MAU22200 2021-2022 Practice Problem Set 6

1. Show that each of the following spaces is compact.

   (a) The set $\{z, \alpha_0, \alpha_1, \ldots\} \subseteq X$ where $(X, \mathcal{T})$ is a topological space, $z \in X$ and $\alpha\colon \mathbf{N} \to X$ such that $\lim_{n\to\infty} \alpha_n = z$.

   (b) $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is the cofinite topology on X.

   (c) The Cantor set.
   Hint: This can be done using the description of the Cantor set from the notes, but it's easier to use the description in terms of intervals from Lecture 12.

2. The version of the Tietze Extension Theorem in the notes applies only to bounded functions. Prove the following related theorem, which has no boundedness assumption.

   > Suppose $(X, \mathcal{T})$ is a normal topological space, $A \in \wp(X)$ is closed and $f\colon A \to \mathbf{R}$ is continuous. Then there is a continuous $g\colon X \to \mathbf{R}$ such that $g(x) = f(x)$ for all $x \in A$.

   Hint: Use the version of the Tietze Extension Theorem you already have, Urysohn's Lemma and the arctangent function.

3. Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is differentiable. Show that f is Lipschitz continuous if and only if f' is bounded.