MAU22200 2021-2022 Practice Problem Set 11

1. (a) Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are locally compact σ-compact Hausdorff topological spaces, $X = X \times Y$ and $\mathcal{T}_Z$ is the product topology on Z. Show that $(Z, \mathcal{T}_Z)$ is a locally compact σ-compact Hausdorff topological space.

   (b) Suppose (X, d) is a metric space such that $\bar{B}(x, r)$ is compact for all $x \in X$ and $r > 0$. Show that $(X, \mathcal{T})$ is a locally compact σ-compact Hausdorff topological space, where $\mathcal{T}$ is the topology induced by the metric.

2. Let $\mathcal{C}$ be the space of compactly supported continuous real valued functions on $\mathbf{R}^2$. Define

$$I_1(g) = \int_c^d \int_a^b g(x, y)\, dx\, dy$$

and

$$I_2(g) = \int_a^b \int_c^d g(x, y)\, dy\, dx$$

for $g \in \mathcal{C}$. These are of course Riemann integrals. a, b, c, and d are such that the support of g is a subset of $(a, b) \times (c, d)$. As long as this condition is satisfied the integrals

$$\int_c^d \int_a^b g(x, y)\, dx\, dy$$

and

$$\int_a^b \int_c^d g(x, y)\, dy\, dx$$

are independent of which a, b, c, and d are chosen, so $I_1(g)$ and $I_2(g)$. There is a version of Fubini's theorem for Riemann integration which implies $I_1(g) = I_2(g)$ for all $g \in \mathcal{C}$. This may or may not have been proved in first year but you may assume it for purposes of this problem.

   (a) Show that there is a unique Radon measure μ on $\mathbf{R}^2$ such that

$$I_1(g) = \int_{(x,y) \in \mathbf{R}^2} g(x, y)\, d\mu(x, y) = I_2(g)$$

   for all $g \in \mathcal{C}$.

   (b) Show that if E and F are Borel sets in $\mathbf{R}$ then

$$\mu(E \times F) = m(E) m(F).$$

   where m is Lebesgue measure on $\mathbf{R}$.
   Note: The cases where one or both sets have zero or infinite measure require somewhat different arguments so assume for simplicity that

$0 < \mu(E) < +\infty$ and $0 < \mu(F) < +\infty$.
Hint: If χ_E and χ_F were compactly supported continuous functions then this would be easy, but that can't happen. Lebesgue measure is a Radon measure so every Borel set has a compact subset which is not much smaller than it and an open superset which is not much larger than it. There is then a compactly supported continuous function which is equal to 1 on the compact set and equal to 0 outside the open set. This function is in some sense a good approximation to the characteristic function of the original set.