MAU22200 2021-2022 Practice Problem Set 4
Solutions

1. Suppose that X and Y are non-empty sets. Show that there is a surjection $f\colon X \to Y$ or a surjection $g\colon Y \to X$.
   Hint: While it is possible to prove this using Zorn's Lemma, I don't really recommend using it directly. It's considerably easier to use results you already have from the notes.
   Solution: By Proposition 2.7.1 from the notes there is an injection $i\colon X \to Y$ or an injection $j\colon Y \to X$. If there is an injection $i\colon X \to Y$ then there is, Proposition 2.1.1, a function $g\colon Y \to X$, such that $g \circ i$ is the identity on X. So if $x \in X$ then there is a $y \in Y$ such that $g(y) = x$, namely $y = i(x)$. So g is a surjection. If there is an injection $j\colon Y \to X$ then there is, Proposition 2.1.1, a function $f\colon X \to Y$, such that $f \circ j$ is the identity on Y. So if $y \in Y$ then there is an $x \in X$ such that $f(x) = y$, namely $x = j(y)$. So f is a surjection.

2. Which of the following are countable? Justify your answers.

   (a) The set of subsets of the set of rational numbers.
   Solution: No, this set is not countable. By Proposition 2.8.2r $\#\mathbf{Q} \leq \#\wp(\mathbf{Q})$ but $\#\mathbf{Q} \neq \#\wp(\mathbf{Q})$. By Proposition 2.9.4 $\mathbf{Q}$ is countable. In other words $\#\mathbf{Q} \leq \#\mathbf{N}$. But $\mathbf{N} \subseteq \mathbf{Q}$ so $\#\mathbf{N} \leq \#\mathbf{Q}$ by Proposition 2.9.3b. Therefore $\#\mathbf{Q} = \#\mathbf{N}$ by Proposition 2.8.2f. Therefore $\#\mathbf{N} \leq \#\wp(\mathbf{Q})$ but $\#\mathbf{N} \neq \#\wp(\mathbf{Q})$. If $\wp(\mathbf{Q})$ were countable we would have $\#\wp(\mathbf{Q}) \leq \#\mathbf{N}$, violating Proposition 2.8.2f.

   (b) The set of empty subsets of the set of rational numbers.
   Solution: With the terminology used in the notes, yes. There's an injection from $\{\varnothing\}$ to $\mathbf{N}$. For example $f(\varnothing) = 0$ works.

   There's an alternate terminology in which "countable" means what I've called "countably infinite". In that terminology the answer would be no, since $\{\varnothing\}$ is not infinite.

   (c) The set non-empty subsets of the set of rational numbers.
   Solution: No, this set is not countable. If it were then the set of all rational numbers would be the union of the set and the one from the previous part and so would be countable by Proposition 2.9.3f.

   (d) The set finite subsets of the set of rational numbers.
   Solution: Yes, this set is countable. The easiest way to see this is to note that it is the union of the sets of subsets of $\{0, 1, \ldots, n\}$. Each of these sets is finite and so countable. There are countably many of them, so the union is countable by Proposition 2.9.3e.

3. (a) Find A°, $\overline{(A^\circ)}$ and $\left(\overline{(A^\circ)}\right)^\circ$ where

$$A = (-1, 0) \cup (0, 1)$$

Solution: A is already open, so $A^\circ = A$ by Proposition 3.3.2f.

$$\overline{(A^\circ)} = [-1, 1].$$

To see this note that it must contain A by Proposition 3.2.2b and it must be contained in $[-1, 1]$ by Proposition 3.2.2e, since $[-1, 1]$ is closed. The only question is which of the points -1, 0 and 1 it contains. It must contain all of them, by Proposition 3.2.2l, since every neighbourhood of any of those points has a non-empty intersection with A.

$$\left(\overline{(A^\circ)}\right)^\circ = (-1, 1).$$

To see this note that it must be contained in $[-1, 1]$ by Proposition 3.2.2b and it must contain $(-1, 1)$ by Proposition 3.2.2d, since $(-1, 1)$ is open. The only question is which of the points -1 and 1 it contains. It can't contain either of them by Proposition 3.2.2k, because there is no neighbourhood of either which is contained in $[-1, 1]$.

(b) Find $\overline{A}$ and $\left(\overline{A}\right)^\circ$ where

$$A = \mathbf{Q} \cap [-1, 1]$$

Solution:

$$\overline{A} = [-1, 1].$$

It can't be any larger by Proposition 3.2.2e. It can't be any smaller because every neighbourhood of every point in $[-1, 1]$ contains a rational point $[-1, 1]$ and so belongs to $\overline{A}$ by Proposition 3.2.2l.

$$\left(\overline{A}\right)^\circ = (-1, 1)$$

because we already found the interior of $[-1, 1]$ in the previous part.

4. (a) Show that

$$A^\circ \subseteq \left(\overline{(A^\circ)}\right)^\circ.$$

Solution: $A^\circ \subseteq \overline{(A^\circ)}$ By Proposition 3.2.2.b, so $(A^\circ)^\circ \subseteq \left(\overline{(A^\circ)}\right)^\circ$ by Proposition 3.2.2c. But $(A^\circ)^\circ = A^\circ$ by Proposition 3.2.2h. So

$$A^\circ \subseteq \left(\overline{(A^\circ)}\right)^\circ.$$

(b) Show that

$$\overline{\left((\overline{A})^{\circ}\right)} \subseteq \overline{A}.$$

Solution: $(\overline{A})^{\circ} \subseteq \overline{A}$ by Proposition 3.2.2b, so $\overline{\left((\overline{A})^{\circ}\right)} \subseteq \overline{(\overline{A})}$ by Proposition 3.3.2c. But $\overline{(\overline{A})} = \overline{A}$ by Proposition 3.2.2h, so

$$\overline{\left((\overline{A})^{\circ}\right)} \subseteq \overline{A}.$$

(c) Show that

$$\overline{(A^{\circ})} = \overline{\left(\left(\overline{(A^{\circ})}\right)^{\circ}\right)}.$$

Solution: From Part (a) and Proposition 3.2.2c it follows that

$$\overline{(A^{\circ})} \subseteq \overline{\left(\left(\overline{(A^{\circ})}\right)^{\circ}\right)}.$$

Applying Part (b) with A° in place of A gives

$$\overline{\left(\left(\overline{(A^{\circ})}\right)^{\circ}\right)} \subseteq \overline{(A^{\circ})}.$$

Combining the two gives

$$\overline{(A^{\circ})} = \overline{\left(\left(\overline{(A^{\circ})}\right)^{\circ}\right)}.$$