MAU22200 2021-2022 Practice Problem Set 8

1. Define $f\colon [-\pi/2, \pi/2] \to [-\infty, +\infty]$ by

$$f(x) = \begin{cases} -\infty & \text{if } x = -\pi/2, \\ \tan(x) & \text{if } -\pi/2 < x < \pi/2, \\ +\infty & \text{if } x = \pi/2. \end{cases}$$

   Show that f is continuous and has a continuous inverse.
   Hint: You can save yourself some time by using Proposition 3.6.3 from the notes and proving the following lemma:

   > Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f\colon X \to Y$ is a function. If $x \in X$, $U \in \mathcal{O}(x)$ and the restriction of f to U is continuous then f is continuous at x.

2. Suppose that S is a set, $f\colon S \to [0, +\infty]$ is a function and

$$\sum_{s \in S} f(s) < +\infty.$$

   Show that for every $\delta > 0$ the set

$$G_\delta = \{s \in S\colon f(s) > \delta\}$$

   is finite.

3. Suppose that S is a set, $f\colon S \to [0, +\infty]$ is a function and

$$\sum_{s \in S} f(s) < +\infty.$$

   Show that the set
$$P = \{s \in S\colon f(s) > 0\}$$
   is countable.
   Hint: Use the result of the previous problem.

4. Suppose that S is a set, $f\colon S \to \mathbf{R}$ is a function and that

$$\sum_{s \in S} f(s)$$

   converges (in $\mathbf{R}$). Show that the set

$$\{s \in S\colon f(s) \neq 0\}$$

   is countable.
   Hint: Use the result of the previous problem.