MAU22200 2021-2022 Practice Problem Set 6

Solutions

1. Show that each of the following spaces is compact.

   (a) The set $\{z, \alpha_0, \alpha_1, \ldots\} \subseteq X$ where $(X, \mathcal{T})$ is a topological space, $z \in X$ and $\alpha \colon \mathbf{N} \to X$ such that $\lim_{n\to\infty} \alpha_n = z$.
   Solution: Let $S = \{z, \alpha_0, \alpha_1, \ldots\} \subseteq X$. Suppose $\mathcal{G}$ is an open cover of S. $z \in S$ so there is a $U \in \mathcal{G}$ such that $z \in U$. $\lim_{n\to\infty} \alpha_n = z$ and U is open so there is an $m \in \mathbf{N}$ such that $\alpha_n \in U$ for all $n \geq m$. For each $j < m$ we have $\alpha_j \in V_j$ for some $V_j \in \mathcal{G}$. If $x \in S$ then $x = z$, $x = \alpha_n$ for some $n \geq m$ or $x = \alpha_j$ for some $j < m$. In either of the first two cases $x \in U$. In the last case $x \in V_j$. So $\{U, V_0, V_1, \ldots, V_{m-1}\}$ is a finite subcover.

   (b) $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is the cofinite topology on X.
   Solution: Suppose $\mathcal{G}$ is an open cover of $(X, \mathcal{T})$. If X is empty then $\varnothing$ is a finite subcover. If X is non-empty choose some $x \in X$. There is a $U \in \mathcal{G}$ such that $x \in U$. U is not empty so $X \setminus U$ is finite by the definition of the cofinite topology. For each $y \in X \setminus U$ there is a $V_y \in \mathcal{G}$ such that $y \in V_y$. Then every point in X is in U or in one of the V_y's for $y \in X \setminus U$ so U and these V_y's form a finite subcover.

   (c) The Cantor set.
   Hint: This can be done using the description of the Cantor set from the notes, but it's easier to use the description in terms of intervals from Lecture 12.
   Solution: As described in Lecture 12, the Cantor set is the intersection of the sets

$$\begin{aligned}
C_0 &= [0, 1], \\
C_1 &= [0, 1/3] \cup [2/3, 1], \\
C_2 &= [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1], \\
&\vdots
\end{aligned}$$

   It was shown in the notes that intervals of the form $[a, b]$ are compact (Proposition 3.12.3) and that unions of finitely many compact sets are compact (Proposition 3.12.7), so each C_j is compact. We also have

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$$

   so the intersection of all of them is compact by the remark after Proposition 3.12.4 of the notes.

2. The version of the Tietze Extension Theorem in the notes applies only to bounded functions. Prove the following related theorem, which has no boundedness assumption.

Suppose $(X, \mathcal{T})$ is a normal topological space, $A \in \wp(X)$ is closed and $f: A \to \mathbf{R}$ is continuous. Then there is a continuous $g: X \to \mathbf{R}$ such that $g(x) = f(x)$ for all $x \in A$.

Hint: Use the version of the Tietze Extension Theorem you already have, Urysohn's Lemma and the arctangent function.
Solution: Define $\tilde{f}: A \to [-\pi/2, \pi/2]$ by

$$\tilde{f}(x) = \arctan(f(x)).$$

The composition of continuous functions is continuous so $\tilde{f}$ is continuous. By the Tietze Extension Theorem it has a continuous extension $\tilde{g}: X \to [-\pi/2, \pi/2]$. We would like to define

$$g(x) = \tan(\tilde{g}(x)),$$

which certainly satisfies $g(x) = f(x)$ for $x \in A$ but this doesn't quite work since $\tilde{g}$ could take the values $\pm\pi/2$ at other points in x and the tangent is not defined at $\pm\pi/2$.

To avoid the problem above, let

$$B = \tilde{g}^*(\{-\pi/2, \pi/2\}).$$

B is closed because $\tilde{g}$ is continuous. $A \cap B = \varnothing$ because $\tilde{f}(x) \in (-\pi/2, \pi/2)$ for all $x \in A$. By Urysohn's Lemma there is a continuous function $h: X \to [0, 1]$ such that $h(x) = 0$ if $x \in B$ and $h(x) = 1$ if $x \in A$. Then $\tilde{g}h$ is continuous since it is the product of two continuous functions. Also $\tilde{g}(x)h(x) \in (-\pi/2, \pi/2)$ for all $x \in X$. For $x \in B$ this is true because $h(x) = 0$. For $x \notin B$ it's true because $|\tilde{g}(x)| < \pi/2$ and $|h(x)| \leq 1$. Note also that if $x \in A$ then

$$\tilde{g}(x)h(x) = \tilde{f}(x)1 = \arctan(f(x)).$$

We can then define

$$g(x) = \tan(\tilde{g}(x)h(x)).$$

This is well defined because $\tilde{g}(x)h(x)$ is in the domain of tan, continuous since it's the composition of continuous functions, and $g(x) = f(x)$ when $x \in A$ because $\tan(\arctan(y)) = y$ for all $y \in \mathbf{R}$.

3. Suppose $f: \mathbf{R} \to \mathbf{R}$ is differentiable. Show that f is Lipschitz continuous if and only if f' is bounded.
Solution: Suppose f' is bounded, i.e. that there is a K such that

$$|f'(x)| \leq K$$

for all x. $K \geq 0$ because absolute values cannot be negative. Suppose $s, t \in \mathbf{R}$. Let $a = \min(s, t)$ and $b = \max(s, t)$. If $s \neq t$ then $a < b$ and the Mean Value Theorem shows that there is an $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

and hence

$$|f'(x)| = \frac{|f(b) - f(a)|}{|b - a|} = \frac{|f(s) - f(t)|}{|s - t|}.$$

So

$$|f(s) - f(t)| \leq K|s - t|.$$

This inequality also holds trivially if $s = t$ so

$$|f(s) - f(t)| \leq K|s - t|$$

for all $s, t \in \mathbf{R}$. In terms of the usual metric on $\mathbf{R}$ this is

$$d(f(s), f(t)) \leq K d(s, t),$$

so f is Lipschitz continuous.

Suppose, conversely, that f is Lipschitz continuous, i.e. that there is a $K \geq 0$ such that

$$d(f(s), f(t)) \leq K d(s, t).$$

Then

$$\left| \frac{f(s) - f(t)}{s - t} \right| \leq K$$

if $s \neq t$. It follows that

$$|f'(t)| = \left| \lim_{s \to t} \frac{f(s) - f(t)}{s - t} \right| \leq K.$$

So f' is bounded.