MAU22200 2021-2022 Practice Problem Set 2
Solutions

1. Lemma 1.9.4 in the notes gives the following two inclusions of sets. Give an example in each case to show that the inclusion cannot be replaced by equality in general.

   (a) $\varphi_*(A \cap B) \subseteq \varphi_*(A) \cap \varphi_*(B)$.
   Solution: Suppose $\varphi\colon \{a, b\} \to \{c\}$ is defined by $\varphi(a) = \varphi(b) = c$. Let $A = \{a\}$ and $B = \{b\}$ Then $\varphi_*(A) = \{c\}$ and $\varphi_*(B) = \{c\}$ so $\varphi_*(A) \cap \varphi_*(B) = \{c\} \cap \{c\} = \{c\}$ but $A \cap B = \varnothing$ so $\varphi_*(A \cap B) = \varnothing$.

   (b) $\varphi_*(A \setminus B) \supseteq \varphi_*(A) \setminus \varphi_*(B)$.
   Solution: With the same φ, A and B as above, $\varphi_*(A) \setminus \varphi_*(B) = \{c\} \setminus \{c\} = \varnothing$ but $A \setminus B = \{a\} \setminus \{b\} = \{a\}$ so $\varphi_*(A \setminus B) = \{a\}$.

2. Suppose $(X, \mathcal{T})$ is a topological space. Show that $\mathcal{T}$ is Hausdorff if and only if for all distinct $x, y \in X$ there are $P \in \mathcal{N}(x)$ and $Q \in \mathcal{N}(y)$ such that $P \cap Q = \varnothing$.
   Solution: Suppose $\mathcal{T}$ is Hausdorff. This means, by Definition 1.11.2 of the notes, that for all distinct $x, y \in X$ there are $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $V \cap W = \varnothing$. By Lemma 1.13.2 V is an open neighbourhood of x and W is an open neighbourhood of y. Open neighbourhoods are neighbourhoods so $V \in \mathcal{N}(x)$ and $W \in \mathcal{N}(y)$. $V \cap W = \varnothing$. So there are $P \in \mathcal{N}(x)$ and $Q \in \mathcal{N}(y)$ such that $P \cap Q = \varnothing$, namely $P = V$ and $Q = W$.

   Suppose, conversely, that for all distinct $x, y \in X$ there are $P \in \mathcal{N}(x)$ and $Q \in \mathcal{N}(y)$ such that $P \cap Q = \varnothing$. P is a neighbourhood of x so by Definition 1.13.1 there is a $V \in \mathcal{T}$ such that $x \in V$ and $V \subseteq P$. Similarly Q is a neighbourhood of y so there is a $W \in \mathcal{T}$ such that $y \in W$ and $W \subseteq Q$. From $V \subseteq P$ and $W \subseteq Q$ it follows that $V \cap W \subseteq P \cap Q$. But $P \cap Q = \varnothing$ so $V \cap W = \varnothing$. So we've just seen that for all distinct $x, y \in X$ there are $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $V \cap W = \varnothing$. In other words $\mathcal{T}$ is Hausdorff.

3. If X is a set then the *cofinite* topology on X is the set $\mathcal{T}$ consisting of those $U \in \wp(X)$ such that $U = \varnothing$ or $X \setminus U$ is finite.

   (a) Show that $\mathcal{T}$ is indeed a topology on X.
   Solution: The three conditions we need to check are listed in Definition 1.11.1.
   $\varnothing \in \mathcal{T}$. $X = X \setminus \varnothing$ and $\varnothing$ is finite so $X \in \mathcal{T}$. This establishes 1.11.1a.
   Suppose $V \in \mathcal{T}$ and $W \in \mathcal{T}$. If either $V = \varnothing$ or $W = \varnothing$ then $V \cap W = \varnothing$ and so $V \cap W \in \mathcal{T}$. The only remaining case is that $X \setminus V$ and $X \setminus W$ are both finite. But

$$= X \setminus (V \cap W)(X \setminus V) \cup (X \setminus W)$$

and the union of two finite sets is finite so $V \cap W \in \mathcal{T}$. This establishes 1.11.1b.

Suppose $\mathcal{E} \subseteq T$. If $\mathcal{E} = \varnothing$ or $\mathcal{E} = \{\varnothing\}$ then $\bigcup_{V \in \mathcal{E}} V = \varnothing$ and so $\bigcup_{V \in \mathcal{E}} V \in \mathcal{T}$. Otherwise $\mathcal{E}$ must contain a non-empty element of T. Call this element W. Then $X \setminus W$ is finite because of how $\mathcal{T}$ was defined. Now $W \in \mathcal{E}$ so

$$W \subseteq \bigcup_{V \in \mathcal{E}} V$$

and hence

$$X \setminus \bigcup_{V \in \mathcal{E}} V \subseteq X \setminus W.$$

Subsets of finite sets are finite so $X \setminus \bigcup_{V \in \mathcal{E}} V$ is finite and hence $\bigcup_{V \in \mathcal{E}} V \in \mathcal{T}$. This establishes 1.11.1c.

(b) Show that $\mathcal{T}$ is Hausdorff if and only if X is finite.

Solution: Suppose X is finite. If $A \in \wp(X)$ then $X \setminus A$ is finite, since subsets of finite sets are finite, so $A \in \mathcal{T}$. So $\wp(X) \subseteq \mathcal{T}$. But $\mathcal{T} \subseteq \wp(X)$. Therefore $\mathcal{T} = \wp(X)$. In other words, τ is the discrete topology, which we already know to be Hausdorff.

Suppose, conversely, that $\mathcal{T}$ is Hausdorff. If X has at most one element then it's certainly finite. If it has at least two elements then we choose distinct $x, y \in X$. Because X is Hausdorff there are $V, W \in \mathcal{T}$ such that $x \in V$, $y \in W$ and $V \cap W = \varnothing$. $x \in V$ so $V \neq \varnothing$ and hence $X \setminus V$ is finite. Similarly, $X \setminus W$ must be finite. Then

$$X = X \setminus \varnothing = X \setminus (V \cap W)(X \setminus V) \cup (X \setminus W)$$

and the union of two finite sets is finite so X is finite.