MAU22200 2021-2022 Practice Problem Set 12
Solutions

1. Suppose X is a set, $\mathcal{B}$ is a σ-algebra on X, and $E, F \in \mathcal{B}$ are such that $E \cup F = X$ and $E \cap F = \varnothing$. Suppose μ_E and μ_F are measures on E and F, respectively. Show that there is a measure μ on X such that

$$\int_{s\in X} g(x)\,d\mu(s) = \int_{s\in E} g(x)\,d\mu_E(s) + \int_{s\in F} g(x)\,d\mu_F(s),$$

for all g which are integrable on X.
Note: While it's certainly possible to do this by hand I think it's easier to deduce it from the propositions in Section 9.5 on extension and restriction of measures.
Solution: Let ν_E and ν_F be the extensions of μ_E and μ_F by zero. In other words,

$$\nu_E(H) = \mu_E(E \cap H)$$

and

$$\nu_F(H) = \mu_F(F \cap H)$$

for all $H \in \mathcal{B}$. Then

$$\int_{s\in E} \chi_E(s)g(s)\,d\mu_E(s) = \int_{s\in X} \chi_E(s)g(s)\,d\nu_E(s)$$

and

$$\int_{s\in F} \chi_F(s)g(s)\,d\mu_F(s) = \int_{s\in X} \chi_F(s)g(s)\,d\nu_F(s)$$

by Proposition 9.5.2. Equivalently,

$$\int_{s\in E} g(s)\,d\mu_E(s) = \int_{s\in X} \chi_E(s)g(s)\,d\nu_E(s)$$

and

$$\int_{s\in F} g(s)\,d\mu_F(s) = \int_{s\in X} \chi_F(s)g(s)\,d\nu_F(s)$$

because $\chi_E(s)g(s) = g(s)$ for $s \in E$ and $\chi_F(s)g(s) = g(s)$ for $s \in F$. Define $\mu\colon \mathcal{B} \to [0, +\infty]$ by

$$\mu(H) = \nu_E(H) + \nu_F(H)$$

for all $H \in \mathcal{B}$. Then

$$\mu(H) = \mu_E(E \cap H) + \mu_F(F \cap H).$$

In particular,

$$\mu(H) = \mu_E(H)$$

if $H \subseteq E$ and

$$\mu(H) = \mu_F(H)$$

if $H \subseteq F$. So μ_E is the restriction of μ to E and μ_F is the restriction of μ to F. So

$$\int_{s\in E} g(s)\, d\mu_E(s) = \int_{s\in X} \chi_E(s) g(s)\, d\mu(s)$$

and

$$\int_{s\in F} g(s)\, d\mu_F(s) = \int_{s\in X} \chi_F(s) g(s)\, d\mu(s)$$

by Proposition 9.5.3. Now

$$g(s) = \chi_E(s)g(s) + \chi_F(s)g(s)$$

so

$$\int_{s\in X} g(x)\, d\mu(s) = \int_{s\in X} \chi_E(s)g(s)\, d\mu(s) + \int_{s\in X} \chi_F(s)g(s)\, d\mu(s).$$

Combining all the equations above,

$$\int_{s\in X} g(x)\, d\mu(s) = \int_{s\in E} g(s)\, d\mu_E(s) + \int_{s\in F} g(s)\, d\mu_F(s).$$

Strictly speaking, we're not quite done yet, because we still need to check that μ is a measure. This is easy though.

$$\mu(\varnothing) = \nu_E(\varnothing) + \nu_F(\varnothing) = 0 + 0 = 0$$

and

$$\begin{aligned}
\mu\left(\bigcup_{i=0}^{\infty} H_i\right) &= \nu_E\left(\bigcup_{i=0}^{\infty} H_i\right) + \nu_F\left(\bigcup_{i=0}^{\infty} H_i\right) \\
&= \sum_{i=0}^{\infty} \nu_E(H_i) + \sum_{i=0}^{\infty} \nu_F(H_i) \\
&= \sum_{i=0}^{\infty} (\nu_E(H_i) + \nu_F(H_i)) \\
&= \sum_{i=0}^{\infty} \mu(H_i)
\end{aligned}$$

if E_0, E_1, ... are disjoint elements of $\mathcal{B}$. So μ is indeed a measure.

2. (a) Suppose U is a non-empty subset of $\mathbf{R}$. Show that $m(U) > 0$. Here, as usual m denotes Lebegue measure.
Solution: U is non-empty so there's an $x \in U$. U is open so there's an $r > 0$ such that $B(x, r) \subseteq U$. $B(x, r)$ is the interval $(x - r, x - r)$.

The length of this interval is $2r$, so its Jordan content is $2r$ and therefore its Lebesgue measure is $2r$. From $B(x,r) \subseteq U$ it follows that $m(B(x,r)) \le m(U)$, so

$$m(U) \ge 2r > 0.$$

(b) Show that if f and g are continuous functions from $\mathbf{R}$ to $\mathbf{R}$ and $f(x) = g(x)$ for almost all $x \in \mathbf{R}$, with respect to Lebesgue measure, then $f(x) = g(x)$ for all x.
Solution: Let $h = f - g$. Then h is continuous and

$$\{x \in \mathbf{R}\colon f(x) \ne g(x)\} = h^*((-\infty, 0) \cup (0, +\infty))$$

is open.

$$m(\{x \in \mathbf{R}\colon f(x) \ne g(x)\}) = 0$$

by assumption so the set must be empty, by the preceding part.

(c) Give an example of a Borel measure μ on $\mathbf{R}$ which does not have the property from the first part of the question, i.e. one for which there is a non-empty open set U with $\mu(U) = 0$.
Solution: Let

$$\mu(E) = \begin{cases} 1 & \text{if } 0 \in E, \\ 0 & \text{if } 0 \notin E, \end{cases}$$

i.e. Dirac measure at 0. Then $(0, +\infty) \ne \varnothing$ and $\mu((0, +\infty)) = 0$.

3. Suppose $(X, \mathcal{B}, \mu)$ is a measurable space and E_0, E_1, … is a sequence of elements of $\mathcal{B}$ such that

$$\sum_{i=0}^{\infty} \mu(E_i) < +\infty.$$

Let F_j be the set of x such that $x \in E_i$ for at least j values of i. Show that

$$\mu(F_j) \le \frac{1}{j} \sum_{i=0}^{\infty} \mu(E_i).$$

Hint: Apply Markov's Inequality and the Monotone Convergence Theorem.
Solution: Let

$$g(x) = \sum_{i=0}^{\infty} \chi_{E_i}(x).$$

Then $x \in F_j$ if and only if $g(x) \ge j$. By Markov's inequality then

$$\mu(F_j) \le \frac{1}{j} \int_{x \in X} g(x)\, d\mu(x).$$

By the Monotone Convergence Theorem we have

$$\begin{aligned}
\int_{x\in X} g(x)\,d\mu(x) &= \int_{x\in X} \lim_{n\to\infty} \sum_{i=0}^{n} \chi_{E_i}(x)\,d\mu(x) \\
&= \lim_{n\to\infty} \int_{x\in X} \sum_{i=0}^{n} \chi_{E_i}(x)\,d\mu(x) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} \int_{x\in X} \chi_{E_i}(x)\,d\mu(x) \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} \mu(E_i) = \sum_{i=0}^{\infty} \mu(E_i).
\end{aligned}$$