MAU22200 2021-2022 Practice Problem Set 5

Solutions

1. Suppose $(X, \mathcal{T}_X)$ is a topological space, $A \in \wp(X)$ and $\mathcal{T}_A$ is the subspace topology on A.

   (a) Show that if $U \in \mathcal{T}_X$ and $U \subseteq A$ then $U \in \mathcal{T}_A$.
   Solution: $U = A \cap U$ since $A \subseteq U$. So $U \in \mathcal{T}_A$ by Proposition 3.8.2.

   (b) Show that if $X \setminus V \in \mathcal{T}_X$ and $V \subseteq A$ then $A \setminus V \in \mathcal{T}_A$.
   Solution: $A \cap X = A$ and $A \cap V = V$ so

   $$A \setminus V = (A \cap X) \setminus (A \cap V) = A \cap (X \setminus V).$$

   Therefore $A \setminus V \in \mathcal{T}_A$ by Proposition 3.8.2.

   (c) Show that if $A \in \mathcal{T}_X$, $U \in \mathcal{T}_A$ and $U \subseteq A$ then $U \in \mathcal{T}_X$.
   Solution: By Proposition 3.8.2 there is a $W \in \mathcal{T}_X$ such that $U = A \cap W$. But then $U \in \mathcal{T}_X$ because $A \in \mathcal{T}_X$ and $W \in \mathcal{T}_X$.

   (d) Show that if $X \setminus A \in \mathcal{T}_X$, $A \setminus V \in \mathcal{T}_A$ and $V \subseteq A$ so $X \setminus V \in \mathcal{T}_X$ because intersections of open sets are open.
   Solution: By Proposition 3.8.2 there is a $W \in \mathcal{T}_X$ such that $A \cap W = A \setminus V$.

   $$X \setminus V = (X \setminus A) \cup (A \setminus V)(X \setminus A) \cup (A \cap W) = (X \setminus A) \cup W$$

   This is in $\mathcal{T}_X$ because intersections of open sets are open.

2. The cofinite topology on a set X was defined in Practice Problem Set 2, where you proved that it is indeed a topology and that it is a Hausdorff topology if and only if X is finite.

   Proposition 3.10.9 says that $(X, \mathcal{T})$ is Hausdorff if and only if the diagonal Δ_X is closed. If X is infinite then it follows that Δ_X is not closed, so by Proposition 3.2.2 Parts (b) and (g) the closure of Δ_X is strictly larger than Δ_X. What is the closure of Δ_X?
   Solution: The closure is all of $X \times X$. Suppose $(a, b) \in X \times X$ and U is a neighbourhood of (a, b) in $X \times X$. By Proposition 3.10.4 U is a union of sets of the form $V \times W$, where V and W are open sets in X. (a, b) must therefore be contained in such a set. V and W are non-empty since $a \in V$ and $b \in W$ so by the definition of the cofinite topology $X \setminus V$ and $X \setminus W$ are both finite. So

   $$X \setminus (V \cap W) = (X \setminus V) \cup (X \setminus W)$$

   is finite and therefore $V \cap W$ is non-empty. If $c \in V \cap W$ then $(c, c) \in V \times W$ so $(c, c) \in U$. But $(c, c) \in \Delta_X$ so $\Delta_X \cap U \neq \varnothing$. It follows from Proposition 3.2.2 Part (l) that $(a, b) \in \overline{\Delta_X}$. But (a, b) was an arbitrary element of $X \times X$, so $\overline{\Delta_X} = X \times X$.

3. (a) Suppose $A \in \wp(\mathbf{R})$ is connected. First show that if $x < y < z$ and $x, z \in A$ then $y \in A$.

Solution: If $x, z \in A$ and $x < y < z$ but $y \notin A$ then

$$\begin{aligned} A &= A \cap (\mathbf{R} \setminus \{y\}) = A \cap ((-\infty, y) \cup (y, +\infty)) \\ &= (A \cap (-\infty, y)) \cup (A \cap (y, +\infty)) \end{aligned}$$

and

$$\begin{aligned} (A \cap (-\infty, y)) \cap (A \cap (y, +\infty)) &= A \cap ((-\infty, y) \cap (y, +\infty)) \\ &= A \cap \varnothing = \varnothing. \end{aligned}$$

Also $x \in A \cap (-\infty, y)$ and $z \in A \cap (y, +\infty)$ so $A \cap (-\infty, y) \neq \varnothing$ and $A \cap (y, +\infty) \neq \varnothing$. Both $A \cap (-\infty, y)$ and $A \cap (y, +\infty)$ are elements of $\mathcal{T}_A$ by Proposition 3.8.2, so A is disconnected. In other words, assuming $x, z \in A$ and $x < y < z$, if $y \notin A$ then A is disconnected. Equivalently, if $x, z \in A$, $x < y < z$ and A is connected then $y \in A$.

(b) Show that if A is connected then A is an interval.

Solution: If A is empty then it's an an interval. Suppose A is non-empty and bounded. Let

$$a = \inf A, \qquad b = \sup A.$$

If $a < y < b$ then y is neither an upper bound nor a lower bound for A so there are $x < y$ and $z > y$ such that $x, z \in A$. It follows from the previous part that $y \in A$. If $x < a$ then $x \notin A$ because a is a lower bound for A. If $z > b$ then $z \notin A$ because b is an upper bound for A. So

$$(a, b) \subseteq A \subseteq [a, b].$$

The only remaining question is whether $a \in A$ and whether $b \in A$, so the only sets with this property are the intervals (a, b), $(a, b]$, $[a, b)$ and $[a, b]$.

Suppose A is bounded neither from above nor below. If $y \in \mathbf{R}$ then y is neither an upper nor a lower bound for A, so there are $x < y$ and $z > y$ such that $x, z \in A$. It follows from the previous part that $y \in A$. So $A = \mathbf{R}$. $\mathbf{R} = (-\infty, +\infty)$ is an interval.

Suppose A is bounded from below but not from above. Let

$$a = \inf A.$$

If $y > a$ then y is neither an upper nor a lower bound for A so $y \in A$. If $x < a$ then $x \notin A$ since a is a lower bound for A. So

$$(a, +\infty) \subseteq A \subseteq [a, +\infty).$$

The only sets with this property are the intervals $(a, +\infty)$ and $[a, +\infty)$.

Suppose A is bounded from below but not from above. Let

$$b = \sup A.$$

If $y < b$ then y is neither an upper nor a lower bound for A so $y \in A$. If $z > b$ then $z \notin A$ since b is an upper bound for A. So

$$(-\infty, b) \subseteq A \subseteq (-\infty, b].$$

The only sets with this property are the intervals $(-\infty, b)$ and $(-\infty, b]$. The various cases considered above exhaust all the possibilities, so in every case A is an interval.