(-\pi/2, \pi/2)$. If we can show that it's continuous at $-\pi/2$ and $\pi/2$ then it follows from Proposition 3.6.3 that it is continuous. To show that f is continuous at $\pi/2$ we need to show that if V is a neighbourhood of $f(\pi/2) = +\infty$ then $f^*(V)$ is a neighbourhood of $\pi/2$. The neighbourhoods of $+\infty$ in $[-\infty, +\infty]$ are precisely the sets which contain an interval $(a, +\infty]$ where $-\infty < a < +\infty$. The preimage of such a set contains an interval $(\arctan a, \pi/2]$ and so is a neighbourhood of $\pi/2$ in $[-\pi/2, \pi/2]$. The proof of continuity at $-\pi/2$ is similar.

Similarly, for any $y \in (-\infty, +\infty)$ we take $U = (-\infty, +\infty)$ and note that this is an open neighbourhood of y and that the restriction of g to U is the arctangent function, which is known to be continuous. It therefore suffices to show that g is continuous at $-\infty$ and $+\infty$. To show that g is continuous at $+\infty$ we need to show that if V is a neighbourhood of $g(+\infty) = \pi/2$ then $f^*(V)$ is a neighbourhood of $+\infty$. The neighbourhoods of $\pi/2$ in $[-\pi/2, \pi/2]$ are precisely the sets which contain an interval $(a, \pi/2]$ where $-\pi/2 < a < +\pi/2$. The preimage of such a set contains an interval $(\tan a, +\infty]$ and so is a neighbourhood of $+\infty$ in $[-\infty, +\infty]$. The proof of continuity at $-\infty$ is similar.

2. Suppose that S is a set, $f\colon S \to [0, +\infty]$ is a function and

$$\sum_{s \in S} f(s) < +\infty.$$

Show that for every $\delta > 0$ the set

$$G_\delta = \{s \in S\colon f(s) > \delta\}$$

is finite.
Solution: Choose an $n \in \mathbf{N}$ with

$$n > \frac{\sum_{s \in S} f(s)}{\delta}.$$

If G_δ is infinite then it has a subset with n elements. Call this subset F. Then

$$\sum_{s \in S} f(s) \geq \sum_{s \in F} f(s) \geq \sum_{s \in F} \delta = n\delta > \sum_{s \in S} f(s).$$

Therefore $\sum_{s \in S} f(s) > \sum_{s \in S} f(s)$, which is impossible, so G_δ cannot be infinite.

3. Suppose that S is a set, $f\colon S \to [0, +\infty]$ is a function and

$$\sum_{s \in S} f(s) < +\infty.$$

Show that the set

$$P = \{s \in S\colon f(s) > 0\}$$

is countable.
Hint: Use the result of the previous problem.
Solution: If $f(s) > 0$ then $f(s) > 1/2^n$ for some $n \in \mathbf{N}$ so

$$P = \bigcup_{n \in \mathbf{N}} G_{1/2^n}.$$

By the preceding problem $G_{1/2^n}$ is finite and hence countable. So P is a countable union of countable sets and is therefore countable.

4. Suppose that S is a set, $f\colon S \to \mathbf{R}$ is a function and that

$$\sum_{s \in S} f(s)$$

converges (in $\mathbf{R}$). Show that the set

$$\{s \in S\colon f(s) \neq 0\}$$

is countable.
Hint: Use the result of the previous problem.
Solution: As shown in the notes, if $\sum_{s \in S} f(s)$ converges then

$$\sum_{s \in S} |f(s)| < +\infty.$$

It then follows from the previous problem that

$$\{s \in S\colon |f(s)| > 0\}$$

is countable. But this is the same as

$$\{s \in S\colon f(s) \neq 0\}.$$