MAU22200 2021-2022 Practice Problem Set 1
Solutions

The following is a theorem in Real Analysis:

> Suppose f and g are functions from $\mathbf{R}$ to $\mathbf{R}$, that $w \in \mathbf{R}$ and that $z \in \mathbf{R}$. Suppose further that $f(x) \leq g(x)$ for all $x \in \mathbf{R}$ and that $\lim_{x\to w} f(x)$ and $\lim_{x\to w} g(x)$ exist. Then $\lim_{x\to w} f(x) \leq \lim_{x\to w} g(x)$.

Here is a proof:

> Suppose $\lambda > 0$. Let $\epsilon = \lambda/2$. Then $\epsilon > 0$. Let
>
> $$z_1 = \lim_{x\to w} f(x)$$
>
> and
>
> $$z_2 = \lim_{x\to w} g(x).$$
>
> By the definition of limits of real valued functions of a real variable there is a $\delta_1 > 0$ such that if $0 < |x - w| < \delta_1$ then $|f(x) - z_1| < \epsilon$ and a $\delta_2 > 0$ such that if $0 < |x - w| < \delta_2$ then $|g(x) - z_2| < \epsilon$. Let
>
> $$\delta = \min(\delta_1, \delta_2).$$
>
> Set $x = w + \delta/2$. Then $0 < |x - w| < \delta$ so $0 < |x - w| < \delta_1$ and $0 < |x - w| < \delta_2$. It follows that $|f(x) - z_1| < \epsilon$ and $|g(x) - z_2| < \epsilon$. Therefore
>
> $$z_1 < f(x) + \epsilon \leq g(x) + \epsilon < z_2 + 2\epsilon = z_2 + \lambda.$$
>
> So $\lim_{x\to w} f(x) \leq \lambda + \lim_{x\to w} f(g)$ for all positive λ, which is possible only if $\lim_{x\to w} f(x) \leq \lim_{x\to w} f(g)$.

1. State and prove the corresponding theorem for functions from $\mathbf{R}^m$ to $\mathbf{R}$, where $m > 0$.
   Solution: The corresponding theorem is

   > Suppose f and g are functions from $\mathbf{R}^m$ to $\mathbf{R}$, $\mathbf{w} \in \mathbf{R}^m$ and $z \in \mathbf{R}$. If $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^m$ and that $\lim_{\mathbf{x}\to\mathbf{w}} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{w}} g(\mathbf{x})$ exist. Then $\lim_{\mathbf{x}\to\mathbf{w}} f(\mathbf{x}) \leq \lim_{\mathbf{x}\to\mathbf{w}} g(\mathbf{x})$.

   and its proof is

   > Suppose $\lambda > 0$. Let $\epsilon = \lambda/2$. Then $\epsilon > 0$. Let
   >
   > $$z_1 = \lim_{\mathbf{x}\to\mathbf{w}} f(\mathbf{x})$$
   >
   > and
   >
   > $$z_2 = \lim_{\mathbf{x}\to\mathbf{w}} g(\mathbf{x}).$$

By the definition of limits of real valued functions of a real variable there is a $\delta_1 > 0$ such that if $0 < \|\mathbf{x} - \mathbf{w}\| < \delta_1$ then $|f(\mathbf{x}) - z_1| < \epsilon$ and a $\delta_2 > 0$ such that if $0 < \|\mathbf{x} - \mathbf{w}\| < \delta_2$ then $|g(\mathbf{x}) - z_2| < \epsilon$. Let

$$\delta = \min(\delta_1, \delta_2).$$

Set $\mathbf{x} = \mathbf{w} + \frac{\delta}{2}\mathbf{u}$, where $\mathbf{u} = (1, 0, \ldots, 0)$. Then $0 < \|\mathbf{x} - \mathbf{w}\| < \delta$ so $0 < \|\mathbf{x} - \mathbf{w}\| < \delta_1$ and $0 < \|\mathbf{x} - \mathbf{w}\| < \delta_2$. It follows that $|f(\mathbf{x}) - z_1| < \epsilon$ and $|g(\mathbf{x}) - z_2| < \epsilon$. Therefore

$$z_1 < f(\mathbf{x}) + \epsilon \leq g(\mathbf{x}) + \epsilon < z_2 + 2\epsilon = z_2 + \lambda.$$

So $\lim_{\mathbf{x}\to\mathbf{w}} f(\mathbf{x}) \leq \lambda + \lim_{\mathbf{x}\to\mathbf{w}} f(g)$ for all positive λ, which is possible only if $\lim_{\mathbf{x}\to\mathbf{w}} f(\mathbf{x}) \leq \lim_{\mathbf{x}\to\mathbf{w}} f(g)$.

2. State and prove the corresponding theorem for functions from a normed vector space to $\mathbf{R}$.
Solution: The corresponding theorem is

> Suppose that (X, p) is a normed vector space, that X is not the zero vector space, that f and g are functions from X to $\mathbf{R}$, that $\mathbf{w} \in X$ and that $z \in \mathbf{R}$. Suppose further that $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in X$ and that $\lim_{\mathbf{x}\to\mathbf{w}} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{w}} g(\mathbf{x})$ exist. Then $\lim_{\mathbf{x}\to\mathbf{w}} f(\mathbf{x}) \leq \lim_{\mathbf{x}\to\mathbf{w}} g(\mathbf{x})$.

and its proof is

Suppose $\lambda > 0$. Let $\epsilon = \lambda/2$. Then $\epsilon > 0$. Let

$$z_1 = \lim_{\mathbf{x}\to\mathbf{w}} f(\mathbf{x})$$

and

$$z_2 = \lim_{\mathbf{x}\to\mathbf{w}} g(\mathbf{x}).$$

By the definition of limits of real valued functions of a real variable there is a $\delta_1 > 0$ such that if $0 < p(\mathbf{x} - \mathbf{w}) < \delta_1$ then $|f(\mathbf{x}) - z_1| < \epsilon$ and a $\delta_2 > 0$ such that if $0 < p(\mathbf{x} - \mathbf{w}) < \delta_2$ then $|g(\mathbf{x}) - z_2| < \epsilon$. Let

$$\delta = \min(\delta_1, \delta_2).$$

X is not the zero vector space so there is $\mathbf{v} \neq \mathbf{0}$ in X. Set

$$\mathbf{u} = \frac{1}{p(\mathbf{v})}\mathbf{v}$$

Then $p(\mathbf{u}) = 1$. Set $\mathbf{x} = \mathbf{w} + \frac{\delta}{2}\mathbf{u}$. Then

$$p(\mathbf{x} - \mathbf{w}) = p\left(\frac{\delta}{2}\mathbf{u}\right) = \left|\frac{\delta}{2}\right| p(\mathbf{u}) = \frac{\delta}{2}$$

so $0 < p(\mathbf{x}-\mathbf{w}) < \delta$ and therefore $0 < p(\mathbf{x}-\mathbf{w}) < \delta_1$ and $0 < p(\mathbf{x}-\mathbf{w}) < \delta_2$. It follows that $|f(\mathbf{x})-z_1| < \epsilon$ and $|g(\mathbf{x})-z_2| < \epsilon$. Therefore

$$z_1 < f(\mathbf{x}) + \epsilon \leq g(\mathbf{x}) + \epsilon < z_2 + 2\epsilon = z_2 + \lambda.$$

So $\lim_{\mathbf{x}\to\mathbf{w}} f(\mathbf{x}) \leq \lambda + \lim_{\mathbf{x}\to\mathbf{w}} f(g)$ for all positive λ, which is possible only if $\lim_{\mathbf{x}\to\mathbf{w}} f(\mathbf{x}) \leq \lim_{\mathbf{x}\to\mathbf{w}} f(g)$.

3. State and prove the corresponding theorem for functions from a subset of a metric space to $\mathbf{R}$.
Solution: The corresponding theorem is

> Suppose that (X, d) is a metric space, that $U \subseteq X$, that f and g are functions from X to R, that $w \in X$ is a limit point of U and that $z \in R$. Suppose further that $f(x) \leq g(x)$ for all $x \in U$ and that $\lim_{x\to w} f(x)$ and $\lim_{x\to w} g(x)$ exist. Then $\lim_{x\to w} f(x) \leq \lim_{x\to w} g(x)$.

and its proof is

> Suppose $\lambda > 0$. Let $\epsilon = \lambda/2$. Then $\epsilon > 0$. Let
>
> $$z_1 = \lim_{x\to w} f(x)$$
>
> and
>
> $$z_2 = \lim_{x\to w} g(x).$$
>
> By the definition of limits of real valued functions of a real variable there is a $\delta_1 > 0$ such that if $0 < d(x, w) < \delta_1$ then $|f(x) - z_1| < \epsilon$ and a $\delta_2 > 0$ such that if $0 < d(x, w) < \delta_2$ then $|g(x) - z_2| < \epsilon$. Let
>
> $$\delta = \min(\delta_1, \delta_2).$$
>
> By assumption w is a limit point of U so there is an $x \in U$ such that $0 < d(x, w) < \delta$, and therefore $0 < d(x, w) < \delta_1$ and $0 < d(x, w) < \delta_2$. It follows that $|f(x) - z_1| < \epsilon$ and $|g(x) - z_2| < \epsilon$. Therefore
>
> $$z_1 < f(x) + \epsilon \leq g(x) + \epsilon < z_2 + 2\epsilon = z_2 + \lambda.$$
>
> So $\lim_{x\to w} f(x) \leq \lambda + \lim_{x\to w} f(g)$ for all positive λ, which is possible only if $\lim_{x\to w} f(x) \leq \lim_{x\to w} f(g)$.