MAU22200 2021-2022 Practice Problem Set 11
Solutions

1. (a) Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are locally compact σ-compact Hausdorff topological spaces, $X = X \times Y$ and $\mathcal{T}_Z$ is the product topology on Z. Show that $(Z, \mathcal{T}_Z)$ is a locally compact σ-compact Hausdorff topological space.

   Solution: X is locally compact so for each $x \in X$ there is compact neighbourhood of X, i.e. a compact K such that there is an open U such that $x \in U$ and $U \subseteq K$. Y is locally compact so for each $y \in Y$ there is compact neighbourhood of Y, i.e. a compact L such that there is an open V such that $y \in V$ and $V \subseteq L$. Then $(x, y) \in U \times V$, $U \times V \subseteq K \times L$, $U \times V$ is open in Z and $K \times L$ is compact. So $K \times L$ is a compact neighbourhood of (x, y). Every $(x, y) \in Z$ has such a neighbourhood so Z is locally compact.

   X is σ-compact so there are compact K_0, K_1, ... such that $X = \bigcup_{i=0}^{\infty} K_i$. Y is σ-compact so there are compact L_0, L_1, ... such that $Y = \bigcup_{j=0}^{\infty} L_j$. If $(x, y) \in Z$ then $x \in X$ and $y \in Y$ so there are K_i and L_j such that $x \in K_i$, $y \in L_j$ and $(x, y) \in K_i \times L_j$. So $Z = \bigcup_{(i,j)\in\mathbf{N}^2} K_i \times L_j$. $K_i \times L_j$ is compact and $\mathbf{N}^2$ is countable so Z is σ-compact.

   The product of Hausdorff topological spaces is Hausdorff so Z is Hausdorff.

   (b) Suppose (X, d) is a metric space such that $\bar{B}(x, r)$ is compact for all $x \in X$ and $r > 0$. Show that $(X, \mathcal{T})$ is a locally compact σ-compact Hausdorff topological space, where $\mathcal{T}$ is the topology induced by the metric.

   Solution: For any x and r we have $x \in B(x, r)$ and $B(x, r) \subseteq \bar{B}(x, r)$. $B(x, r)$ is open so $\bar{B}(x, r)$ is a compact neighbourhood of x. Every x has such a neighbourhood so X is locally compact.

   If $X = \varnothing$ then X is the union of an empty set of compact sets and the empty set is countable so X is σ-compact. If $X \neq \varnothing$ then there is an $x \in X$. Let $K_n = \bar{B}(x, n)$. Then K_n is compact. If $y \in X$ then there is an n such that $d(x, y) \leq n$ and hence $y \in K_n$. Therefore $y \in \bigcup_{j=0}^{\infty} K_j$. This holds for all $y \in X$ so $X \subseteq \bigcup_{j=0}^{\infty} K_j$. The reverse inclusion also holds because $K_j \in X$ for each j. So X is σ-compact.

   All metric spaces are Hausdorff so X is Hausdorff.

2. Let $\mathcal{C}$ be the space of compactly supported continuous real valued functions on $\mathbf{R}^2$. Define

$$I_1(g) = \int_c^d \int_a^b g(x, y)\, dx\, dy$$

and

$$I_2(g) = \int_a^b \int_c^d g(x, y)\, dy\, dx$$

for $g \in \mathcal{C}$. These are of course Riemann integrals. a, b, c, and d are such that the support of g is a subset of $(a,b)\times(c,d)$. As long as this condition is satisfied the integrals

$$\int_c^d \int_a^b g(x,y)\,dx\,dy$$

and

$$\int_a^b \int_c^d g(x,y)\,dy\,dx$$

are independent of which a, b, c, and d are chosen, so $I_1(g)$ and $I_2(g)$. There is a version of Fubini's theorem for Riemann integration which implies $I_1(g) = I_2(g)$ for all $g \in \mathcal{C}$. This may or may not have been proved in first year but you may assume it for purposes of this problem.

(a) Show that there is a unique Radon measure μ on $\mathbf{R}^2$ such that

$$I_1(g) = \int_{(x,y)\in\mathbf{R}^2} g(x,y)\,d\mu(x,y) = I_2(g)$$

for all $g \in \mathcal{C}$.
Solution: $\mathbf{R}^2$ is a locally compact σ-compact Hausdorff topological space so we can apply the Riesz Representation Theorem. To do this we need to check that I_1 is linear and that $I_1(g) \geq$ if $g(x,y) \geq 0$ for all (x,y). These follow easily from familiar properties of the Riemann integral. If $p, q \in \mathbf{R}$ and $f, g \in \mathcal{C}$ then

$$\begin{aligned} I_1(pf+qg) &= \int_c^d \int_a^b (pf(x,y) + qg(x,y))\,dx\,dy \\ &= \int_c^d \left(p\int_a^b f(x,y)\,dx + q\int_a^b g(x,y)\,dx \right) dy \\ &= p\int_c^d \int_a^b f(x,y)\,dx\,dy + q\int_c^d \int_a^b g(x,y)\,dx\,dy \\ &= pI_1(f) + qI_1(g). \end{aligned}$$

If $g(x,y) \geq 0$ for all (x,y) then

$$\int_a^b g(x,y)\,dx \geq 0$$

for all y so

$$\int_c^d \int_a^b g(x,y)\,dx\,dy \geq 0.$$

The Riesz Representation Theorem then gives the existence of a unique measure μ such that

$$I_1(g) = \int_{(x,y)\in\mathbf{R}^2} g(x,y)\,d\mu(x,y).$$

Combining this with Fubini's Theorem gives

$$I_1(g) = \int_{(x,y)\in\mathbf{R}^2} g(x,y)\,d\mu(x,y) = I_2(g).$$

(b) Show that if E and F are Borel sets in $\mathbf{R}$ then

$$\mu(E \times F) = m(E)m(F).$$

where m is Lebesgue measure on $\mathbf{R}$.
Note: The cases where one or both sets have zero or infinite measure require somewhat different arguments so assume for simplicity that $0 < \mu(E) < +\infty$ and $0 < \mu(F) < +\infty$.
Hint: If χ_E and χ_F were compactly supported continuous functions then this would be easy, but that can't happen. Lebesgue measure is a Radon measure so every Borel set has a compact subset which is not much smaller than it and an open superset which is not much larger than it. There is then a compactly supported continuous function which is equal to 1 on the compact set and equal to 0 outside the open set. This function is in some sense a good approximation to the characteristic function of the original set.
Solution: Suppose $0 < p < m(E) < q$ and $0 < r < m(F) < s$. m is a Radon measure so there are compact K and L and open U and V in $\mathbf{R}$ such that

$$m(K) > p,$$
$$m(U) < q,$$
$$m(L) > r,$$

and

$$m(V) < s.$$

By the variant of Urysohn's Lemma from the notes there are compactly supported continuous functions $g\colon \mathbf{R} \to [0,1]$ and $h\colon \mathbf{R} \to [0,1]$ such that $g(x) = 1$ if $x \in K$, $g(x) = 0$ if $x \notin U$, $h(y) = 1$ if $y \in L$ and $h(y) = 0$ if $y \notin V$. Then

$$\chi_K(x) \le g(x) \le \chi_U(x)$$

for all x and

$$\chi_L(y) \le h(y) \le \chi_V(y)$$

for all y. Define f by

$$f(x,y) = g(x)h(y).$$

f is compactly supported and continuous so

$$I_1(f) = \int_{(x,y)\in\mathbf{R}^2} f(x,y)\,d\mu(x,y).$$

Now

$$
\begin{aligned}
I_1(f) &= \int_c^d \int_a^b f(x,y)\,dx\,dy \\
&= \int_c^d \int_a^b g(x)h(y)\,dx\,dy \\
&= \int_c^d h(y) \int_a^b g(x)\,dx\,dy \\
&= \int_a^b g(x)\,dx \int_c^d h(y)\,dy.
\end{aligned}
$$

The Lebesgue measure m was defined so as to make the Lebesgue integral of a compactly supported continuous function equal to the Riemann integral so

$$
\int_a^b g(x)\,dx = \int_{x\in\mathbf{R}} g(x)\,dm(x)
$$

and

$$
\int_c^d h(y)\,dy = \int_{y\in\mathbf{R}} h(y)\,dm(y).
$$

Also, by the monotonicity properties of the integral,

$$
\begin{aligned}
p < m(K) &= \int_{x\in\mathbf{R}} \chi_K(x)\,dm(x) \le \int_{x\in\mathbf{R}} g(x)\,dm(x) \\
&\le \int_{x\in\mathbf{R}} \chi_U(x)\,dm(x) = m(U) < q
\end{aligned}
$$

and

$$
\begin{aligned}
r < m(L) &= \int_{y\in\mathbf{R}} \chi_L(y)\,dm(y) \le \int_{y\in\mathbf{R}} h(y)\,dm(y) \\
&\le \int_{y\in\mathbf{R}} \chi_V(y)\,dm(y) = m(V) < s
\end{aligned}
$$

Combining the equations and inequalities above we get

$$
pr \le m(K)m(L) \le \int_{(x,y)\in\mathbf{R}^2} f(x,y)\,d\mu(x,y) \le m(U)m(V) \le qs.
$$

Now

$$
\chi_{K\times L}(x,y) \le f(x,y) \le \chi_{U\times V}(x,y)
$$

so

$$\begin{aligned}\mu(K \times L) &= \int_{(x,y)\in\mathbf{R}^2} \chi_{K\times L}(x,y)\,d\mu(x,y)\\ &\le \int_{(x,y)\in\mathbf{R}^2} f(x,y)\,d\mu(x,y)\\ &\le \int_{(x,y)\in\mathbf{R}^2} \chi_{K\times L}(x,y)\,d\mu(x,y)\\ &= \mu(U \times V).\end{aligned}$$

We also have

$$K \times L \subseteq E \times F \subseteq U \times V$$

so

$$\mu(K \times L) \le \mu(E \times F) \le \mu(U \times V).$$

All of the the above hold in particular for $p = \kappa m(E)$, $q = m(E)/\kappa$, $r = \kappa m(F)$ and $s = m(F)/\kappa$, where $\kappa \in (0,1)$. Then

$$\begin{aligned}\kappa^2 m(E)m(F) \le m(K)m(L) &\le \int_{(x,y)\in\mathbf{R}^2} f(x,y)\,d\mu(x,y)\\ &\le m(U)m(L) \le \frac{m(E)m(F)}{\kappa^2},\end{aligned}$$

so

$$m(E\times F) \le \mu(U\times V) \le \frac{1}{\kappa^2}\int_{(x,y)\in\mathbf{R}^2} f(x,y)\,d\mu(x,y) \le \frac{1}{\kappa^4} m(E)m(F)$$

and

$$m(E\times F) \ge \mu(K\times L) \ge \kappa^2 \int_{(x,y)\in\mathbf{R}^2} f(x,y)\,d\mu(x,y) \ge \kappa^4 m(E)m(F).$$

These hold for all $\kappa \in (0,1)$ so

$$m(E \times F) \le m(E)m(F)$$

and

$$m(E \times F) \ge m(E)m(F).$$