MAU22200 2021-2022 Practice Problem Set 0

1. The Fibonacci sequence $\varphi\colon \mathbf{N} \to \mathbf{Q}$ is defined inductively by

$$\varphi_0 = 0, \quad \varphi_1 = 1, \quad \varphi_{n+2} = \varphi_{n+1} + \varphi_n$$

for all n. Define another sequence $\alpha\colon \mathbf{N} \to \mathbf{Q}$ by

$$\alpha_n = \frac{\varphi_n}{\varphi_{n+1}}.$$

(a) Prove that the following hold for all $n \in \mathbf{N}$.

i.
$$\varphi_n \geq 0,$$

ii.
$$\varphi_{n+1} \geq \varphi_n$$

iii.
$$\varphi_{n+1}\varphi_{n+2} \geq 2\varphi_n\varphi_{n+1},$$

iv.
$$\varphi_{n+1}\varphi_{n+2} \geq 2^{n+1},$$

v.
$$\varphi_{n+1}^2 \geq 2^n$$

vi.
$$\varphi_{n+1}^2 - \varphi_n\varphi_{n+1} - \varphi_n^2 = (-1)^n$$

(b) Prove that

$$\alpha_{n+1} - \alpha_n = \frac{(-1)^n}{\varphi_{n+1}\varphi_{n+2}}$$

and

$$1 - \alpha_n - \alpha_n^2 = \frac{(-1)^n}{\varphi_{n+1}^2}.$$

(c) Prove that α is a Cauchy sequence.

(d) Prove that α is not convergent. *Hint:* Don't forget that α was defined as a sequence of rationals, not of reals.

2. Suppose that (V, p) and (W, q) are normed vector spaces and that V is finite dimensional. Show that every linear transformation from V to W is bounded.
Hint: Use the equivalence of norms on finite dimensional normed vector spaces, Proposition 5.3.1.