MAU22200 2021-2022 Practice Problem Set 0
Solutions

1. The Fibonacci sequence $\varphi\colon \mathbf{N} \to \mathbf{Q}$ is defined inductively by

$$\varphi_0 = 0, \quad \varphi_1 = 1, \quad \varphi_{n+2} = \varphi_{n+1} + \varphi_n$$

for all n. Define another sequence $\alpha\colon \mathbf{N} \to \mathbf{Q}$ by

$$\alpha_n = \frac{\varphi_n}{\varphi_{n+1}}.$$

(a) Prove that the following hold for all $n \in \mathbf{N}$.

i.
$$\varphi_n \geq 0,$$

ii.
$$\varphi_{n+1} \geq \varphi_n$$

iii.
$$\varphi_{n+1}\varphi_{n+2} \geq 2\varphi_n\varphi_{n+1},$$

iv.
$$\varphi_{n+1}\varphi_{n+2} \geq 2^{n+1},$$

v.
$$\varphi_{n+1}^2 \geq 2^n$$

vi.
$$\varphi_{n+1}^2 - \varphi_n\varphi_{n+1} - \varphi_n^2 = (-1)^n$$

Solution:

i. We prove that for each n we have $\varphi_j \geq 0$ for all $j \leq n$. This is true for $n = 1$. If it's true for n then

$$\varphi_{n+1} = \varphi_n + \varphi_{n-1} \geq 0$$

since $n \leq n$ and $n - 1 \leq n$. Therefore $\varphi_j \geq 0$ for $j = n + 1$ and for $j \leq n$ and hence for all $j \leq n+1$. By induction then we have $\varphi_j \geq 0$ for all $j \leq n$ for all n and hence $\varphi_n \geq 0$ for all n.

ii. For $n = 0$ we can check this directly. For $n \geq 1$ we have

$$\varphi_{n+1} = \varphi_n + \varphi_{n-1}$$

and $\varphi_{n-1} \geq 0$ by the previous subpart.

iii.
$$\begin{aligned}\varphi_{n+1}\varphi_{n+2} &= \varphi_{n+1}(\varphi_{n+1} + \varphi_n)\\ &= \varphi_{n+1}(\varphi_{n+1} - \varphi_n + 2\varphi_n)\\ &= 2\varphi_n\varphi_{n+1} + \varphi_{n+1}(\varphi_{n+1} - \varphi_n)\end{aligned}$$

It follows from the previous two subparts that both factors of $\varphi_{n+1}(\varphi_{n+1} - \varphi_n)$ are non-negative so

$$\varphi_{n+1}\varphi_{n+2} \geq 2\varphi_n\varphi_{n+1}.$$

iv. This is true for $n = 0$. If it's true for a given value of n then

$$\varphi_{n+2}\varphi_{n+3} \geq 2\varphi_{n+1}\varphi_{n+2} \geq 2 \cdot 2^{n+1} = 2^{(n+1)+2}$$

so it's true with $n+1$ in place of n. By induction it's therefore true for all n.

v. If $n = 0$ then this is true. For $n > 0$ we can apply the previous part with $n-1$ in place of n to get

$$\varphi_n\varphi_{n+1} \geq 2^n.$$

But $\varphi_{n+1} \geq \varphi_n$ and $\varphi_{n+1} \geq 0$ so

$$\varphi_{n+1}^2 \geq \varphi_n\varphi_{n+1}.$$

vi. This is true for $n = 0$. If it's true for a given value of n then

$$\begin{aligned}
\varphi_{n+2}^2 - \varphi_{n+1}\varphi_{n+2} - \varphi_{n+1}^2 &= (\varphi_{n+2} - \varphi_{n+1})\varphi_{n+2} - \varphi_{n+1}^2 \\
&= \varphi_n(\varphi_{n+1} + \varphi_n) - \varphi_{n+1}^2 \\
&= -(\varphi_{n+1}^2 - \varphi_n\varphi_{n+1} - \varphi_n)^2 \\
&= -(-1)^n = (-1)^{n+1}
\end{aligned}$$

so it's true with $n+1$ in place of n. By induction it therefore holds for all n.

(b) Prove that

$$\alpha_{n+1} - \alpha_n = \frac{(-1)^n}{\varphi_{n+1}\varphi_{n+2}}$$

and

$$1 - \alpha_n - \alpha_n^2 = \frac{(-1)^n}{\varphi_{n+1}^2}.$$

Solution:

$$\begin{aligned}
\varphi_{n+1}^2 - \varphi_n\varphi_{n+2} &= \varphi_{n+1}^2 - \varphi_n\left(\varphi_{n+1} + \varphi_n\right) \\
&= \varphi_n^2 - \varphi_n\varphi_{n+1} - \varphi_n^2 = (-1)^n
\end{aligned}$$

Dividing by $\varphi_{n+1}\varphi_{n+2}$ gives

$$\alpha_{n+1} - \alpha_n = \frac{(-1)^n}{\varphi_{n+1}\varphi_{n+2}}.$$

$$\varphi_n^2 - \varphi_n\varphi_{n+1} - \varphi_n^2 = (-1)^n.$$

Dividing by φ_{n+1}^2 gives

$$1 - \alpha_n - \alpha_n^2 = \frac{(-1)^n}{\varphi_{n+1}^2}.$$

(c) Prove that α is a Cauchy sequence.
Solution: From

$$\alpha_{n+1} - \alpha_n = \frac{(-1)^n}{\varphi_{n+1}\varphi_{n+2}}$$

and

$$\varphi_{n+1}\varphi_{n+2} \geq 2^{n+1},$$

we get

$$|\alpha_{n+1} - \alpha_n| \leq \frac{1}{2^{n+1}}.$$

If $j \leq k$ then

$$\begin{aligned}
|\alpha_k - \alpha_j| &\leq \left| \sum_{n=j}^{k-1} \alpha_{n+1} - \alpha_n \right| \\
&\leq \sum_{n=j}^{k-1} |\alpha_{n+1} - \alpha_n| \\
&\leq \sum_{n=j}^{k-1} \frac{1}{2^{n+1}} \\
&= \frac{1}{2^j}\left(1 - \frac{1}{2^{k-j}}\right) < \frac{1}{2^j} = \frac{1}{2^{\min(j,k)}}
\end{aligned}$$

The same works with j and k reversed so we also have

$$|\alpha_k - \alpha_j| \leq \frac{1}{2^{\min(j,k)}}$$

if $j \geq k$ and hence for any j and k. If $\epsilon > 0$ and m is chosen sufficiently large that $2^m\epsilon \geq 1$ then

$$|\alpha_k - \alpha_j| < \epsilon$$

whenever $j, k \geq m$. So α satisfies the Cauchy criterion.

(d) Prove that α is not convergent. *Hint:* Don't forget that α was defined as a sequence of rationals, not of reals.
Solution: If α were convergent then there would be a $z \in \mathbf{Q}$ such that

$$\lim_{n\to\infty} \alpha_n = z.$$

We would then also have

$$\lim_{n\to\infty} \alpha_{n+1} = z.$$

Taking limits in

$$1 - \alpha_n - \alpha_n^2 = \frac{(-1)^n}{\varphi_{n+1}^2}$$

and using the fact that

$$\varphi_{n+1}^2 \geq 2^n,$$

which implies $\lim_{n\to\infty} \frac{(-1)^n}{\varphi_{n+1}^2} = 0$, we find that

$$1 - z - z^2 = 0.$$

Then

$$(2z+1)^2 = 4z^2 + 4z + 1 = 5 - 4(1 - z - z^2) = 5.$$

So $2z+1$ is a rational number whose square is 5. There is no such number.

2. Suppose that (V, p) and (W, q) are normed vector spaces and that V is finite dimensional. Show that every linear transformation from V to W is bounded.
Hint: Use the equivalence of norms on finite dimensional normed vector spaces, Proposition 5.3.1.
Solution: Define $r\colon W \to \mathbf{R}$ by

$$r(\mathbf{x}) = p(\mathbf{x}) + q(A\mathbf{x}).$$

I claim that r is a norm.

First of all,

$$r(\mathbf{x}) \geq 0$$

for all $\mathbf{x} \in V$. Also $r(\mathbf{x}) = 0$ if and only if $p(\mathbf{x}) = 0$ and $q(A\mathbf{x}) = 0$. This happens if and only if $\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{0}$. The second of these conditions is redundant. So $r(\mathbf{x}) \geq 0$ and $r(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

Next,

$$\begin{aligned} r(\alpha\mathbf{x}) &= p(\alpha\mathbf{x}) + q(A\alpha\mathbf{x}) = p(\alpha\mathbf{x}) + q(\alpha A\mathbf{x}) \\ &= |\alpha| p(\mathbf{x}) + |\alpha| q(A\mathbf{x}) = |\alpha|(p(\mathbf{x}) + q(A\mathbf{x})) = |\alpha| r(\mathbf{x}). \end{aligned}$$

Finally,

$$\begin{aligned} r(\mathbf{x}+\mathbf{y}) &= p(\mathbf{x}+\mathbf{y}) + q(A(\mathbf{x}+\mathbf{y})) = p(\mathbf{x}+\mathbf{y}) + q(A\mathbf{x} + A\mathbf{y}) \\ &\leq p(\mathbf{x}) + p(\mathbf{y}) + q(A\mathbf{x}) + q(A\mathbf{y}) = r(\mathbf{x}) + \mathbf{r}(y). \end{aligned}$$

So r is a norm, as claimed.

By Proposition 5.3.1 the norms p and r on the finite dimensional space V are equivalent. It follows that there is a $C > 0$ such that

$$r(\mathbf{x}) \leq C p(\mathbf{x}).$$

But

$$r(\mathbf{x}) = p(\mathbf{x}) + q(A\mathbf{x})$$

and $p(\mathbf{x}) \geq 0$ so

$$r(\mathbf{x}) \geq q(A\mathbf{x}).$$

Therefore

$$q(A\mathbf{x}) \leq Cp(\mathbf{x}).$$

In other words, C is a bound for A, which is therefore bounded.