MAU22200 2021-2022 Practice Problem Set 12

1. Suppose X is a set, $\mathcal{B}$ is a σ-algebra on X, and $E, F \in \mathcal{B}$ are such that $E \cup F = X$ and $E \cap F = \varnothing$. Suppose μ_E and μ_F are measures on E and F, respectively. Show that there is a measure μ on X such that

$$\int_{s\in X} g(x)\, d\mu(s) = \int_{s\in E} g(x)\, d\mu_E(s) + \int_{s\in F} g(x)\, d\mu_F(s),$$

for all g which are integrable on X.
Note: While it's certainly possible to do this by hand I think it's easier to deduce it from the propositions in Section 9.5 on extension and restriction of measures.

2. (a) Suppose U is a non-empty subset of $\mathbf{R}$. Show that $m(U) > 0$. Here, as usual m denotes Lebegue measure.
   (b) Show that if f and g are continuous functions from $\mathbf{R}$ to $\mathbf{R}$ and $f(x) = g(x)$ for almost all $x \in \mathbf{R}$, with respect to Lebesgue measure, then $f(x) = g(x)$ for all x.
   (c) Give an example of a Borel measure μ on $\mathbf{R}$ which does not have the property from the first part of the question, i.e. one for which there is a non-empty open set U with $\mu(U) = 0$.

3. Suppose $(X, \mathcal{B}, \mu)$ is a measurable space and E_0, E_1, ... is a sequence of elements of $\mathcal{B}$ such that

$$\sum_{i=0}^{\infty} \mu(E_i) < +\infty.$$

Let F_j be the set of x such that $x \in E_i$ for at least j values of i. Show that

$$\mu(F_j) \le \frac{1}{j} \sum_{i=0}^{\infty} \mu(E_i).$$

Hint: Apply Markov's Inequality and the Monotone Convergence Theorem.