MAU22200 2021-2022 Practice Problem Set 10

1. Suppose E_0, E_1, ... is a sequence of elements of a σ-algebra on a set X. Let F be the set of $x \in X$ such that $x \in E_n$ for all but finitely many $n \in \mathbf{N}$. Let G be the set of all $x \in X$ such that $x \in E_n$ for infinitely many $n \in \mathbf{N}$

   (a) Show that $F \subseteq G$.

   (b) Show that $F \in \mathcal{B}$.
   Hint: What is $\bigcup_{m \geq 0} \bigcap_{n \geq m} E_n$?

   (c) Show that $G \in \mathcal{B}$.
   Hint: What is $\bigcap_{m \geq 0} \bigcup_{n \geq m} E_n$?

2. Suppose that $(X, \mathcal{B}, \mu)$ is a measure space an E_0, E_1, ... is a sequence of elements of $\mathcal{B}$ such that
$$\lim_{n \to \infty} \mu(E_n) = 0.$$
Let F be the set of $x \in X$ such that $x \in E_n$ for all but finitely many $n \in \mathbf{N}$. Show that $\mu(F) = 0$.
Hint: Apply Fatou's Lemma to the sequence of characteristic functions of the E's.

3. Suppose that $(X, \mathcal{B}, \mu)$ is a measure space an E_0, E_1, ... is a sequence of elements of $\mathcal{B}$ such that
$$\sum_{i=0}^{\infty} \mu(E_i) < +\infty.$$
Let G be the set of all $x \in X$ such that $x \in F_n$ for infinitely many $n \in \mathbf{N}$ Show that $\mu(G) = 0$.
Hint: Apply the Monotone Convergence Theorem to a sequence of sums of characteristic functions.