MAU22200 2021-2022 Practice Problem Set 3
Solutions

1. For any set X and any $x \in X$ the set
$$\mathcal{F} = \{A \in \wp(X) : x \in A\}$$
is called the principal filter of X at x. Show that it is indeed a filter.
Solution: We check the four conditions.
   (a) $\mathcal{F} \neq \varnothing$ because $\{x\} \in \mathcal{F}$.
   (b) $\varnothing \notin \mathcal{F}$ because $x \notin \varnothing$.
   (c) If $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then $x \in A$ and $x \in B$ so $x \in C$ where $C = A \cap B$. So $C \in \mathcal{F}$. $A \supseteq C$ and $B \supseteq C$.
   (d) If $A \subseteq B$ and $A \in \mathcal{F}$ then $x \in A$ so $x \in B$ and hence $B \in \mathcal{F}$.

2. Show that the neighbourhood filter $\mathcal{N}(x)$ is a subset of the principal filter at x.
Solution: If $U \in \mathcal{N}(x)$ then $x \in U$, by the definition of a neighbourhood, so U is an element of the principal filter. Every element of the neighbourhood filter is an element of the principal filter, so the neighbourhood filter is a subset of the principal filter.

3. Suppose X is infinite. The cofinite filter on X is defined to be the set of subsets of A of X such that $X \setminus A$ is finite. Show that it is indeed a filter.
Solution: Again, we check the four conditions.
   (a) $X \in \mathcal{F}$ since $X \setminus X$ is finite.
   (b) $\varnothing \notin \mathcal{F}$ because $X \setminus \varnothing$ is not finite.
   (c) If $A, B \in \mathcal{F}$ then $X \setminus A$ and $X \setminus B$ are finite. Let $C = A \cap B$. Then $A \supseteq C$ and $B \supseteq C$. Also
$$X \setminus C = X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$$
is finite so $C \in \mathcal{F}$.

4. Show that the cofinite filter is not contained in any principal filter.
Solution: For any $x \in X$ the set $\{x\}$ is finite so the set $X \setminus \{x\}$ belongs to the cofinite filter. It does not belong to the principal filter at x.

5. List all the filters on the set $\{1, 2, 3\}$.
Solution: Let $X = \{1, 2, 3\}$. There are 3 elements in X, 8 in $\wp(X)$ and 256 in $\wp(\wp(X))$. Since every filter on X is an element of $\wp(\wp(X))$ we could, in theory, list them and check which ones satisfy the four required conditions, but that would be very unpleasant, so it's better to choose a different approach.

   Suppose $\{1\} \in \mathcal{F}$. $\varnothing \notin \mathcal{F}$ because $\mathcal{F}$ is a filter. None of the sets $\{2\}$, $\{3\}$ or $\{2, 3\}$ belongs to $\mathcal{F}$ because for none of them is there a $C \neq \varnothing$ which is a subset of it and $\{1\}$. The sets $\{1,2\}$, $\{1,3\}$ and $\{1,2,3\}$ all belong to $\mathcal{F}$ because they are supersets of $\{1\}$. We've now accounted for all the subsets, so the filter could only be

$$\mathcal{F} = \{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}.$$

This is indeed a filter.

Similarly, if $\{2\} \in \mathcal{F}$ then

$$\mathcal{F} = \{\{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$$

and if $\{3\} \in \mathcal{F}$ then

$$\mathcal{F} = \{\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Suppose $\{1,2\} \in \mathcal{F}$. As usual $\varnothing \notin \mathcal{F}$. We've covered the cases where $\{1\} \in \mathcal{F}$, $\{2\} \in \mathcal{F}$ and $\{3\} \in \mathcal{F}$ above so only need to consider the case where $\{1\} \notin F$, $\{2\} \notin F$ and $\{3\} \notin F$, The only sets which are subsets of both $\{1,2\}$ and $\{1,3\}$ are $\varnothing$ and $\{1\}$, neither of which is in $\mathcal{F}$. So $\{1,3\} \notin \mathcal{F}$. Similarly, $\{2,3\} \notin \mathcal{F}$. On the other hand $\{1,2,3\} \in \mathcal{F}$ because it is a superset of $\{1,2\}$. So the only possibility is

$$\mathcal{F} = \{\{1,2\}, \{1,2,3\}\}$$

which is indeed a filter.

Similarly if $\{1,3\} \in \mathcal{F}$ then

$$\mathcal{F} = \{\{1,3\}, \{1,2,3\}\}$$

and if $\{2,3\} \in \mathcal{F}$ then

$$\mathcal{F} = \{\{2,3\}, \{1,2,3\}\}.$$

Suppose $\{1,2,3\} \in \mathcal{F}$. As always $\varnothing \notin \mathcal{F}$ and we've already considered the cases when any non-empty proper subsets are in $\mathcal{F}$ so the only remaining possibility is

$$\mathcal{F} = \{\{1,2,3\}\}.$$

$\mathcal{F} \neq \varnothing$ and $\varnothing \notin \mathcal{F}$ so $\mathcal{F}$ must contain some non-empty subset and we've enumerated all the possibilities above so the filters listed above are the only filters on $\{1,2,3\}$.